218(4): General Spiral from Fractal Conical Section

In general:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos(x\theta)\right) \quad -(1)$$

$$= \frac{1}{\alpha}\left(1+\epsilon\left(1-\frac{x^2\theta^2}{2!}+\frac{x^4\theta^4}{4!}-\frac{x^6\theta^6}{6!}+\dots\right)\right)$$

for

$$x\theta < 1, \quad -(2)$$

for example if: $x \ll 1, \quad 0 \leq \theta \leq 2\pi. \quad -(3)$

It is seen that the conical section is given by an infinite sum of spirals of increasing order in θ.

The force law from eq. (1) is:

$$F(r) = -\frac{kx^2}{r^2} - \frac{k\alpha}{r^3}\left(1-x^2\right) \quad -(4)$$

$$= -x^2\frac{mMG}{r^2} - \frac{(1-x^2)L^2}{mr^3} \quad -(5)$$

and this must also be the force law for the Maclaurin expansion of $\cos(x\theta)$.

This shows that the general fractal conical sections can be expanded in a Maclaurin series of hyperbolic spirals, provided that eq. (2) is true.

So:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{x^2}{\alpha} + \frac{12x^4\theta^2}{4!\,\alpha} - \frac{30}{6!}\frac{x^6\theta^4}{\alpha} + \dots$$

— (6).

In the limit

$$x \to 0, \; x \neq 0 \quad -(7)$$

$$\frac{1}{r} \to \frac{1}{\alpha}\left(1 + \epsilon\left(1 - \frac{x^2\theta^2}{2}\right)\right) - (8)$$

and

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{x^2\epsilon}{\alpha}. \quad -(9)$$

Now use:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2}F(r)$$

— (10)

so

$$F(r) = -\frac{L^2}{mr^2}\left(-\frac{x^2\epsilon}{\alpha} + \frac{1}{r}\right) \quad -(11)$$

$$= -\frac{kx^2}{r^2} - \frac{k\alpha}{r^3}\left(1 - x^2\right)$$

i.e

$$\frac{\epsilon L^2}{\alpha m} = -k \;; \quad x \to 0 \quad -(12)$$

using

$$\alpha = \frac{L^2}{mk} \quad -(13)$$

then

$$\epsilon = -1 \quad -(14)$$