214(7): Calculation of Acceleration in the $(r, \beta)$ Coordinate System

As in UFT 196 the acceleration is calculated using:

$$\underline{a} = \frac{d\underline{v}}{dt} = (\ddot{r} - r\dot{\beta}^2)\,\underline{e}_r + (r\ddot{\beta} + 2\dot{r}\dot{\beta})\,\underline{e}_\beta \quad -(1)$$

with

$$\dot{r} = \left(\frac{L\epsilon}{md}\right)\sin\theta, \quad \dot{\theta} = \frac{L}{mr^2} \quad -(2)$$

Therefore:

$$\ddot{r} = \left(\frac{L\epsilon}{md}\right)\frac{d}{dt}(\sin\beta), \quad \ddot{\beta} = \frac{L}{m}\frac{d}{dt}\left(\frac{1}{r^2}\right) \quad -(3)$$

Now use the chain rules:

$$\frac{df(r)}{d\beta} = \frac{df(r)}{dr}\frac{dr}{d\beta} \quad -(4)$$

$$\frac{df(\beta)}{dr} = \frac{df(\beta)}{d\beta}\frac{d\beta}{dr} \quad -(5)$$

to find:

$$\frac{d}{dt}\left(\frac{1}{r^2}\right) = -\frac{2}{r^3}\frac{dr}{dt} \quad -(6)$$

$$\frac{d}{dt}(\sin\beta) = \cos\beta\,\frac{d\beta}{dt} \quad -(7)$$

Therefore:

$$\ddot{r} = \left(\frac{\epsilon L^2}{m^2 d}\right)\frac{1}{r^2}\cos\beta, \quad \ddot{\beta} = -\left(\frac{2L^2\epsilon}{m^2 d}\right)\frac{\sin\beta}{r^3}. \quad -(8)$$

It follows that:

$$\ddot{r} - r\dot{\beta}^2 = \frac{\epsilon MG}{r^2}\cos\beta - \frac{L^2}{m^2 r^3} \quad -(9)$$

2) where $G$ is Newton's constant and:

$$\alpha = \frac{L^2}{m^2 M G}. \quad — (10)$$

Similarly: $$r\ddot{\beta} + 2\dot{r}\dot{\beta} = 0, \quad — (11)$$

so the force is:

$$\underset{\sim}{F} = m\left(\frac{\epsilon M G}{r^2}\cos\beta - \frac{L^2}{m^2 r^3}\right)\underline{e}_r. \quad — (12)$$

Now use: $$\epsilon\cos\beta = \frac{\alpha}{r} - 1, \quad — (13)$$

so the force is:

$$\boxed{\begin{aligned}\underline{F} &= \left(-\frac{mMG}{r^2} + \frac{L^2}{mr^3} - \frac{L^2}{mr^3}\right)\underline{e}_r \quad — (14)\\ &= -\frac{mMG}{r^2}\underline{e}_r\end{aligned}}$$

<u>Therefore the precessing ellipse is given by the inverse square law and the centrifugal forces cancel.</u>