2(8(8)): Transformation into the Hyperbolic Spiral.

Consider the generalized fractal conical section:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos\ \theta(x(\theta))\right) \quad - (1)$$

then:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\sin(x\theta)\left(x + \theta\frac{dx}{d\theta}\right) - (2)$$

where

$$\sin(x\theta) = \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} - (3)$$

For the hyperbolic spiral:

$$\frac{1}{r} = \frac{\theta}{r_0}, \quad \frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{1}{r_0}. \quad - (4)$$

So:

$$x + \theta\frac{dx}{d\theta} = -\frac{\alpha}{\epsilon r_0 \sin(x\theta)}. \quad - (5)$$

This is a differential equation for $dx/d\theta$ and can be integrated numerically:

$$\boxed{\frac{dx}{d\theta} = -\frac{x}{\theta} - \frac{\alpha}{\epsilon r_0 \theta \sin(x\theta)}} - (6)$$

Eq. (6) can be transformed using:

$$\frac{dx}{d\theta} = \frac{dx}{dr}\frac{dr}{d\theta}, \quad - (7)$$

$$= -\frac{r^2}{r_0}\frac{dx}{dr},$$

so in eq. (5):

2)

$$\theta \frac{dx}{d\theta} = -r \frac{dx}{dr} \quad -(8)$$

and

$$\frac{dx}{dr} = \frac{x}{r} + \frac{\alpha}{\epsilon r r_0}\left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{-1/2}$$

$-(9)$

i.e.

$$\boxed{\frac{dx}{dr} = \frac{x}{r} + \frac{\alpha}{r_0\left(\epsilon^2 r^2 - (\alpha - r)^2\right)^{1/2}}}$$

$-(10)$

Under the conditions (6) and (10):

$$\boxed{r = \frac{r_0}{\theta} = \frac{\alpha}{1 + \epsilon\cos(\theta x(\theta))}} \quad -(11)$$

If

$$r_0 = \alpha \quad -(12)$$

then:

$$\frac{dx}{d\theta} = -\frac{1}{\theta}\left(x + \frac{1}{\epsilon \sin(x\theta)}\right) \quad -(13)$$

and

$$\frac{dx}{dr} = \frac{x}{r} + \frac{1}{\left(\epsilon^2 r^2 - (\alpha - r)^2\right)^{1/2}}$$

$-(14)$