213(5): Some Self Consistency Checks and Derivation

In proving that the inhomogeneous term is zero it must be proven that:

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}}\right) = 0, \quad -(1)$$

in the transformation:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\lambda}}\right). -(2)$$

There is no way of proving this without realizing the existence of torsion, and without using the concept of tangent spacetime. The latter results in the tetrad postulate:

$$D_{\mu} q^{a}_{\lambda} = \partial_{\mu} q^{a}_{\lambda} + \omega^{a}_{\mu b} q^{b}_{\lambda} - \Gamma^{\nu}_{\mu\lambda} q^{a}_{\nu} = 0 \quad -(3)$$

$$= \partial_{\mu} q^{a}_{\lambda} + \omega^{a}_{\mu\nu} - \Gamma^{a}_{\mu\nu}$$

where

$$\Gamma^{a}_{\mu\nu} = \Gamma^{\nu}_{\mu\lambda} q^{a}_{\nu} \quad -(4)$$

and

$$q^{a}_{\nu} = \frac{\partial x^{a}}{\partial x^{\nu}}. \quad -(5)$$

Therefore:

$$\frac{\partial x^{\nu'}}{\partial x^{\lambda}} = \frac{\partial}{\partial x^{\lambda}}\left(x^{a'}\frac{\partial x^{\nu'}}{\partial x^{a'}}\right) = 0 \quad -(6)$$

because

$$\frac{\partial}{\partial x^{\lambda}} = \frac{\partial x^{a'}}{\partial x^{\lambda}}\frac{\partial x^{\nu'}}{\partial x^{a'}} + x^{a'}\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial x^{\nu'}}{\partial x^{a'}}\right) -(7)$$

By definition there is no functional dependence

2) of $x^{a'}$ and x^{λ}. The result:

$$x^{\mu'} = \frac{dx^{\mu'}}{dx^{a'}} x^{a'} = q^{\mu'}_{\ a'} x^{a'} \quad -(8)$$

follows from the definition of the tetrad and:

$$q^{\mu'}_{\ a'} = \frac{dx^{\mu'}}{dx^{\mu}} \frac{dx^{a}}{dx^{a'}} q^{\mu}_{\ a} \quad -(9)$$

$$\neq 0.$$

As soon as eq. (4) is introduced it is seen that:

$$\boxed{\Gamma^{\nu'}_{\mu'\lambda'} = \frac{dx^{\mu}}{dx^{\mu'}} \frac{dx^{\lambda}}{dx^{\lambda'}} \frac{dx^{\nu'}}{dx^{\nu}} \Gamma^{\nu}_{\mu\lambda}} \quad -(10)$$

and the transformation of the connection does not generate a symmetric inhomogeneous term. This was the reason for assuming a symmetric connection. In the standard model it was possible to transform $\Gamma^{\nu}_{\mu\lambda}$ to zero when in eq. (2)

$$\frac{dx^{\nu'}}{dx^{\nu}} \Gamma^{\nu}_{\mu\lambda} = \frac{d}{dx^{\mu}}\left(\frac{dx^{\nu'}}{dx^{\lambda}}\right) \quad -(11)$$

i.e. when

$$\Gamma^{\nu}_{\mu\lambda} = \frac{dx^{\nu}}{dx^{\nu'}} \frac{d}{dx^{\mu}}\left(\frac{dx^{\nu'}}{dx^{\lambda}}\right) \quad -(12)$$

then

$$\Gamma^{\nu'}_{\mu'\lambda'} = 0. \quad -(13)$$

This kind of transformation to zero is not

3)

possible in Cartan geometry because in eq. (12):

$$\Gamma^{\nu}_{\mu\lambda} = 0. \quad -(14)$$

This result is self consistent with eq. (10) because ~~when~~ the connection is zero in all frames if it is zero in one frame. The idea of transforming a connection to zero makes no sense because it would imply that the torsion and curvature could be transformed to zero. In fact the torsion transforms as:

$$T^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} T^{\nu}_{\mu\lambda} \quad -(15)$$

and if non-zero in one frame is non zero in all frames. It is defined as:

$$T^{\nu}_{\mu\lambda} = \Gamma^{\nu}_{\mu\lambda} - \Gamma^{\nu}_{\lambda\mu} \neq 0. \quad -(16)$$

Cartan geometry also allows important deductions such as:

$$x_a x^a = x_\mu x^\mu. \quad -(17)$$

This result follows from:

$$x^a = q^a_\mu x^\mu \quad -(18)$$

$$x_a = q^\mu_a x_\mu \quad -(19)$$

and

$$x^a x_a = q^a_\mu q^\mu_a x^\mu x_\mu = x^\mu x_\mu \quad -(20)$$

because

$$q^a_\mu q^\mu_a = 1. \quad -(21)$$

4) It follows that:

$$\boxed{p_\mu p^\mu = p_a p^a = m^2 c^2} \quad -(22)$$

which shows immediately that the Einstein energy equation retains its format in general relativity.

In dealing with summation over repeated indices there are results such as:

$$\frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu = \frac{\partial x^{\mu'}}{\partial x^\lambda} V^\lambda \quad -(23)$$

i.e. the summation index μ can be replaced by λ.

This result follows from:

$$\frac{\partial x^{\mu'}}{\partial x^\mu} = \frac{\partial x^{\mu'}}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial x^\mu} \quad -(24)$$

and

$$V^\mu = V^\lambda \frac{\partial x^\mu}{\partial x^\lambda} \quad -(25)$$

Eq. (25) is:

$$V^\mu = q^\mu_{\ \lambda} V^\lambda \quad -(26)$$

where $q^\mu_{\ \lambda}$ is the tetrad in the same manifold.

Finally there are results such as:

$$\frac{\partial x^{a'}}{\partial x^{\mu'}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\mu'}}{\partial x^\lambda}\right) = \frac{\partial x^{a'}}{\partial x^{a'}} \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{a'}}{\partial x^\lambda}\right)$$

$$= \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{a'}}{\partial x^\lambda}\right) \quad -(27)$$

5) This follows from:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^\lambda}\right) = \frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{a'}}{\partial x^\lambda}\frac{\partial x^{\nu'}}{\partial x^{a'}}\right) \quad -(28)$$

$$= \frac{\partial x^{\nu'}}{\partial x^{a'}}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{a'}}{\partial x^\lambda}\right) + \frac{\partial x^{a'}}{\partial x^\lambda}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^{a'}}\right)$$

with:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^{a'}}\right) = 0 \quad -(29)$$

because there is no functional dependence of $q^{\nu'}_{a'}$ on x^μ by definition.

It follows again that:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^\lambda}\right) = 0 \quad -(30)$$

because there is no functional dependence either of $q^{a'}_\lambda$ on x^μ and in fact:

$$q^{a'}_\lambda = 0 \quad -(31)$$

because there is no functional dependence of $x^{a'}$ on x^λ.
