211(6) : Further Consequences of the Antisymmetric Connection

Consider the Riemannian formats of the first and second Cartan structure identities :

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad -(1)$$

and

$$R^\lambda_{\mu\nu\rho} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}. \quad -(2)$$

The Cartan identity is :

$$\begin{aligned} R^\lambda_{\mu\nu\rho} + R^\lambda_{\rho\mu\nu} + R^\lambda_{\nu\rho\mu} := {} & \partial_\mu T^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma_{\nu\rho} \\ & + \partial_\rho T^\lambda_{\mu\nu} + \Gamma^\lambda_{\rho\sigma} T^\sigma_{\mu\nu} \quad -(3) \\ & + \partial_\nu T^\lambda_{\rho\mu} + \Gamma^\lambda_{\nu\sigma} T^\sigma_{\rho\mu} \end{aligned}$$

In note 211(2) it was proven ~~that~~ these basic equations imply :

$$\Gamma^\lambda_{\mu\nu} = -\Gamma^\lambda_{\nu\mu} \quad -(4)$$

otherwise they are self-contradictory.

From eqs. (1), (2) and (4) it follows that :

$$R^\lambda_{\mu\nu\rho} = \frac{1}{2}\left(\partial_\mu T^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma_{\nu\rho} - \left(\partial_\nu T^\lambda_{\mu\rho} + \Gamma^\lambda_{\nu\sigma} T^\sigma_{\mu\rho}\right)\right) \quad -(5)$$

2) Now consider the three structure equations:

$$R^\lambda{}_{\mu\nu\rho} = \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}$$

$$R^\lambda{}_{\rho\mu\nu} = \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\rho\nu} \quad -(6)$$

$$R^\lambda{}_{\nu\rho\mu} = \partial_\nu \Gamma^\lambda_{\rho\mu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\rho\mu} - \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\nu\mu}$$

and the antisymmetry (4) to find that:

$$R^\lambda{}_{\mu\nu\rho} = \frac{1}{2}\left(\partial_\mu T^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma_{\nu\rho} - \partial_\nu T^\lambda_{\mu\rho} - \Gamma^\lambda_{\nu\sigma} T^\sigma_{\mu\rho}\right)$$

$$R^\lambda{}_{\rho\mu\nu} = \frac{1}{2}\left(\partial_\rho T^\lambda_{\mu\nu} + \Gamma^\lambda_{\rho\sigma} T^\sigma_{\mu\nu} - \partial_\mu T^\lambda_{\rho\nu} - \Gamma^\lambda_{\mu\sigma} T^\sigma_{\rho\nu}\right).$$

$$R^\lambda{}_{\nu\rho\mu} = \frac{1}{2}\left(\partial_\nu T^\lambda_{\rho\mu} + \Gamma^\lambda_{\nu\sigma} T^\sigma_{\rho\mu} - \partial_\rho T^\lambda_{\nu\mu} - \Gamma^\lambda_{\rho\sigma} T^\sigma_{\nu\mu}\right) \quad -(7)$$

Use the ECE antisymmetry law to find that:

$$R^\lambda{}_{\mu\nu\rho} = \partial_\mu T^\lambda_{\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^\sigma_{\nu\rho} \quad -(8)$$

$$R^\lambda{}_{\rho\mu\nu} = \partial_\rho T^\lambda_{\mu\nu} + \Gamma^\lambda_{\rho\sigma} T^\sigma_{\mu\nu}$$

$$R^\lambda{}_{\nu\rho\mu} = \partial_\nu T^\lambda_{\rho\mu} + \Gamma^\lambda_{\nu\sigma} T^\sigma_{\rho\mu}$$

Adding gives eq. (1) of note 211(2). The three equations (8) are three Evans identities.

3) The sum can be written as the Cartan identity:

$$R^{\lambda}{}_{\mu\nu\rho} + R^{\lambda}{}_{\rho\mu\nu} + R^{\lambda}{}_{\nu\rho\mu} =: D^*_{\mu} T^{\lambda}{}_{\nu\rho} + D^*_{\rho} T^{\lambda}{}_{\mu\nu} + D^*_{\nu} T^{\lambda}{}_{\rho\mu} \quad -(9)$$

where:

$$D^*_{\mu} T^{\lambda}{}_{\nu\rho} = \partial_{\mu} T^{\lambda}{}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}{}_{\nu\rho} \quad -(10)$$

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The sum on the right hand side is the covariant wedge derivative in Riemann format. This must be distinguished from the covariant derivative of a rank three tensor, which is:

$$D_{\mu} T^{\kappa}{}_{\nu\sigma} = \partial_{\mu} T^{\kappa}{}_{\nu\sigma} + \Gamma^{\kappa}_{\mu\lambda} T^{\lambda}{}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu} T^{\kappa}{}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma} T^{\kappa}{}_{\nu\lambda} \quad -(11)$$

As shown in UFT 209:

$$\partial_{\mu} T^{a}{}_{\nu\rho} + \omega^{a}{}_{\mu b} T^{b}{}_{\nu\rho} = \left(\partial_{\mu} T^{\lambda}{}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}{}_{\nu\rho} \right) q^{a}{}_{\lambda} \quad -(12)$$

Multiplying both sides of eq. (9) by $q^{a}{}_{\lambda}$ gives

$$D \wedge T^{a} := R^{a}{}_{b} \wedge q^{b} \quad -(13)$$

which is the Cartan identity of differential geometry, Q.E.D.

