211(2): Some Implications of the Riemannian type Identity Schwarz Curvature and Torsion

This identity is:

$$R^\lambda{}_{\mu\nu\rho} + R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\nu\rho\mu} := \partial_\mu T^\lambda{}_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma{}_{\nu\rho} + \partial_\rho T^\lambda{}_{\mu\nu} + \Gamma^\lambda_{\rho\sigma} T^\sigma{}_{\mu\nu} + \partial_\nu T^\lambda{}_{\rho\mu} + \Gamma^\lambda_{\nu\sigma} T^\sigma{}_{\rho\mu} \quad -(1)$$

The second structure equation of Cartan is equivalent to the definition of curvature:

$$R^\lambda{}_{\mu\nu\rho} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho} \quad -(2)$$

and eq. (2) is a solution of eq. (1). The first structure equation of Cartan is equivalent to the definition of the Riemannian torsion:

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \quad -(3)$$

Now consider:

$$R^\lambda{}_{\mu\rho\nu} = \partial_\mu \Gamma^\lambda_{\rho\nu} - \partial_\rho \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\rho\nu} - \Gamma^\lambda_{\rho\sigma} \Gamma^\sigma_{\mu\nu} \quad -(4)$$

where:

$$R^\lambda{}_{\mu\nu\rho} = - R^\lambda{}_{\mu\rho\nu} \quad -(5)$$

as is well known.

Substituting (4) from (2):

2)

$$2R^{\lambda}_{\mu\nu\rho} = \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho}$$

$$- \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \partial_\rho \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu} + \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu}$$

$$= \partial_\mu T^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}_{\nu\rho}$$

$$- \left(\partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} - \partial_\rho \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} \right)$$

— (6)

If it is assumed incorrectly that:

$$\Gamma^{\lambda}_{\mu\nu} \stackrel{?}{=} \Gamma^{\lambda}_{\nu\mu} \quad -(7)$$

then

$$R^{\lambda}_{\mu\nu\rho} \stackrel{?}{=} -\frac{1}{2}\left(\partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} - \partial_\rho \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} \right)$$

— (8)

which contradicts eq. (2), reductio ad absurdum.

If it is assumed correctly that:

$$\Gamma^{\lambda}_{\mu\nu} = - \Gamma^{\lambda}_{\nu\mu} \quad -(9)$$

then:

$$\boxed{\begin{aligned} R^{\lambda}_{\mu\nu\rho} = {} & \frac{1}{2}\left(\partial_\mu T^{\lambda}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}_{\nu\rho} \right) \\ & - \frac{1}{4}\left(\partial_\nu T^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\nu\sigma} T^{\sigma}_{\mu\rho} - \partial_\rho T^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\rho\sigma} T^{\sigma}_{\mu\nu} \right) \end{aligned}} \quad -(10)$$

3) because:

$$T^{\lambda}_{\mu\rho} = 2\Gamma^{\lambda}_{\mu\rho} \quad -(11)$$

$$T^{\lambda}_{\mu\nu} = 2\Gamma^{\lambda}_{\mu\nu}. \quad -(12)$$

Eq. (10) correctly expresses the conventional definition (2) in terms of torsion, and eq. (10) shows that if the connection is symmetric, so that the torsion and curvature vanish. This is also the result of the commutator method.