213(1): Symmetry of Mixed Index Connections.

The mixed index connections originate in the tetrad postulate:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda = 0 \quad (1)$$

where $\omega^a_{\mu b}$ is the spin connection and $\Gamma^\lambda_{\mu\nu}$ the Christoffel connection. By definition:

$$\Gamma^a_{\mu\nu} = \Gamma^\lambda_{\mu\nu} q^a_\lambda \quad -(2)$$

$$\omega^a_{\mu\nu} = \omega^a_{\mu b} q^b_\nu \quad -(3)$$

so

$$\Gamma^a_{\mu\nu} = \partial_\mu q^a_\nu + \omega^a_{\mu\nu} \quad -(4)$$

and

$$\Gamma^\lambda_{\mu\nu} = q^\lambda_a \Gamma^a_{\mu\nu}$$

$$= q^\lambda_a \left(\partial_\mu q^a_\nu + \omega^a_{\mu\nu}\right) \quad -(5)$$

$$= q^\lambda_a \partial_\mu q^a_\nu + \omega^\lambda_{\mu\nu}$$

i.e

$$\boxed{\begin{aligned} \Gamma^\lambda_{\mu\nu} - \omega^\lambda_{\mu\nu} &= q^\lambda_a \partial_\mu q^a_\nu \\ &= q^\lambda_a \left(\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}\right) \end{aligned}} \quad -(6)$$

In the definition (2):

$$x^a = q^a_\lambda x^\lambda \quad -(7)$$

So
$$\boxed{\frac{\partial x^a}{\partial x^\lambda} = q^a_\lambda} \quad -(8)$$

A functional dependence has been defined of x^a on x^λ. However, in the definition (2) no functional dependence is defined of x^μ on x^a and of x^ν on x^a. Under general coordinate transformation:

$$\Gamma^{a'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{a'}}{\partial x^a}\Gamma^a_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{a'}}{\partial x^\lambda}\right) \quad -(9)$$

in which
$$\frac{\partial x^{a'}}{\partial x^\lambda} = \frac{\partial x^{a'}}{\partial x^{b'}}\frac{\partial x^{b'}}{\partial x^\lambda}. \quad -(10)$$

In eq. (10), it has been assumed that $x^{a'}$ is a function of $x^{b'}$ which is a function of x^λ. So:
$$\frac{\partial x^{a'}}{\partial x^\lambda} = q^{a'}_{b'}\, q^{b'}_\lambda \quad -(11)$$

In Cartan geometry:
$$q^{a'}_{b'}\, q^{b'}_\lambda = \delta^{a'}_\lambda \quad -(12)$$

where
$$\delta^{a'}_\lambda = 1 \text{ if } a' = \lambda \quad -(13)$$
$$= 0 \text{ if } a' \neq \lambda$$

3) Also:

$$q^{a'}{}_{b'} = q^{a'}{}_{\lambda}\, q^{\lambda}{}_{b'} = \delta^{a'}{}_{b'} \quad -(14)$$

In general

$$V^a = q^a_\mu V^\mu \quad -(15)$$

$$V^a{}_b = q^a_\mu V^\mu{}_b = q^\nu{}_b V^a{}_\nu$$

$$= q^a_\mu q^\nu{}_b V^\mu{}_\nu \quad -(16)$$

and so on. For any mixed index tensors there are general coordinate transformations such as:

$$T^{a'\mu'}{}_{b'\nu'} = \frac{\partial x^{a'}}{\partial x^a} \frac{\partial x^{\mu'}}{\partial x^\mu} \frac{\partial x^b}{\partial x^{b'}} \frac{\partial x^\nu}{\partial x^{\nu'}} T^{a\mu}{}_{b\nu} \quad -(17).$$

Here

$$\Lambda^{a'}{}_a = \frac{\partial x^{a'}}{\partial x^a} \quad -(18)$$

$$\Lambda^b{}_{b'} = \frac{\partial x^b}{\partial x^{b'}} \quad -(19)$$

where $\Lambda^{a'}{}_a$ is the Lorentz transform and $\Lambda^b{}_{b'}$ is the inverse Lorentz transform. From eqs. (11) to (14):

$$\boxed{\frac{\partial x^{a'}}{\partial x^\lambda} = 0 \text{ unless } a' = \lambda} \quad -(20)$$

This result is a direct consequence of the tetrad postulate and of the definition (2).

