214(4) : Hooke / Newton Force Law for the Precessing Ellipse.

The precessing elliptical orbit is defined by:

$$r = \frac{\alpha}{1+\epsilon\cos\beta} \quad -(1)$$

where

$$\beta = x\theta. \quad -(2)$$

This is

Adapt the plane polar coordinate system (r, β) for determining the optimal choice of coordinates for the problem of the force law for the orbit.

Consider the Lagrangian:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\beta}^2) - U(r) \quad -(3)$$

The Euler Lagrange equations are:

$$\frac{\partial\mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} = 0 \quad -(4)$$

and

$$\frac{\partial\mathcal{L}}{\partial\beta} = 0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\beta}}. \quad -(5)$$

The total angular momentum is conserved, so:

$$L = \frac{\partial\mathcal{L}}{\partial\dot{\beta}} = mr^2\dot{\beta} = \text{constant} \quad -(6)$$

and

$$\frac{dL}{dt} = 0. \quad -(7)$$

From eqs. (3) and (4):

$$m(\ddot{r} - r\dot{\beta}^2) = -\frac{\partial U}{\partial r} = F(r) \quad -(8)$$

) Here $F(r)$ is the force. Denote:

$$u = \frac{1}{r} \quad -(9)$$

then

$$\frac{du}{d\beta} = -\frac{1}{r^2}\frac{dr}{d\beta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\beta} \quad -(10)$$

From eq. (6):

$$\dot{\beta} = \frac{L}{mr^2} \quad -(11)$$

so

$$\frac{du}{d\beta} = -\frac{m}{L}\dot{r} \quad -(12)$$

Therefore

$$\frac{d^2u}{d\beta^2} = \frac{d}{d\beta}\left(-\frac{m}{L}\dot{r}\right) = \frac{dt}{d\beta}\frac{d}{dt}\left(-\frac{m\dot{r}}{L}\right)$$

$$= -\frac{m\ddot{r}}{L\dot{\beta}} \quad -(13)$$

$$= -\frac{m^2}{L^2}r^2\ddot{r},$$

and:

$$\ddot{r} = -\frac{L^2}{m^2}u^2\frac{d^2u}{d\beta^2} \quad -(14)$$

$$r\dot{\beta}^2 = \frac{L^2}{m^2}u^3 \quad -(15)$$

So

$$\boxed{\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{m}{L^2}r^2F(r)} \quad -(16)$$

) where

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos\beta\right). \quad -(17)$$

Therefore:

$$\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) = -\frac{\epsilon}{\alpha}\cos\beta \quad -(18)$$

and:

$$\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha} = -\frac{m}{L^2} r^2 F(r) \quad -(19)$$

so

$$F(r) = -\frac{L^2}{m \alpha r^2}. \quad -(20)$$

As in the previous note:

$$\alpha = \frac{L^2}{mk} \quad -(21)$$

so

$$\boxed{F(r) = -\frac{k}{r^2}} \quad -(22)$$

where

$$k = mMG \quad -(23)$$

So in the coordinate system (r, β) the precessing elliptical orbit (1) is obtained from eq. (22). This proves that Einsteinian general relativity is not needed for this problem. Conversely the precessing elliptical orbit does not prove Einsteinian general relativity at all.