4) The covariant wedge derivative of Cartan is:

$$D \wedge T^a = \partial_\mu T^a_{\nu\rho} + \omega^a_{\mu b} T^b_{\nu\rho} + \partial_\rho T^a_{\mu\nu} + \omega^a_{\rho b} T^b_{\mu\nu} + \partial_\nu T^a_{\rho\mu} + \omega^a_{\nu b} T^b_{\rho\mu} \quad -(13)$$

and the wedge product is:

$$R^a_{\ b} \wedge q^b = R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu} \quad -(14)$$

$$= q^b \wedge R^a_{\ b}$$

In Riemann format the covariant wedge derivative is:

$$D \wedge T^\lambda = D^*_\mu T^\lambda_{\nu\rho} + D^*_\rho T^\lambda_{\mu\nu} + D^*_\nu T^\lambda_{\rho\mu} \quad -(15)$$

Raising indices in eq. (8):

$$R^{\lambda\ \ \nu\rho}_{\ \mu} = \partial_\mu T^{\lambda\nu\rho} + \Gamma^\lambda_{\mu\sigma} T^{\sigma\nu\rho} \quad -(16)$$

et cyclicum

When

$$\mu = \nu \quad -(17)$$

$$R^{\lambda\ \ \mu\rho}_{\ \mu} = \partial_\mu T^{\lambda\mu\rho} + \Gamma^\lambda_{\mu\sigma} T^{\sigma\mu\rho} \quad -(18)$$

which is the Evans identity used in "Criticism of the Einstein Field Equation", Q.E.D

5)

From eq. (6):

$$R^{\lambda}{}_{\mu\nu\rho} = \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \quad -(19)$$

$$R^{\lambda}{}_{\mu\rho\nu} = \partial_\mu \Gamma^{\lambda}_{\rho\nu} - \partial_\rho \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} \quad -(20)$$

Subtract (20) from (19):

$$\begin{aligned} 2R^{\lambda}{}_{\mu\nu\rho} = {} & \partial_\mu T^{\lambda}{}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}{}_{\nu\rho} \\ & + \partial_\rho \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \\ & - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} \end{aligned} \quad -(21)$$

If the torsion is zero then:

$$\begin{aligned} 2R^{\lambda}{}_{\mu\nu\rho} = {} & \partial_\rho \Gamma^{\lambda}_{\nu\mu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \\ & - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} \end{aligned} \quad -(22)$$

which is not the structure equation (19). From eqs. (19) and (20) it is seen that eq. (22) is:

$$0 = 0. \quad -(23)$$

The precise structure of the Cartan identity is

$$\begin{aligned} R^{\lambda}{}_{\mu\nu\rho} + R^{\lambda}{}_{\rho\mu\nu} + R^{\lambda}{}_{\nu\rho\mu} := {} & \partial_\mu T^{\lambda}{}_{\nu\rho} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}{}_{\nu\rho} \\ & + \partial_\rho T^{\lambda}{}_{\mu\nu} + \Gamma^{\lambda}_{\rho\sigma} T^{\sigma}{}_{\mu\nu} \\ & + \partial_\nu T^{\lambda}{}_{\rho\mu} + \Gamma^{\lambda}_{\nu\sigma} T^{\sigma}{}_{\rho\mu} \end{aligned} \quad -(24)$$

) When written out in full:

$$R^{\lambda}_{\mu\nu\rho} + R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\rho\mu}$$

$$:= \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\mu\sigma}\left(\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\rho\nu}\right)$$
$$+ \partial_\rho \Gamma^{\lambda}_{\mu\nu} - \partial_\rho \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\rho\sigma}\left(\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu}\right) \quad -(25)$$
$$+ \partial_\nu \Gamma^{\lambda}_{\rho\mu} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\nu\sigma}\left(\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\sigma}_{\mu\rho}\right)$$

It is seen that the structure eq. (19) is obtained as follows:

$$R^{\lambda}_{\mu\nu\rho} = \partial_\mu T^{\lambda}_{\nu\rho} + \partial_\nu T^{\lambda}_{\rho\mu} + \Gamma^{\lambda}_{\mu\sigma} T^{\sigma}_{\nu\rho} + \Gamma^{\lambda}_{\nu\sigma} T^{\sigma}_{\rho\mu}$$
$$+ \partial_\mu \Gamma^{\lambda}_{\rho\nu} - \partial_\nu \Gamma^{\lambda}_{\rho\mu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\rho\mu}$$
$$-(26)$$

The second structure of Cartan is a particular solution of the Cartan identity. It is true if and only if the connection is antisymmetric.

Otherwise the torsion vanishes in eq. (26)

and $$R^{\lambda}_{\mu\nu\rho} =? \; \partial_\mu \Gamma^{\lambda}_{\rho\nu} - \partial_\nu \Gamma^{\lambda}_{\rho\mu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\rho\mu}$$
$$-(27)$$

and this is not the structure equation.

The correct structure equation is equation (19). The sum (25) can be rearranged to give the sum of the three parts of eq. (6) but the first structure equation is also incorporated in eq. (25), which is the precisely correct format of the Cartan identity.

Finally, note that eqs. (19) and (27) can be the same if and only if

$$\Gamma^{\lambda}_{\rho\nu} = \Gamma^{\lambda}_{\nu\rho} \quad -(28)$$

in which case eq. (25) gives:

$$R^{\lambda}_{\mu\nu\rho} + R^{\lambda}_{\rho\mu\nu} + R^{\lambda}_{\nu\rho\mu} = 0. \quad -(29)$$

This would mean that the second Cartan structure equation would be valid only for a symmetric connection, but it is valid for any connection. The three parts of eq. (8) are also valid solutions of eq. (25), and in fact are the natural choice of structure equation. Eqs. (8) show clearly that if the connection is symmetric, then:

$$R^{\lambda}_{\mu\nu\rho} = R^{\lambda}_{\rho\mu\nu} = R^{\lambda}_{\nu\rho\mu} = 0 \quad -(30)$$