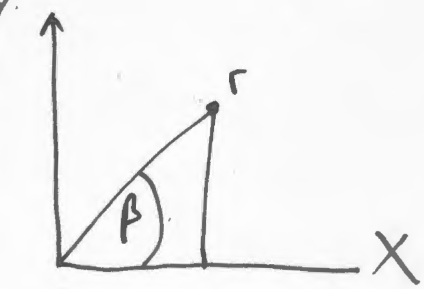
314(5). Meaning of the (r, β) Coordinate System

By definition:

$$X = r\cos\beta, \quad Y = r\sin\beta, \quad -(1)$$

$$\beta = x\theta. \quad -(2)$$



Consider:

$$\theta \rightarrow \theta + 2\pi. \quad -(3)$$

then

$$\beta \rightarrow \beta + 2\pi x. \quad -(4)$$

and

$$X_1 = r\cos(\beta + 2\pi x) = r\left(\cos\beta\cos(2\pi x) - \sin\beta\sin(2\pi x)\right) \quad -(5)$$

$$Y_1 = r\sin(\beta + 2\pi x) = r\left(\sin\beta\cos(2\pi x) + \cos\beta\sin(2\pi x)\right). \quad -(6)$$

Here

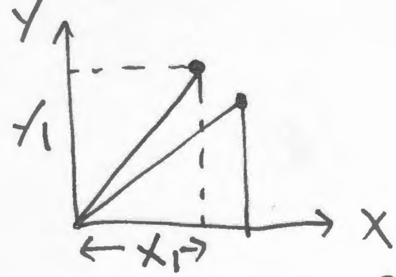
$$r^2 = X^2 + Y^2 = X_1^2 + Y_1^2. \quad -(7)$$

Note that:

$$X_1 < X \quad -(8)$$

so eq (3) rotates the vector anticlockwise after θ has increased by 2π.

↺ r rotated if $\theta \rightarrow \theta + 2\pi$



This rotation is a precession. It is equivalent to rotating the axes clockwise after θ has increased from θ to $\theta + 2\pi$.

2)

So eq. (1) is a cylindrical polar system with axes rotating clockwise.

The rotation of the axes mean that there is a connection present in the geometry. The unit vectors of the system are:

$$\underline{e}_r = \underline{i}\cos\beta + \underline{j}\sin\beta \quad -(9)$$

$$\underline{e}_\beta = -\underline{i}\sin\beta + \underline{j}\cos\beta \quad -(10)$$

so:

$$\underline{e}_\beta = \frac{d\underline{e}_r}{d\beta}, \quad \underline{e}_r = -\frac{d\underline{e}_\beta}{d\beta} \quad -(11)$$

It follows that:

$$d\underline{e}_r = d\beta\,\underline{e}_\beta \quad -(12)$$

$$d\underline{e}_\beta = -d\beta\,\underline{e}_r \quad -(13)$$

and

$$\dot{\underline{e}}_r = \frac{d\underline{e}_r}{dt} = \dot{\beta}\,\underline{e}_\beta \quad -(14)$$

$$\dot{\underline{e}}_\beta = \frac{d\underline{e}_\beta}{dt} = -\dot{\beta}\,\underline{e}_r \quad -(15)$$

The linear velocity is:

$$\underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}\left(r\,\underline{e}_r\right) \quad -(16)$$

$$= \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r$$

$$\boxed{\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\beta}\,\underline{e}_\beta} \quad -(17)$$

3) This leads to the lagrangian used in note 214(4)

$$\mathcal{L} = \frac{1}{2} m v^2 - U(r) \quad - (18)$$

$$\boxed{\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right) - U(r)} \quad - (19)$$

We may now proceed as in section 3 of UFT 196, by considering the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos \beta} \quad - (20)$$

The conserved angular momentum is:

$$L = m r^2 \dot{\beta} = m r^2 \frac{d\beta}{dt} \quad - (21)$$

From eq. (20):

$$\frac{dr}{d\beta} = \frac{\epsilon}{\alpha} r^2 \sin \beta \quad - (22)$$

so

$$\frac{dr}{dt} = \frac{dr}{d\beta} \frac{d\beta}{dt} = \left(\frac{L \epsilon}{m \alpha} \right) \sin \beta \quad - (23)$$

and

$$\boxed{\dot{r} = \left(\frac{L \epsilon}{m \alpha} \right) \sin \beta, \quad \dot{\beta} = \frac{L}{m r^2}} \quad - (24)$$

In the next note we will proceed as in UFT 196 to find the U that gives eq. (20).