213(7): Transformation of the Mixed Index Connection

The mixed index connection $\Gamma^{a}_{\mu\nu}$ is defined by the tetrad postulate.

$$D_\mu q^{a}_{\nu} = \partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu b} q^{b}_{\nu} - \Gamma^{\lambda}_{\mu\nu} q^{a}_{\lambda} = 0 \quad -(1)$$

$$= \partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu\nu} - \Gamma^{a}_{\mu\nu}$$

so

$$\boxed{\Gamma^{a}_{\mu\nu} = \partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu\nu}} \quad -(2)$$

Note that:

$$\frac{\partial x^{\nu'}}{\partial x^{\nu}} = q^{\nu'}_{\nu} = q^{\nu'}_{a'} q^{a'}_{\nu}$$

$$= q^{\nu'}_{a} q^{a}_{\nu} \quad -(3)$$

$$= \delta^{\nu'}_{\nu}$$

therefore

$$\boxed{x^{\nu'} = x^{\nu}}, \quad -(4)$$

if

$$q^{a}_{\nu} = \frac{\partial x^{a}}{\partial x^{\nu}}. \quad -(5)$$

therefore

$$q^{a}_{\nu} = q^{a}_{\nu'} \quad -(6)$$

This result is a consequence of the definition of Cartan geometry, eq. (3). If x^{ν} is

2) defined to be a function (5) of x^a, then eq. (4) follows by definition. If x^{ν} is not defined to be a function of x^a then:

$$x^{\nu'} \neq x^{\nu} \quad -(7)$$

in general.

The mixed index connection transforms as:

$$\Gamma^{a'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \left(\frac{\partial x^{a'}}{\partial x^{a}} \Gamma^{a}_{\mu\lambda} - \frac{\partial x^{a'}}{\partial x^{\nu'}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) \right) \quad -(8)$$

where

$$\frac{\partial x^{a'}}{\partial x^{\nu'}} = \frac{\partial x^{a'}}{\partial x^{a}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} \frac{\partial x^{a}}{\partial x^{\nu}} \quad -(9)$$

i.e.

$$\Gamma^{a'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{a'}}{\partial x^{a}} \left(\Gamma^{a}_{\mu\lambda} - \frac{\partial x^{a}}{\partial x^{\nu}} \frac{\partial}{\partial x^{\mu}} \left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}} \right) \right)$$

$$\boxed{\Gamma^{a'}_{\mu'\lambda'} = q^{\mu}_{\mu'} q^{\lambda}_{\lambda'} q^{a'}_{a} \left(\Gamma^{a}_{\mu\nu} - q^{a}_{\nu} \partial_{\mu} q^{\nu'}_{\lambda} \right)} \quad -(10)$$

Note carefully that x^{μ} is not defined to be a function of x^a. Also, x^{λ} is not defined to be a function of x^a. So:

$$q^{\mu}_{\mu'} \neq q^{\mu}_{a} q^{a}_{\mu'} \quad -(11)$$

$$q^{\lambda}_{\lambda'} \neq q^{\lambda}_{a} q^{a}_{\lambda'} \quad -(12)$$

3) but: $q^{\nu}{}_{\nu'} = q^{\nu}{}_{a}\, q^{a}{}_{\nu'}$ — (13)

where $q^{a}{}_{\nu'} = \dfrac{\partial x^{a}}{\partial x^{\nu'}} = \dfrac{\partial x^{a}}{\partial x^{\nu}} \dfrac{\partial x^{\nu}}{\partial x^{\nu'}}$ — (14)

the reason for this is that only $q^{a}{}_{\nu}$ appears in the definition of $\Gamma^{a}{}_{\mu\nu}$:

$$\Gamma^{a}{}_{\mu\lambda} = \Gamma^{\nu}{}_{\mu\lambda}\, q^{a}{}_{\nu} \quad - (15)$$

i.e

$$\Gamma^{a} = \Gamma^{\nu}\, q^{a}{}_{\nu} \quad - (16)$$

in form notation. In these definitions μ and λ are fixed and have no dependence on a, but ν and a are related by the tetrad. It could be objected that $q^{a}{}_{\lambda}$ is used in eq. (1), so x^{a} would have a dependence on x^{λ}, but λ in eq. (1) is a summation index that could be replaced by ν. Similarly, b in eq. (1) is a summation index that could be replaced by a. So:

$$D_\mu q^{a}{}_{\nu} = \partial_\mu q^{a}{}_{\nu} + \omega^{a}{}_{\mu a}\, q^{a}{}_{\nu} - \Gamma^{\nu}{}_{\mu\nu}\, q^{a}{}_{\nu} \quad - (17)$$

$$= 0$$

i.e. $$\left(\partial_\mu + \omega^{a}{}_{\mu a} - \Gamma^{\nu}{}_{\mu\nu}\right) q^{a}{}_{\nu} = 0 \quad - (18)$$

The covariant derivative may therefore be expressed as:

$$D_\mu = \partial_\mu + \omega^a_{\ \mu a} - \Gamma^\nu_{\mu\nu} \quad -(19)$$

in which case it becomes clear that only the tetrad $q^a_{\ \nu}$ is being considered, and no other tetrad. The format (19) is similar to that used in gauge theory, e.g.:

$$D_\mu = \partial_\mu + i\frac{e}{\hbar} A_\mu \quad -(20)$$

The inhomogeneous term in eq. (10) vanishes because:

$$\partial_\mu q^{\nu'}_{\ \lambda} = \partial_\mu \frac{\partial}{\partial x^\lambda} x^{\nu'} \quad -(21)$$

and

$$x^{\nu'} = x^{a'} \frac{\partial x^{\nu'}}{\partial x^{a'}} \quad -(22)$$

by definition, i.e.:

$$x^{\nu'} = q^{a'}_{\ \nu'} x^{\nu'} - (23)$$

By definition:

$$q^{a'}_{\ \lambda} = \frac{\partial x^{a'}}{\partial x^\lambda} = 0 \quad -(24)$$

and

$$q^{\nu'}_{\ a'} = q^{\nu'}_{\ a} q^{a}_{\ a'} = \delta^{\nu'}_{a'}$$

$$= 1 \quad -(25)$$

5) Self consistently:

$$q^{\mu'}_{\ a'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{a}}{\partial x^{a'}} q^{\mu}_{\ a} \quad -(26)$$

because the tetrad is a rank two mixed index tensor. So:

$$\begin{aligned} q^{\mu'}_{\ a'} &= q^{\mu'}_{\ \mu} q^{a}_{\ a'} q^{\mu}_{\ a} \\ &= q^{a}_{\ a'} \left(q^{\mu'}_{\ \mu} q^{\mu}_{\ a} \right) \\ &= q^{a}_{\ a'} q^{\mu'}_{\ a} \quad -(27) \\ &= q^{\mu'}_{\ a} q^{a}_{\ a'} \end{aligned}$$

which is eq. (25).

The tetrad is defined as an invertible $n \times n$ matrix, so it is defined as:

$$q^{\mu'}_{\ a} q^{a}_{\ \mu'} = 1 \, . \quad -(28).$$

This means that:

$$q^{\mu'}_{\ a} q^{a}_{\ a'} = \delta^{\mu'}_{\ a'} \quad -(29)$$

where

$$\delta^{\mu'}_{\ a'} = 0 \quad \text{if } \mu' \neq a' \quad -(30)$$