213(9): Further Internal Inconsistency in the Standard Transformation of the Connection.

The standard transformation of the connection is:

$$\Gamma^{\nu'}_{\mu'\lambda'} = q^{\mu}_{\mu'} q^{\lambda}_{\lambda'} \left(q^{\nu'}_{\nu} \Gamma^{\nu}_{\mu\lambda} - \partial_\mu q^{\nu'}_{\lambda} \right) \quad -(1)$$

in the notation of previous notes. However, as in Carroll's online notes eq. (2.15):

$$q^{\nu'}_{\lambda} = \frac{\partial x^{\nu'}}{\partial x^{\lambda}} = \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x^{\lambda}} \quad -(2)$$

and in an arbitrary manifold:

$$\boxed{\frac{\partial x^{\nu}}{\partial x^{\lambda}} = \delta^{\nu}_{\lambda}} \quad -(3)$$

$$= 0$$

unless:

$$\nu = \lambda. \quad -(4)$$

This fact shows immediately that the inhomogeneous term is zero unless $\nu = \lambda$.

If eq. (4) is true then:

$$\frac{\partial x^{\nu'}}{\partial x^{\lambda}} = \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} = \frac{\partial x^{\lambda'}}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial x^{\lambda}}$$

$$= 0 \quad -(5)$$

unless

$$\mu = \lambda \quad -(6)$$

2) So the inhomogeneous term is zero unless:

$$\mu = \nu = \lambda. \quad -(7)$$

This is inconsistent with the fact that in general:

$$\mu \neq \nu \neq \lambda. \quad -(8)$$

The only way to resolve this inconsistency is to use an antisymmetric connection, in which case:

$$\nu \neq \lambda \quad -(9)$$

and

$$\boxed{\frac{\partial x^{\nu'}}{\partial x^{\lambda}} = 0} \quad -(10)$$

The standard model uses the idea that there exists a local Lorentz frame in which the spacetime is locally flat. In this locally flat region the connection can vanish. From eq. (1) a zero connection can be obtained in the transformed frame when:

$$q^{\nu'}{}_{\nu} \Gamma^{\nu}_{\mu\lambda} = \partial_\mu q^{\nu'}{}_{\lambda} \quad -(11)$$

i.e.

$$\Gamma^{\nu}_{\mu\lambda} = \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right), \quad -(12)$$

$$\Gamma^{\nu'}_{\mu'\lambda'} = 0. \quad -(13)$$

3) However from eq. (10) and eq. (12), for the arbitrary manifold:

$$\Gamma^{\sim}_{\mu\lambda} = 0, \quad -(14)$$

if

$$\Gamma^{\sim'}_{\mu'\lambda'} = 0. \quad -(15)$$

If the connection is zero in one frame it is zero in all frames. This has profound implications because the idea of Riemann normal coordinates can no longer be used.

The Einstein equivalence principle depends on this idea, and must also be abandoned.

The original 1869 definition of the connection by E. B. Christoffel is similar to eq. (12). The latter equation would mean that:

$$\Gamma^{\sim}_{\mu\lambda} =? \Gamma^{\sim}_{\lambda\mu} \quad -(16)$$

because

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\sim'}}{\partial x^{\lambda}}\right) = \frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial x^{\sim'}}{\partial x^{\mu}}\right) \quad -(17)$$

From eq. (10) it is seen that:

$$\Gamma^{\sim}_{\mu\lambda} = \Gamma^{\sim}_{\lambda\mu} = 0 \quad -(18)$$

i.e. the symmetric connection is zero.

4) The antisymmetric connection is the only non-zero connection, and transforms as:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \partial^{\mu}_{\mu'}\,\partial^{\lambda}_{\lambda'}\,\partial^{\nu'}_{\nu}\,\Gamma^{\nu}_{\mu\lambda} \quad -(19)$$

The torsion is defined as:

$$T^{\nu}_{\mu\lambda} = \Gamma^{\nu}_{\mu\lambda} - \Gamma^{\nu}_{\lambda\mu} \quad -(20)$$

and transforms as:

$$T^{\nu'}_{\mu'\lambda'} = \partial^{\mu}_{\mu'}\,\partial^{\lambda}_{\lambda'}\,\partial^{\nu'}_{\nu}\,T^{\nu}_{\mu\lambda}. \quad -(21)$$

The next step is to consider the effect of an antisymmetric connection on the geodesic equation.
