211(5): Coordinate Transformation of the Christoffel Connection

The Christoffel connection is defined by:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \quad -(1)$$

Under the general coordinate transformation:

$$D_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} D_\mu V^\nu \quad -(2)$$

Using the Leibniz theorem:

$$D_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} V^\nu \frac{\partial}{\partial x^\mu} \frac{\partial x^{\nu'}}{\partial x^\nu}$$

$$+ \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda \quad -(3)$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} V^\lambda$$

so:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} \quad -(4)$$

$$- \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}.$$

Partial derivatives commute, so the second

2)
term in eq. (4) vanishes by symmetry:

$$\boxed{\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\lambda}} = 0} \quad -(5)$$

where there is summation over repeated indices. The antisymmetric connection therefore transforms as a tensor:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} \, . \quad -(6)$$

In the obsolete literature:

$$\Gamma^{\nu}_{\mu\lambda} = ? \; \Gamma^{\nu}_{\lambda\mu} \quad -(7)$$

and in consequence the Christoffel connection did not transform as a tensor. In the correct geometry:

$$\Gamma^{\nu}_{\mu\lambda} = - \Gamma^{\nu}_{\lambda\mu} \quad -(8)$$

and the connection does not have a symmetric component in any frame of reference. Eq. (5) is a fundamental constraint on geometry.