211(7): Further Proof of the Antisymmetry of the Connection.

Assume that the Christoffel connection is in general asymmetric:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\left(\Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\nu\mu}\right) + \frac{1}{2}\left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) \quad -(1)$$

with symmetric and antisymmetric components. Eq. (1) is a well known general theorem of matrices.

Now note that the Cartan identity is true for eq. (1).

The Cartan identity is:

$$R^\lambda_{\ \mu\nu\rho} + R^\lambda_{\ \rho\mu\nu} + R^\lambda_{\ \nu\rho\mu} := \partial_\mu T^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma_{\nu\rho} + \partial_\rho T^\lambda_{\mu\nu} + \Gamma^\lambda_{\rho\sigma} T^\sigma_{\mu\nu} + \partial_\nu T^\lambda_{\rho\mu} + \Gamma^\lambda_{\nu\sigma} T^\sigma_{\rho\mu}. \quad -(2)$$

Assume that:

$$R^\lambda_{\ \mu\nu\rho} = \partial_\mu T^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma_{\nu\rho} \quad -(3)$$

et cyclicum

the three equations (3) are mathematical solutions of eq. (2).

Note that:

$$[D_\mu, D_\nu] V^\rho = R^\rho_{\ \sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} D_\lambda V^\rho \quad -(4)$$

where:

$$R^\rho_{\ \sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \quad -(5)$$

and

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad -(6)$$

2)

In general, the connection in eqs. (4) to (6) is defined by eq. (1). This is also true for eq. (2).

This set of equations must be rigorously self-consistent.

The tensor defined by eq. (3) and eq. (5) must be the same tensor therefore, otherwise the geometry is not self consistent. Therefore:

$$R^{\lambda}_{\ \mu\nu\rho} = \partial_\mu T^{\lambda}_{\ \nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}_{\ \nu\rho} \quad -(7)$$

$$= \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho}.$$

If the connection is symmetric:

$$R^{\lambda}_{\ \mu\nu\rho} = 0. \quad -(8)$$

Therefore

$$\boxed{\Gamma^{\lambda}_{\mu\nu} = -\Gamma^{\lambda}_{\nu\mu}} \quad -(9)$$

<u>Q.E.D.</u>

If the connection is symmetric, then:

$$\mu = \nu \quad -(10)$$

in which case

$$[D_\mu, D_\nu] = 0 \quad -(11)$$

and from eq. (4):

$$R^{\lambda}_{\ \mu\nu\rho} = 0, \quad T^{\lambda}_{\ \mu\nu} = 0, \quad -(12)$$

<u>reductio</u> <u>ad</u> <u>absurdum</u>.

3) Reduction to Absurdity, Reductio ad Absurdum

Note that eq. (2) implies:

$$R^\lambda_{\mu\nu\rho} = \partial_\mu T^\lambda_{\nu\rho} + \partial_\nu T^\lambda_{\rho\mu} + \Gamma^\lambda_{\mu\sigma} T^\sigma_{\nu\rho} + \Gamma^\lambda_{\nu\sigma} T^\sigma_{\rho\mu}$$

$$+ \partial_\mu \Gamma^\lambda_{\rho\nu} - \partial_\nu \Gamma^\lambda_{\rho\mu} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\rho\nu} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\rho\mu} \quad -(13)$$

$$= \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}$$

a) Eq (13) implies that eq. (5) is true simultaneously with a non-zero torsion defined by eq. (6), because in both cases the connection is asymmetric in general. If the connection is symmetric, both eqs. (3) and eq (4) show that the curvature vanishes, reductio ad absurdum.

b) The assumption of a symmetric connection ~~shows~~ implies that eq. (3) cannot be true, because eq. (3) would result in zero curvature. However, if eq. (3) were not true eq. (13) would reduce to:

$$R^\lambda_{\mu\nu\rho} = R^\lambda_{\mu\nu\rho} \quad -(14)$$

and the only possible solution of eq. (2) would be eq. (5) with a symmetric connection and eq. (6) with zero torsion. This is an absurdity because eq. (2) is true for any connection, reductio ad absurdum.

From eqs. (7) and (9) :

$$\partial_\mu T^\lambda{}_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma{}_{\nu\rho} \qquad -(15)$$

$$= \frac{1}{2}\left(\partial_\mu T^\lambda{}_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma{}_{\nu\rho} - \left(\partial_\nu T^\lambda{}_{\mu\rho} + \Gamma^\lambda_{\nu\sigma} T^\sigma{}_{\mu\rho}\right)\right)$$

so:

$$\boxed{\partial_\mu T^\lambda{}_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma{}_{\nu\rho} = -\left(\partial_\nu T^\lambda{}_{\mu\rho} + \Gamma^\lambda_{\nu\sigma} T^\sigma{}_{\mu\rho}\right)} \quad -(16)$$

<u>et</u> <u>cyclicum</u>.

In differential form notation: $\quad -(17)$

$$\boxed{\begin{gathered}\partial_\mu T^a{}_{\nu\rho} + \omega^a{}_{\mu b} T^b{}_{\nu\rho} = -\left(\partial_\nu T^a{}_{\mu\rho} + \omega^a{}_{\nu b} T^b{}_{\mu\rho}\right)\\ \text{et cyclicum}\end{gathered}}$$

$-(17)$

<u>The equations</u> <u>are again</u> <u>antisymmetric</u> <u>under</u> the <u>exchange</u> of <u>μ and ν</u>.