4)

If $$a' = \lambda \quad -(21)$$

then
$$\frac{\partial x^{a'}}{\partial x^{\lambda}} = 1 \quad -(22)$$

and
$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{a'}}{\partial x^{\lambda}}\right) = 0. \quad -(23)$$

Therefore in eq. (9):

$$\boxed{\Gamma^{a'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\Lambda^{a'}_{a}\,\Gamma^{a}_{\mu\lambda}} \quad -(24)$$

where
$$\Lambda^{a'}_{a} = \frac{\partial x^{a'}}{\partial x^{a}} \quad -(25)$$

is a Lorentz transform because the tangent space of Cartan is a Minkowski space.

The tetrad transforms as:

$$q^{\lambda'}_{a'} = \frac{\partial x^{\lambda'}}{\partial x^{\lambda}}\frac{\partial x^{a}}{\partial x^{a'}}\,q^{\lambda}_{a} \quad -(26)$$

So:
$$\Gamma^{\nu'}_{\mu'\lambda'} = q^{\nu'}_{a'}\,\Gamma^{a'}_{\mu'\lambda'}$$

$$= \frac{\partial x^{\nu'}}{\partial x^{\nu}}\frac{\partial x^{a}}{\partial x^{a'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{a'}}{\partial x^{a}}\,\Gamma^{\nu}_{\mu\lambda} \quad -(27)$$

5)

i.e.

$$\boxed{\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \Gamma^{\nu}_{\mu\lambda}} \quad -(28)$$

For the first time in the history of differential geometry it is seen that the Christoffel connection transforms as a tensor.

From eq. (9) it is seen that its symmetric part is zero, because its symmetric part is symmetric in any frame of reference and by commutativity of partial derivatives:

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{a'}}{\partial x^{\lambda}}\right) = \frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial x^{a'}}{\partial x^{\mu}}\right) \quad -(29)$$

$$= 0.$$

Under symmetric interchange of μ and λ the connection vanishes. So its only component is antisymmetric:

$$\Gamma^{\lambda}_{\mu\nu} = -\Gamma^{\lambda}_{\nu\mu} \quad -(30)$$

and

$$\Gamma^{\lambda'}_{\mu'\nu'} = -\Gamma^{\lambda'}_{\nu'\mu'} \quad -(31)$$

in any frame of reference.

6) From eq. (6):

$$\omega^a_{\ \mu\nu} = -\,\omega^a_{\ \nu\mu} \quad -(32)$$

and

$$\omega^\lambda_{\ \mu\nu} = -\,\omega^\lambda_{\ \nu\mu}. \quad -(33)$$

From eq. (28) it is seen that if the connection had a symmetric component it would remain symmetric in any reference frame, but the same equation shows that its symmetric inhomogeneous term is zero. In the standard model it is always assumed that:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^\lambda}\right) \quad -(34)$$

and that

$$\Gamma^\nu_{\mu\lambda} \overset{?}{=} \Gamma^\nu_{\lambda\mu} \quad -(35)$$

because:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^\lambda}\right) = \frac{\partial}{\partial x^\lambda}\left(\frac{\partial x^{\nu'}}{\partial x^\mu}\right). \quad -(36)$$

So it is assumed that $\Gamma^\nu_{\mu\lambda}$ is not a tensor and that torsion vanishes. <u>These</u> <u>assumptions</u> are <u>incorrect</u>.