211(3) : Corrections to the Old Second Bianchi Identity.

The old second Bianchi identity is :

$$d \wedge R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b := 0 \quad -(1)$$

which when written out in full is :

$$\partial_\rho R^a{}_{b\mu\nu} + \omega^a{}_{\rho c} R^c{}_{b\mu\nu} - R^a{}_{c\rho\mu} \omega^c{}_{\nu b} + \dots := 0 \quad -(2)$$

or

$$D_\rho R^a{}_{b\mu\nu} + D_\nu R^a{}_{b\rho\mu} + D_\mu R^a{}_{b\nu\rho} := 0 \quad -(3)$$

The tensor valued two form $R^a{}_{b\mu\nu}$ is a two form in the base manifold and for each μ and ν is a tensor in the tangent space at point P to the base manifold. For each μ and ν :

$$D_\rho R^a{}_{b\mu\nu} = \partial_\rho R^a{}_{b\mu\nu} + \omega^a{}_{\rho c} R^c{}_{b\mu\nu} - R^a{}_{c\rho\mu} \omega^c{}_{\nu b} \quad -(4)$$

i.e. D_ρ is the covariant derivative of $R^a{}_b$.

By definition :

$$R^a{}_{b\mu\nu} = q^\sigma{}_b R^a{}_{\sigma\mu\nu} \quad -(5)$$

and

$$D_\rho R^a{}_{b\mu\nu} = (D_\rho q^\sigma{}_b) R^a{}_{\sigma\mu\nu} + q^\sigma{}_b D_\rho R^a{}_{\sigma\mu\nu} \quad -(6)$$

2) The tetrad postulate is:

$$D_\rho q^\sigma_{\ b} = 0 \quad -(7)$$

so

$$D_\rho R^a_{\ b\mu\nu} = q^\sigma_{\ b} D_\rho R^a_{\ \sigma\mu\nu} \quad -(8)$$

and eq. (3) becomes:

$$D_\rho R^a_{\ \sigma\mu\nu} + D_\nu R^a_{\ \sigma\rho\mu} + D_\mu R^a_{\ \sigma\nu\rho} := 0 \quad -(9)$$

Finally, using $R^a_{\ \sigma\mu\nu} = R^\lambda_{\ \sigma\mu\nu} q^a_{\ \lambda}$ $-(10)$ the old second Bianchi identity is obtained in the base manifold in Riemannian format:

$$D_\rho R^\lambda_{\ \sigma\mu\nu} + D_\nu R^\lambda_{\ \sigma\rho\mu} + D_\mu R^\lambda_{\ \sigma\nu\rho} := 0 \quad -(11)$$

However, the old second Bianchi identity assumed a symmetric connection. This is seen from page 80, eq (3.87) of the 1997 online notes by Carroll, where he derives eq. (11) using Riemann normal coordinates. He uses the formula linking the metric to the symmetric connection. It is now known that the connection is antisymmetric, so eq. (11) is obsolete.

3) Eq. (1) is just a translation of eq. (11). In the much studied UFT 88 the correct identity was worked out:

$$D \wedge (D \wedge T) := D \wedge (R \wedge q) \quad -(12)$$

From eq. (10) of note 211(2) it is seen that the cyclic sum (11) is not zero in general, it is related to the Riemann format of the Cartan identity, eq. (1) of note 211(2):

$$R^\lambda{}_{\mu\nu\rho} + \dots := \partial_\mu T^\lambda{}_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma{}_{\nu\rho} + \dots \quad -(13)$$

Using the correct:

$$\Gamma^\sigma_{\mu\nu} = - \Gamma^\sigma_{\nu\mu} \quad -(14)$$

it was shown in note 211(2) that:

$$R^\lambda{}_{\mu\nu\rho} = \frac{1}{2} \left(\partial_\mu T^\lambda{}_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma{}_{\nu\rho} \right) - \frac{1}{4} \left(\partial_\nu T^\lambda{}_{\mu\rho} + \Gamma^\lambda_{\nu\sigma} T^\sigma{}_{\mu\rho} - \partial_\rho T^\lambda{}_{\mu\nu} - \Gamma^\lambda_{\rho\sigma} T^\sigma{}_{\mu\nu} \right) \quad -(15)$$

So:

$$D_\kappa R^\lambda{}_{\mu\nu\rho} = \frac{1}{2} D_\kappa \left(\partial_\mu T^\lambda{}_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma{}_{\nu\rho} \right) - \frac{1}{4} D_\kappa \left(\partial_\nu T^\lambda{}_{\mu\rho} + \Gamma^\lambda_{\nu\sigma} T^\sigma{}_{\mu\rho} - \partial_\rho T^\lambda{}_{\mu\nu} - \Gamma^\lambda_{\rho\sigma} T^\sigma{}_{\mu\nu} \right) \quad -(16)$$

4) and in general:

$$D_\kappa R^{\lambda}_{\ \mu\nu\rho} + D_\rho R^{\lambda}_{\ \mu\kappa\nu} + D_\nu R^{\kappa}_{\ \mu\rho\kappa} \neq 0 \quad -(17)$$

Q.E.D

The Einstein field equation was based on eq. (11) and is incorrect.