213(4): Further Proof of the Tensorial Nature of the Christoffel Connection

In note 213(3) it was proven that:

$$\Gamma^{a'}_{\mu'\lambda'} = \frac{\partial x^{a'}}{\partial x^{a}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \Gamma^{a}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{a'}}{\partial x^{\nu'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) \quad -(1)$$

In defining:

$$\Gamma^{a}_{\mu\lambda} = \Gamma^{\nu}_{\mu\lambda} q^{a}_{\nu} \quad -(2)$$

it has been assumed that there is the following functional dependence of a on ν:

$$q^{a}_{\nu} = \frac{\partial x^{a}}{\partial x^{\nu}} . \quad -(3)$$

Therefore:

$$q^{a'}_{\nu'} = \frac{\partial x^{a'}}{\partial x^{\nu'}} = \frac{\partial x^{a'}}{\partial x^{a}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} q^{a}_{\nu} \neq 0 . \quad -(4)$$

However

$$q^{a'}_{\nu'} = \frac{\partial x^{a'}}{\partial x^{a}} \frac{\partial x^{a}}{\partial x^{\nu'}} \quad -(5)$$

$$= \Lambda^{a'}_{a} q^{a}_{\nu} q^{\nu}_{\nu'}$$

$$= 0$$

unless

$$a = \nu' . \quad -(6)$$

2) There is a contradiction between eqs. (4) and (5) because it has been assumed that there is a functional dependence of x^a on $x^{\nu'}$. There is no functional dependence of this type, so the chain rule (5) cannot be used.

Using the relabelling of repeated indices:

$$\frac{\partial x^{a'}}{\partial x^{\nu'}} \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}}\right) = \frac{\partial x^{a'}}{\partial x^{c'}} \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{a'}}{\partial x^{\lambda}}\right)$$

$$= \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{a'}}{\partial x^{\lambda}}\right) \quad -(7)$$

because ν' runs from 0 to 3 and a runs from 0 to 3.

The following rule is valid:

$$\frac{\partial x^{a'}}{\partial x^{\lambda}} = \frac{\partial x^{a'}}{\partial x^{a}} \frac{\partial x^{a}}{\partial x^{\lambda}} \quad -(8)$$

$$= \Lambda^{a'}{}_{a}\, q^{a}{}_{\lambda}$$

$$= q^{a'}{}_{a}\, q^{a}{}_{\lambda} = \delta^{a'}{}_{\lambda}$$

$$= 0 \quad -(9)$$

unless $a' = \lambda$

and if $a' = \lambda$ $\quad -(10)$

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{a'}}{\partial x^{\lambda}}\right) = 0 \quad -(11)$$

The rules of Cartan geometry are:

$$q^{a}{}_{\mu}\, q^{\mu}{}_{b} = \delta^{a}{}_{b} \quad -(12)$$

$$q^{a}{}_{\mu}\, q^{\nu}{}_{a} = \delta^{\nu}{}_{\mu} \quad -(13)$$

So:

$$q^{a}{}_{\mu}\, q^{\mu}{}_{b} = q^{a}{}_{b}\, q^{b}{}_{\mu}\, q^{\mu}{}_{b}$$

$$= q^{a}{}_{b} = \delta^{a}{}_{b} \quad -(14)$$

So

$$q^{a'}{}_{\lambda} = \Lambda^{a'}{}_{a}\, q^{a}{}_{a'}\, q^{a'}{}_{\lambda}$$

$$= \Lambda^{a'}{}_{a}\, \Lambda^{a}{}_{a'}\, q^{a'}{}_{\lambda} \quad -(15)$$

i.e

$$\Lambda^{a'}{}_{a}\, \Lambda^{a}{}_{a'} = 1 \quad -(16)$$

which means that the Lorentz transform and inverse Lorentz transform are inverse matrices.

By definition:

$$\boxed{q^{a}{}_{b} = \delta^{a}{}_{b}} \quad -(17)$$

So:

$$\boxed{\begin{aligned} \Gamma^{a'}{}_{\mu'\lambda'} &= \frac{\partial x^{a'}}{\partial x^{a}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \Gamma^{a}{}_{\mu\lambda} \\ \Gamma^{\nu'}{}_{\mu'\lambda'} &= \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \Gamma^{\nu}{}_{\mu\lambda} \end{aligned}} \quad -(18)$$

QED