218(2): Force Law for the Hyperbolic Spiral Orbit.

In this case:

$$r = \frac{r_0}{\theta} \quad -(1)$$

so

$$\frac{1}{r} = \frac{\theta}{r_0} \quad -(2)$$

The force law for this orbit is found from the Lagrangian equation:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2} F(r). \quad -(3)$$

From eq. (2):

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = 0 \quad -(4)$$

so

$$F(r) = -\frac{L^2}{mr^3}. \quad -(5)$$

The universal force law is:

$$F(r) = -\frac{mMGx^2}{r^2} - \frac{L^2}{mr^3}\left(1 - x^2\right) \quad -(6)$$

Therefore eq. (5) is found from eq. (6) as follows.

1) If $x < 1$ (ellipse):

$$x \to 0 \quad -(7)$$

and eq. (6) reduces to eq. (5)

2) If $x > 1$ (hyperbola), then eq. (7) gives:

$$r = -\frac{r_0}{\theta} \quad -(8)$$

2) The force law for eq. (1) is:

$$F(r) = - \frac{L^2 \theta^3}{m r_0^3} \quad - (9)$$

and the force law for eq. (8) is:

$$F(r) = \frac{L^2 \theta^3}{m r_0^3} \quad - (10)$$

Therefore this procedure transforms the fractal conical section:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad - (11)$$

into the hyperbolic spirals:

$$r = \pm \frac{r_0}{\theta} . \quad - (12)$$

* So it transforms solar system orbits into galactic orbits using the same universal force law (6).

Note that the ellipticity ϵ does not appear in the universal force law. Therefore this procedure is true for all ϵ.

A circular orbit can be represented by a constant r. So a circular orbit

3) is obtained from:

$$x \to 0 \quad \text{---} \quad (13)$$

giving:

$$r = \frac{\alpha}{1+\epsilon} = \text{constant} \quad - (14)$$

For a circle:

$$\epsilon = 0 \quad - (15)$$

$$r = \alpha \quad - (16)$$

so

the force law for a circle is:

$$F(r) = -\frac{L^2}{mr^3} = -\frac{L^2}{m\alpha^3} \quad - (17)$$

$$= \underline{\text{constant.}}$$