218(10): Lagrangian theory of Three Dimensional Orbits

Using the cylindrical polar coordinates (r, θ, z):

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \qquad -(1)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \qquad -(2)$$

$$\frac{\partial \mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}} \qquad -(3)$$

where:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - U(r, z) \qquad -(4)$$

From eqs. (1) and (2):

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2}F(r) \qquad -(5)$$

where:

$$L = mr^2\frac{d\theta}{dt} \qquad -(6)$$

$$F(r) = -\frac{\partial U}{\partial r} \qquad -(7)$$

From eqs. (3) and (4):

$$m\ddot{z} = F(z) \qquad -(8)$$

where

$$F(z) = -\frac{\partial U}{\partial z} \qquad -(9)$$

2) The radial vector is defined as:

$$\underset{\sim}{R} = r\,\underline{e}_r + Z\,\underline{e}_z \quad -(10)$$

so

$$R^2 = r^2 + Z^2 \quad -(11)$$

The distance between mass m and $\underline{M}$ is $|\underline{R}|$.
So the new universal potential is:

$$U(R) = -\frac{x^2 m M G}{R} + \frac{(x^2-1)L^2}{2mR^2} \quad -(12)$$

$$= -\frac{x^2 m M G}{(r^2+Z^2)^{1/2}} + \frac{(x^2-1)L^2}{2m(r^2+Z^2)^{3/2}}$$

Now define:

$$\underline{R} = r\,\underline{e}_r + Z\,\underline{e}_z := R\,\underline{e}_r' \quad -(13)$$

where $\underline{e}_r'$ is aligned between m and $\underline{M}$. Then:

$$\underline{v} = \underline{\dot{R}} = \dot{R}\,\underline{e}_r' + R\dot{\theta}_1\,\underline{e}_\theta' \quad -(14)$$

and

$$v^2 = \dot{R}^2 + R^2\dot{\theta}_1^2. \quad -(15)$$

The Lagrangian is therefore:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}_1^2\right) - U(R) \quad -(16)$$

and the potential $U(R)$ is defined by eq. (12).

3) The Euler Lagrange equation is:

$$\frac{\partial \mathcal{L}}{\partial R} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{R}} \quad -(17)$$

Eqs. (2) and (17) give:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{R}\right) + \frac{1}{R} = -\frac{mR^2}{L_1^2}F(R). \quad -(18)$$

where

$$L_1 = mR^2\frac{d\theta_1}{dt}. \quad -(19)$$

The orbit corresponding to eqs. (18) and (19) is:

$$\boxed{R = \frac{\alpha}{1+\epsilon\cos(x\theta_1)}} \quad -(20)$$

where:

$$\alpha = \frac{L_1^2}{mk}, \quad \epsilon = \left(1 + \frac{2EL_1^2}{mk^2}\right)^{1/2}, \quad k = mMG \quad -(21)$$

where

$$E = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\theta}_1^2\right) + U(R) \quad -(22)$$

For an ellipse, $E < 0$, $0 < \epsilon < 1$; for a hyperbola $E > 0$, $\epsilon > 1$; for a parabola $E = 0$, $\epsilon = 1$.

4) So:

$$R = \left(r^2 + Z^2\right)^{1/2} = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(23)$$

which is a processing elliptical spiral, i.e a precessing helix.

The transformation used in deriving this result is:

$$\underline{R} = R\,\underline{e}_r' = r\,\underline{e}_r + Z\,\underline{e}_Z \quad -(24)$$

so

$$\underline{v} = \dot{\underline{R}} = \dot{R}\,\underline{e}_r' + R\,\dot{\underline{e}}_r' \quad -(25)$$

where:

$$\dot{\underline{e}}_r' = \dot{\theta}_1\,\underline{e}_\theta' \quad -(26)$$

i.e

$$\underline{v} = \dot{R}\,\underline{e}_r' + R\dot{\theta}_1\,\underline{e}_\theta' \quad -(27)$$

$$= \dot{r}\,\underline{e}_r + \dot{Z}\,\underline{e}_Z + r\dot{\theta}\,\underline{e}_\theta .$$

Therefore:

$$v^2 = \dot{R}^2 + R^2\dot{\theta}_1^2 \quad -(28)$$

$$= \dot{r}^2 + \dot{Z}^2 + r^2\dot{\theta}^2$$

However:

$$\dot{R}^2 = \dot{r}^2 + \dot{Z}^2 \quad -(29)$$

so:

$$R^2 \theta_1^2 = r^2 \theta^2 \quad - (30)$$

$$\theta_1 = \frac{r}{R} \theta \quad - (31)$$

Final Result

The orbit is:

$$R = \frac{\alpha}{1 + \epsilon \cos(x \theta_1)} \quad - (32)$$

where

$$\theta_1 = \frac{r}{R} \theta. \quad - (32)$$

In the original plane:

$$Z = 0 \quad - (33)$$

and

$$r = \frac{\alpha}{1 + \epsilon \cos(x \theta)}. \quad - (34)$$

The conserved angular momentum is:

$$L_1 = m R^2 \frac{d\theta_1}{dt} = \text{constant} \quad - (35)$$

Using the result:

$$R^2 = \text{constant} \quad - (36)$$