218(9): Transformation into the General Function $f(\theta)$

In this case:

$$\frac{1}{r} = f(\theta) = \frac{1}{\alpha}\left(1 + \epsilon\cos\left(\theta x(\theta)\right)\right) \quad -(1)$$

so

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = f'(\theta) = -\frac{\epsilon}{\alpha}\left(x + \theta\frac{dx}{d\theta}\right)\sin(x\theta) \quad -(2)$$

and

$$\boxed{\frac{dx}{d\theta} = -\frac{1}{\theta}\left(x + \frac{\alpha}{\epsilon}\frac{f'(\theta)}{\sin(x\theta)}\right)} \quad -(3)$$

This can be transformed further using:

$$\frac{dx}{d\theta} = \frac{dx}{dr}\frac{dr}{d\theta}, \quad -(4)$$

so:

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{f(\theta)}\right) = -\frac{f'(\theta)}{f^2(\theta)}; \quad -(5)$$

i.e.

$$\boxed{\frac{dr}{d\theta} = -r^2 f'(\theta)} \quad -(6)$$

From eqs. (3) and (6):

$$\boxed{\frac{dx}{dr} = \frac{1}{\theta}\left(\frac{x}{r^2 f'(\theta)} + \frac{\alpha}{r\left(\epsilon^2 r^2 - (\alpha - r)^2\right)^{1/2}}\right)} \quad -(7)$$

Eq. (7) transforms the generalized fractal conical section (GCS) into $f(\theta)$.

In eq. (7):

$$f'(\theta) = \frac{df(\theta)}{d\theta} \qquad -(8)$$

and θ is defined by:

$$\theta = f^{-1}\left(\frac{1}{r}\right), \qquad -(9)$$

i.e. from the inverse of f:

$$\frac{1}{r} = f(\theta). \qquad -(10)$$

<u>Theorem</u>

Any function $f(\theta)$ can be expressed as a GCS using the transformation eq. (7).

This theorem is akin to the Fourier theorem, where any function $f(\theta)$ can be expressed as a Fourier series.

<u>Example</u>, <u>Hyperbolic Spiral</u>

Here:

$$\theta r = r_0 \qquad -(11)$$

$$\frac{1}{r} = \frac{\theta}{r_0}, \quad f'(\theta) = \frac{1}{r_0}, \qquad -(12)$$

so:

$$\frac{dx}{dr} = \frac{x}{r} + \frac{\alpha}{r_0\left(\epsilon^2 r^2 - (\alpha - r)^2\right)^{1/2}}$$

as in note 218(8), QED.