214(3): The Effect of a Precessing Elliptical orbit on Kepler's Law.

The precessing ellipse is:

$$r = \frac{\alpha}{1 + \epsilon \cos \beta} \quad -(1)$$

where

$$\beta = x\theta. \quad -(2)$$

Here α is the half right magnitude, ϵ is the ellipticity and x the precession constant. The coordinate system is (r, θ) in the plane:

$$dZ = 0. \quad -(3)$$

Write:

$$\beta(r) = \int \frac{dr}{r^2 \left(2m \left(E + \frac{k}{r} - \frac{L^2}{2mr^2} \right) \right)^{1/2}} \quad -(4)$$

where m, k, L and E are constants. Then:

$$\cos \beta = \left(\frac{L^2}{2mk} - 1 \right) \left(1 + \frac{2EL^2}{mk^2} \right)^{-1/2} \quad -(5)$$

Eq. (1) and (5) are the same if:

$$\alpha = \frac{L^2}{nk}, \quad \epsilon = \left(1 + \frac{2EL^2}{nk^2} \right)^{1/2}. \quad -(6)$$

Now write

$$\frac{d\beta}{dr} = \frac{d\beta}{dt} \frac{dt}{dr} \quad -(7)$$

2) where:
$$\frac{dr}{dt} = \left(\frac{2}{m}\left(E - U\right) - \frac{L^2}{m^2 r^2} \right)^{1/2}, \quad -(8)$$

then:
$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} + U(r). \quad -(9)$$

Now define:
$$L = mr^2 \frac{d\beta}{dt}, \quad -(10)$$

then:
$$E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right) + U(r). \quad -(11)$$
$$= \frac{1}{2} m v^2 + U(r)$$

where the linear velocity is
$$v = \left(\dot{r}^2 + r^2 \dot{\beta}^2 \right)^{1/2} \quad -(12)$$

therefore:
$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \dot{\beta}^2 \quad -(13)$$

In eq. (11) E is the total energy and U the potential energy.

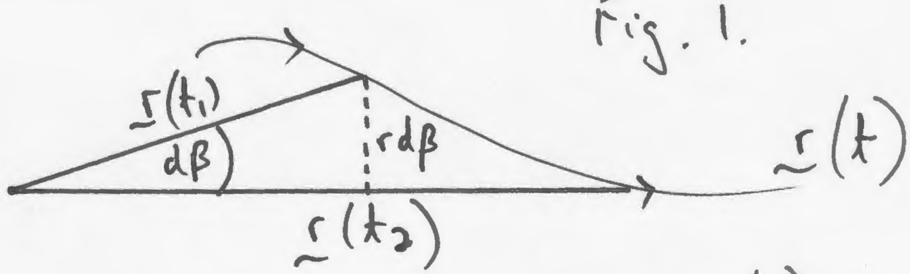
Define the Lagrangian:
$$\mathcal{L} = \frac{1}{2} m v^2 - U(r) \quad -(14)$$

then the total angular momentum L is:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\beta}} = mr^2 \dot{\beta} = mr^2 \frac{d\beta}{dt} \quad -(15)$$

$$\boxed{L = mr^2 \frac{d\beta}{dt}} \quad -(16)$$

Fig. 1.



With reference to Fig 1, then for any curve $\underset{\sim}{r}(t)$:

$$\boxed{dA = \frac{1}{2} r^2 d\beta} \quad -(17)$$

which is a general law of geometry. So:

$$\boxed{\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\beta}{dt} = \frac{L}{2m}} \quad -(18)$$

In eq. (16):

$$L = x m r^2 \frac{d\theta}{dt} \quad -(19)$$

and in eq. (18):

$$\frac{dA}{dt} = \frac{1}{2} x r^2 \frac{d\theta}{dt} \quad -(20)$$

4) If $$x = 1 \quad -(21)$$
then eq. (20) is the second law of Kepler (1609), equal area in equal times.

Provided that the total angular momentum is defined as in eq. (19), the Kepler law is unchanged by the precession of the ellipse.
The area law (17) is:
$$\boxed{\frac{dA}{d\theta} = \frac{x r^2}{2}} \quad -(22)$$

Using the Euler Lagrange equation:
$$\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \quad -(23)$$
with the lagrangian (14) it is found that:
$$\frac{d^2}{d\beta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{m r^2}{L} F(r) \quad -(24)$$
where
$$F(r) = -\frac{\partial U}{\partial r} \quad -(25)$$

Therefore if:
$$r = \frac{\alpha}{1 + \epsilon \cos\beta} \quad -(26)$$
then from eq. (24):

$$F(r) = -\frac{mMG}{r^2} \quad -(27)$$

where

$$k = mMG \quad -(28)$$

In this description:

$$\frac{dA}{d\beta} = \frac{dA}{dr}\frac{dr}{d\beta} \quad -(29)$$

where

$$\frac{dr}{d\beta} = \left(\frac{\epsilon\beta}{\alpha}\right) r^2 \sin\beta \quad -(30)$$

and

$$\frac{dA}{dt} = \frac{dA}{d\beta}\frac{d\beta}{dt} \quad -(31)$$

$$= \frac{1}{2} r^2 \frac{d\beta}{dt},$$

$$\frac{dA}{d\beta} = \frac{1}{2} r^2 \quad -(32)$$

Therefore in eq. (29)

$$\frac{1}{2} r^2 = \frac{dA}{dr}\left(\frac{\epsilon\beta}{\alpha}\right) r^2 \sin\beta \quad -(33)$$

$$\boxed{\frac{dA}{dr} = \frac{1}{2}\left(\frac{\alpha}{\epsilon\beta}\right)\frac{1}{\sin\beta}} \quad -(34)$$

Eq. (34) is the correct observational law.

6) The Einstein theory does not give the correct result for $\sin\beta$ so will not give the correct
$\frac{dA}{dt}$. The correct Kepler law is:

$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{d\beta}\frac{d\beta}{dt} \quad -(35)$$

and the Einstein theory will give an incorrect dA/dr, and incorrect $dr/d\beta$, and an incorrect $d\beta/dt$.

Finally note that the Hooke inverse square law (27) will give a precessing ellipse provided that the coordinate system is (r, β) and not (r, θ). So there is no need of EGR at all.