214(8): Rotation Generator and Connection for Rotating Coordinate System

If a coordinate system is described by:

$$X = r\cos\theta, \quad Y = r\sin\theta \quad -(1)$$

then the passive rotation of axes is described by:

$$\underline{i}' = \underline{i}\cos\theta - \underline{j}\sin\theta \quad -(2)$$

$$\underline{j}' = \underline{i}\sin\theta + \underline{j}\cos\theta, \quad -(3)$$

$$\begin{bmatrix} \underline{i}' \\ \underline{j}' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix}. \quad -(4)$$

Similarly, if a coordinate system is described by:

$$X = r\cos\beta, \quad Y = r\sin\beta \quad -(5)$$

then the passive rotation of axes is described by:

$$\underline{i}' = \underline{i}\cos\beta - \underline{j}\sin\beta \quad -(6)$$

$$\underline{j}' = \underline{i}\sin\beta + \underline{j}\cos\beta \quad -(7)$$

$$\begin{bmatrix} \underline{i}' \\ \underline{j}' \end{bmatrix} = \begin{bmatrix} \cos(x\theta) & -\sin(x\theta) \\ \sin(x\theta) & \cos(x\theta) \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix}. \quad -(8)$$

Denote:

$$R_z = \begin{bmatrix} \cos(x\theta) & -\sin(x\theta) \\ \sin(x\theta) & \cos(x\theta) \end{bmatrix} \quad -(9)$$

then

$$\left.\frac{dR_z}{d\theta}\right|_{\theta=0} = x\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad -(10)$$

) Therefore:

$$J_{ij} = -ix\epsilon_{ij} \quad -(11)$$

Defining the antisymmetric unit tensor by:

$$\epsilon_{ijk} = \epsilon_{ij}\epsilon_k \quad -(12)$$

then

$$\epsilon_{ij} = \epsilon^k_{ij}\epsilon_k \quad -(13)$$

and

$$\boxed{\Gamma^k_{ij} = \frac{x}{r}\epsilon^k_{ij}} \quad -(14)$$

This is the Christoffel connection for the rotation defined by:

$$X = r\cos(x\theta), \; Y = r\sin(x\theta). \quad -(15)$$

It is concluded that the precessing elliptical orbit can be described by the Christoffel connection (14) and coordinate system (15).
