218(5): Force Law for Combined Fractal Conical Section and Hyperbolic Spiral or Any Orbit.

In this case:

$$r = \frac{\alpha}{\theta} = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(1)$$

so

$$x = \frac{r}{\alpha} \cos^{-1}\left(\frac{\alpha - r}{\epsilon r}\right) \quad -(2)$$

i.e. x is a function of r.

Therefore:

$$\frac{dx}{d\theta} = \frac{dx}{dr}\frac{dr}{d\theta}. \quad -(3)$$

From Lagrangian theory the force law is:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right) \quad -(4)$$

where

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos(x\theta)\right). \quad -(5)$$

Therefore:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{\epsilon}{\alpha}\frac{d}{d\theta}\cos(x\theta) \quad -(6)$$

$$= -\left(\frac{d(x\theta)}{d\theta}\sin(x\theta)\right)\frac{\epsilon}{\alpha}$$

and:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\frac{d}{d\theta}\left(\frac{d(x\theta)}{d\theta}\sin(x\theta)\right) \quad -(7)$$

2)

i.e.:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\left[\frac{d^2(x\theta)}{d\theta^2}\sin(x\theta) + \frac{d(x\theta)}{d\theta}\frac{d}{d\theta}\left(\sin(x\theta)\right)\right]$$

$$= -\frac{\epsilon}{\alpha}\left[\frac{d^2(x\theta)}{d\theta^2}\sin(x\theta) + \left(\frac{d(x\theta)}{d\theta}\right)^2\cos(x\theta)\right] \quad -(8)$$

Here:

$$\frac{d(x\theta)}{d\theta} = x + \theta\frac{dx}{d\theta} \quad -(9)$$

$$= x + \theta\frac{dx}{dr}\frac{dr}{d\theta},$$

$$\frac{d^2(x\theta)}{d\theta^2} = \frac{d}{d\theta}\left(x + \theta\frac{dx}{d\theta}\right) = \frac{dx}{d\theta} + \frac{dx}{d\theta} + \theta\frac{d^2x}{d\theta^2}$$

$$= 2\frac{dx}{d\theta} + \theta\frac{d}{d\theta}\left(\frac{dx}{d\theta}\right) \quad -(10)$$

From the equation of the hyperbolic spiral:

$$r = \frac{\alpha}{\theta}. \quad -(11)$$

Therefore:

$$\theta\frac{dr}{d\theta} = -r \quad -(12)$$

From eqs. (9) and (12):

$$\frac{d(x\theta)}{d\theta} = x - r\frac{dx}{dr} \quad -(13)$$

$$:= f(r)$$

5) Therefore:

$$\frac{d^2(x\theta)}{d\theta^2} = \frac{df}{d\theta} = \frac{df}{dr}\frac{dr}{d\theta}$$

$$= -\alpha r^2 \frac{df}{dr} \qquad -(14)$$

Finally:

$$\frac{df}{dr} = \frac{dx}{dr} - \frac{d}{dr}\left(r\frac{dx}{dr}\right)$$

$$= \frac{dx}{dr} + \frac{dx}{dr} - r\frac{d^2x}{dr^2}$$

$$\frac{d^2(x\theta)}{d\theta^2} = 2\frac{dx}{dr} - r\frac{d^2x}{dr^2} \qquad -(15).$$

Therefore the force law is:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{1}{r} + \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)\right) \qquad -(16)$$

where:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{d}\left(A\sin(x\theta) + B\cos(x\theta)\right), \qquad -(17)$$

$$A = 2\frac{dx}{dr} - r\frac{d^2x}{dr^2} \qquad -(18)$$

$$B = \left(x - r\frac{dx}{dr}\right)^2 \qquad -(19)$$

4)

$$x = \frac{r}{\alpha} \cos^{-1}\left(\frac{\alpha - r}{\epsilon r}\right) \quad -(20)$$

$$\frac{dx}{dr} = \frac{d}{dr}\left(\frac{r}{\alpha} \cos^{-1}\left(\frac{\alpha - r}{\epsilon r}\right)\right) \quad -(21)$$

$$\frac{d^2 x}{dr^2} = \frac{d}{dr}\left(\frac{dx}{dr}\right) \quad -(22)$$

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(23)$$

$$\sin(x\theta) = \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad -(24)$$

In general, for x a function of r:

$$\frac{d(x\theta)}{d\theta} = x + \theta \frac{dx}{dr}\frac{dr}{d\theta} \quad -(25)$$

$$\frac{d^2(x\theta)}{d\theta^2} = 2\frac{dx}{d\theta} + \theta \frac{d}{d\theta}\frac{dx}{d\theta} \quad -(26)$$

$$\frac{dx}{d\theta} = \frac{dx}{dr}\frac{dr}{d\theta} \quad -(27)$$

So in general:

$$\boxed{r = f(\theta) = \frac{\alpha}{1 + \epsilon\cos(x\theta)}} \quad -(28)$$

and any function $f(\theta)$ can be expressed as a fractal conical section, i.e <u>any orbit</u>.