213(2): Further Proof of Zero Inhomogeneous Term

This proof rests on:

$$\frac{\partial x^{a'}}{\partial x^{\lambda}} = \frac{\partial x^{a'}}{\partial x^{a}} \frac{\partial x^{a}}{\partial x^{\lambda}} = q^{a'}_{\ a} q^{a}_{\ \lambda} \quad -(1)$$

Here

$$q^{a'}_{\ a} = \Lambda^{a'}_{\ a} \quad -(2)$$

where $\Lambda^{a'}_{\ c}$ is the Lorentz transform matrix. Also:

$$q^{a'}_{\ a} q^{a}_{\ \lambda} = \delta^{a'}_{\ \lambda} \quad -(3)$$

$$= 1 \text{ if } a' = \lambda$$
$$0 \text{ if } a' \neq \lambda$$

In general:

$$q^{a'}_{\ a} \neq 0, \; q^{a}_{\ \lambda} \neq 0. \quad -(4)$$

but if

$$x^{a'} \neq x^{\lambda} \quad -(5)$$

then

$$\frac{\partial x^{a'}}{\partial x^{\lambda}} = 0 \quad -(6)$$

and if

$$x^{a'} = x^{\lambda} \quad -(7)$$

$$\frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{a'}}{\partial x^{\lambda}} \right) = 0. \quad -(8)$$

Therefore $\Gamma^{a}_{\mu\lambda}$ and $\Gamma^{\sim}_{\mu\lambda}$ transform as tensors. The symmetric inhomogeneous term is zero.

2) In general:

$$\Gamma^{\sim}_{\mu\lambda} = \Gamma^{\sim\,(S)}_{\mu\lambda} + \Gamma^{\sim\,(A)}_{\mu\lambda} \quad -(9)$$

where

$$\Gamma^{\sim\,(S)}_{\mu\lambda} = \frac{1}{2}\left(\Gamma^{\sim}_{\mu\lambda} + \Gamma^{\sim}_{\lambda\mu}\right)$$

$$\Gamma^{\sim\,(A)}_{\mu\lambda} = \frac{1}{2}\left(\Gamma^{\sim}_{\mu\lambda} - \Gamma^{\sim}_{\lambda\mu}\right) \quad -(10)$$

In any other frame:

$$\Gamma^{\sim'}_{\mu'\lambda'} = \Gamma^{\sim'\,(S)}_{\mu'\lambda'} + \Gamma^{\sim'\,(A)}_{\mu'\lambda'} \quad -(11)$$

so

$$\Gamma^{\sim\,(S)}_{\mu\lambda} \rightarrow \Gamma^{\sim'\,(S)}_{\mu'\lambda'} \quad -(12)$$

$$\Gamma^{\sim\,(A)}_{\mu\lambda} \rightarrow \Gamma^{\sim'\,(A)}_{\mu'\lambda'} \quad -(13)$$

The fact that the inhomogeneous term is zero means that the antisymmetric connection remains antisymmetric in any frame of reference.

If use is made only of Riemann geometry then the hypothetical symmetric connection transforms as

$$\Gamma^{\sim'}_{\mu'\lambda'} = \Gamma^{\sim'\,(1)}_{\mu'\lambda'} + \Gamma^{\sim'\,(2)}_{\mu'\lambda'} \quad -(14)$$

3) By assumption:

$$\Gamma^{\nu'}_{\mu'\lambda'} \overset{?}{=} \Gamma^{\nu'}_{\lambda'\mu'} \quad -(15)$$

which means that:

$$\Gamma^{\nu'}_{\mu'\lambda'}{}^{(1)} \overset{?}{=} \Gamma^{\nu'}_{\lambda'\mu'}{}^{(1)} \quad -(16)$$

$$\Gamma^{\nu'}_{\mu'\lambda'}{}^{(2)} \overset{?}{=} \Gamma^{\nu'}_{\lambda'\mu'}{}^{(2)} \quad -(17)$$

Eq. (14) can be written as:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \left(\Gamma^{\nu'(1)} + \Gamma^{\nu'(2)}\right)_{\mu'\lambda'} \quad -(18)$$

and the i.by consideration of Cartan geometry:

$$\Gamma^{\nu'}_{\mu'\lambda'}{}^{(2)} = \Gamma^{\nu'}_{\lambda'\mu'}{}^{(2)} = 0 \quad -(19)$$

meaning that the connection cannot be non-zero and symmetric under interchange of μ' and λ'