218(7): Generalized Force Law in Polar Representation

A polar plot of the force law can be made in terms of the general $x(\theta)$ as follows:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{1}{r} + \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\right) \quad -(1)$$

This equation is transformed into $F(\theta)$ using:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon\cos(x\theta)\right) \quad -(2)$$

and:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\left(\left(x + \theta\frac{dx}{d\theta}\right)^2\cos(\theta x(\theta)) + \left(2\frac{dx}{d\theta} + \theta\frac{d^2x}{d\theta^2}\right)\sin(\theta x(\theta))\right) \quad -(3)$$

Then for any given $x(\theta)$, F can be plotted as a function of θ in a polar plot using eq. (3).

The Special Case of Constant x.

In this case:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}x^2\cos(\theta x) \quad -(4)$$

2) and:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{1}{r} - \frac{\epsilon x^{+1+2}}{\alpha}\cos(x\theta)\right) \quad -(5)$$

where $$\alpha = \frac{L^2}{m^2 MG}. \quad -(6)$$

So: $$F(r) = -\frac{L^2}{mr^3} + \frac{\epsilon m M G x^2 \cos(x\theta)}{r^2}. \quad -(7)$$

Here: $$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(8)$$

So: $$F(r) = -\frac{L^2}{mr^3} + \frac{mMG x^2 \alpha}{r^3} - \frac{x^2 mMG}{r^2}$$

$$F(r) = -x^2\frac{mMG}{r^2} + (x^2-1)\frac{L}{mr^3} \quad -(9)$$

In polar representation:

$$F(\theta) = \frac{1}{\alpha^2}\left(1+\epsilon\cos(x\theta)\right)^2\left(-x^2 mMG + \frac{(x^2-1)L}{m\alpha}\left(1+\epsilon\cos(x\theta)\right)\right) \quad -(10)$$

i.e. for the constant x fractal conical sections:

3)

$$F(\theta) = \left(1+\epsilon\cos(x\theta)\right)^2\left[\frac{(x^2-1)L}{m\alpha^3}\left(1+\epsilon\cos(x\theta)\right) - \frac{x^2 mMG}{\alpha^3}\right] \quad -(11)$$

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Any planar orbit can be described by the general function $x(\theta)$ in eq. (2). The general orbit is:

$$r = f(\theta) = \frac{\alpha}{1+\epsilon\cos(\theta\, x(\theta))} \quad -(12)$$