213(16): The Idea of Parallel Transport and the Geodesic Equation.

Consider the position vector $\underset{\sim}{r}$ in three dimensional Euclidean space. In Cartesian coordinates:

$$\underset{\sim}{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(1)$$

$$= \underset{\sim}{r}(X, Y, Z)$$

The tangent vectors are:

$$\underline{e}_1 = \partial \underline{r} / \partial X \quad -(2)$$

$$\underline{e}_2 = \partial \underline{r} / \partial Y \quad -(3)$$

$$\underline{e}_3 = \partial \underline{r} / \partial Z \quad -(4)$$

So

$$\underline{e}_1 = \underline{i} \; ; \; \underline{e}_2 = \underline{j} \; ; \; \underline{e}_3 = \underline{k} . \quad -(5)$$

In this case the unit vectors are the tangent vectors.

In curvilinear coordinates the unit tangent vectors are defined as in VAPS 1096 ("Vector Analysis Problem Solver"):

$$\underline{e}_i = \frac{\partial \underline{r}}{\partial u_i} \Big/ \left| \frac{\partial \underline{r}}{\partial u_i} \right| \quad -(6)$$

$$= \underset{\sim}{r}(u_1, u_2, u_3)$$

In the case of Cartesian coordinates:

$$|\underline{i}| = |\underline{j}| = |\underline{k}| = 1 . \quad -(7)$$

2) It is seen that $\underline{r}$ is parameterised by X, Y and Z, or more generally u_1, u_2 and u_3.

In the arbitrary manifold, consider:

$$x^\mu = x^\mu(\lambda). \quad -(8)$$

In four dimensional spacetime for example:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, X, Y, Z). \quad -(9)$$
$$= (ct, \underset{\sim}{r})$$

The parameter λ can be t, X, Y, or Z, or the proper time τ. In this notation the tangent vector is denoted:

$$\boxed{k^\mu = \frac{dx^\mu}{d\lambda}} \quad -(10)$$

Now consider a scalar quantity T in Minkowski space. By the chain rule:

$$\frac{dT}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial T}{\partial x^\mu}. \quad -(11)$$

The idea of parallel transport means that:

$$\boxed{\frac{dT}{d\lambda} = 0} \quad -(12)$$

Now consider the vector:

$$\underset{\sim}{T} = T_x \underline{i} + T_y \underline{j} + T_z \underline{k} \quad -(13)$$

$$= \underset{\sim}{T}(X, Y, Z)$$

Then:

$$\frac{d\underset{\sim}{T}}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial \underline{T}}{\partial x^\mu} \quad -(14)$$

where

$$\lambda = X, Y, Z. \quad -(15)$$

Parallel transport means:

$$\frac{dT_x}{dX} = \frac{dT_y}{dY} = \frac{dT_z}{dZ} = 0 \quad -(16)$$

and

$$\underline{\nabla} \cdot \underline{T} = 0. \quad -(17)$$

Consider next the vector

$$T = (T^0, T^1, T^2, T^3). \quad -(18)$$

then:

$$\frac{dT^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial T^\nu}{\partial x^\mu} \quad -(19)$$

In general it is seen that the operator:

$$\boxed{\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu} \quad -(20)$$

in the arbitrary manifold. The basis:

$$e_\mu = \partial_\mu \quad -(21)$$

4) is the coordinate basis. In Cartan geometry:

$$e_\mu = q^a_{\ \mu}\, e_a \quad -(22)$$

where e_a is the arbitrary orthogonal basis of the tangent space. A special case of eq. (20) is:

$$\frac{dx^\nu}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial x^\nu}{\partial x^\mu} \quad -(23)$$

in which case

$$\boxed{\frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu} \quad -(24)$$

as used in note 213(9).

In general the four derivative in these equations must be replaced by the covariant derivative, so:

$$\boxed{\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} D_\mu} \quad -(25)$$

For the arbitrary tensor:

$$\boxed{\left(\frac{DT}{d\lambda}\right)^{\mu_1, \mu_2, \ldots, \mu_k}_{\nu_1, \nu_2, \ldots, \nu_k} = \frac{dx^\sigma}{d\lambda} D_\sigma T^{\mu_1, \ldots, \mu_k}_{\nu_1, \ldots, \nu_k} = 0} \quad -(26)$$

for parallel transport to occur.

For the four vector T^μ:

$$\frac{dx^\sigma}{d\lambda}\left(\partial_\sigma T^\mu + \Gamma^\mu_{\sigma\rho} T^\rho\right) = 0 \quad -(27)$$

i.e.

$$\boxed{\frac{dT^\mu}{d\lambda} + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda} V^\rho = 0} \quad -(28)$$

which is the equation of parallel transport. This equation is true for the anti-symmetric connection:

$$\Gamma^\mu_{\sigma\rho} = -\Gamma^\mu_{\rho\sigma} \quad -(29)$$

The metric compatibility condition is:

$$D_\sigma g_{\mu\nu} = 0 \quad -(30)$$

so:

$$\frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\sigma}{d\lambda} D_\sigma g_{\mu\nu} = 0 \quad -(31)$$

The metric is always parallel transported.

The tangent vector to a path $x^\mu(\lambda)$ is defined by eq. (10). If this is parallel transported then:

$$\frac{D k^\mu}{d\lambda} = 0 \quad -(32)$$

$$\boxed{= \frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = \frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0}$$

6) This is the geodesic equation. In a flat space:

$$\frac{d^2 x^\mu}{d\lambda^2} = 0 \quad - (33)$$

In the Euclidean space:

$$x^i = (x^0, x^1, x^2) \quad - (34)$$
$$= (X, Y, Z)$$

and

$$\frac{d}{d\lambda}\left(\frac{dX}{d\lambda}\right) = 0 \quad - (35)$$

and so on.

So eq. (33) is the Euclidean space is that of the straight line.

In general relativity free particles move along geodesics, i.e.

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \quad - (36)$$

but in the presence of a force:

$$\boxed{\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = \frac{F^\mu}{m}} \quad - (37)$$

where m is mass.

The next note will reduce this to the Newton equation with the correct antisymmetry of the connection.