218(3): Transition from Fractal Conical Section to Hyperbolic Spiral

In order to define this transition consider the class of fractal conical sections with:

$$\epsilon < 0 , \quad - (1)$$

for example:

$$\epsilon = -1 . \quad - (2)$$

Then:

$$r = \frac{\alpha}{1 - \cos(x\theta)} \quad - (3)$$

In the limit:

$$x\theta << 1 \quad - (4)$$

$$\cos(x\theta) = 1 - \frac{x^2\theta^2}{2} + \dots \quad - (5)$$

so

$$r = \frac{2\alpha}{x^2 \theta^2} \quad - (6)$$

which is a hyperbolic spiral of type:

$$r = \frac{r_0}{\theta^2} . \quad - (7)$$

Now apply the lagrangian equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L_1^2} F(r) \quad - (8)$$

to eq. (7). From eqs. (7) and (8):

$$F(r) = -\frac{L_1^2}{mr^2}\left(\frac{2}{r_0} + \frac{1}{r}\right) \quad - (9)$$

2) which is of the type:

$$F(r) = -x^2 \frac{mMG}{r^2} - (1-x^2)\frac{L^2}{m\, r^3} \quad -(10)$$

if:

$$x << 1, \quad x \neq 0. \quad -(11)$$

Comparing eqs. (9) and (10):

$$x^2 mMG = \frac{2L_1^2}{m r_0} \quad -(12)$$

and

$$(1-x^2)\frac{L^2}{m^2} = \frac{L_1^2}{m^2} \quad -(13)$$

i.e.

$$\boxed{\begin{aligned} L_1^2 &= (1-x^2)L^2 \\ &= \frac{1}{2} x^2 m^2 MG r_0 \end{aligned}} \quad -(14)$$

Graphical Work

Investigate systematically the class of frontal conical sections with:

$$E < 0. \quad -(15)$$

PS: In eq (15): $r_0 = \frac{2d}{x^2} = \frac{2L^2}{x^2 mMG}$

so $L_1^2 = L^2 \quad -(16)$

This result is self consistent if $x \to 0$