208(6): Development of Log. Spiral and Archimedes Spiral Orbits.

Log Spiral

This is:

$$r = r_0 e^{\alpha\theta} \quad -(1)$$

so

$$\frac{dr}{d\theta} = \alpha r \; ; f = r^2 \left(\frac{d\theta}{dr}\right)^2 = \frac{1}{\alpha^2} \quad -(2)$$

The equation of motion is:

$$\frac{d\omega}{d\theta} = -\omega \frac{d^2 f / d\theta^2}{df / d\theta}, \quad -(3)$$

so:

$$\frac{d\omega}{d\theta} = -\omega \left(\frac{0}{0}\right) \quad -(4)$$

and is mathematically indeterminate. If it is assumed that:

$$\frac{0}{0} \rightarrow 1 \quad -(5)$$

then

$$\boxed{\frac{d\omega}{d\theta} = -\omega} \quad -(6)$$

The arc length is:

$$r = \int_{r_1}^{r_2} (1 + f)^{1/2} dr \quad -(7)$$

$$r = \left(1 + \frac{1}{\alpha^2}\right)^{1/2} (r_2 - r_1) \quad -(8)$$

and:

2)

$$r \to \infty, \quad r_2 \to \infty, \quad r_1 \to 0. \qquad -(9)$$

However, for the logarithmic spiral there is no torsion. Therefore the solution of eqn (6) is:

$$\omega = 0. \qquad -(10)$$

Spiral of Archimedes

This is:

$$r = a + b\theta \qquad -(11)$$

and

$$f = \left(\frac{r}{b}\right)^2 = \frac{1}{b^2}(a + b\theta)^2 \qquad -(12)$$

$$= \frac{1}{b^2}\left(a^2 + 2ab\theta + b^2\theta^2\right).$$

So

$$\frac{df}{d\theta} = 2\theta\left(1 + \frac{a}{b}\right) \qquad -(13)$$

$$\frac{d^2 f}{d\theta^2} = 2\left(1 + \frac{a}{b}\right) \qquad -(14)$$

and

$$\boxed{\frac{d\omega}{d\theta} = -\omega\theta.} \qquad -(15)$$

The arc length is:

$$r = \int_{r_1}^{r_2}\left(1 + \frac{r^2}{b^2}\right)^{1/2} dr. \qquad -(16)$$

In the case:

$$a = 0 \qquad -(17)$$

the spiral is:

$$r = b\theta \qquad -(18)$$

the solution of eq. (15) is:

$$\int \frac{d\omega}{\omega} = -\int \theta \, d\theta \quad \text{--- (19)}$$

in which:

$$r = b\theta \, , \quad \frac{dr}{d\theta} = b \quad \text{--- (20)}$$

so

$$\int \frac{d\omega}{\omega} = -\int \frac{r}{b^2} \, dr \quad \text{--- (21)}$$

i.e.

$$\log_e \omega = -\frac{r^2}{2b^2} \quad \text{--- (22)}$$

assuming that the constant of integration is zero.

For correct dimensionality:

$$\log_e \omega = -\frac{r^2}{2b^2} + \log_e \omega_0 \quad \text{--- (23)}$$

so

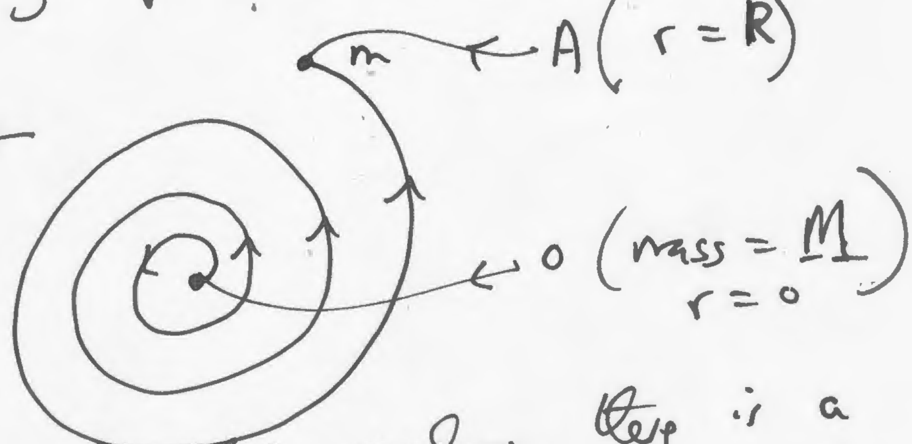
$$\log_e \frac{\omega}{\omega_0} = -\frac{r^2}{2b^2} \quad \text{--- (24)}$$

and

$$\omega = \omega_0 \exp\left(-\frac{r^2}{2b^2}\right) \quad \text{--- (25)}$$

4) in which ω_0 is a characteristic frequency so that the units of eq. (23) are correct.

Spiral Galaxy



At the centre of the galaxy there is a region spinning with angular velocity ω_0 at $r = 0$. So from equation (23):

$$\omega = \omega_0 \ (\text{initially}) \quad \cancel{-(24)}$$

At the point A the angular velocity has slowed to:

$$\omega = \omega_0 \exp\left(-\left(\frac{R}{b}\right)^2\right) \quad \cancel{-(25)}$$

where

$$R = \int_0^R \left(1 + \left(\frac{r}{b}\right)^2\right)^{1/2} dr \quad -(26)$$

The torsion is:

$$T^1{}_{01} = \frac{1}{c(1+f)} \frac{df}{dt} = \frac{\omega}{2c(1+f)} \frac{df}{d\theta}, \quad -(27)$$

$$f = \left(\frac{r}{b}\right)^2, \quad \frac{df}{d\theta} = 2\frac{r}{b}$$

so:

$$T^1{}_{01} = \frac{\omega r/b}{c\left(1+\frac{r^2}{b^2}\right)} = \frac{\omega}{c}\left(\frac{x}{1+x^2}\right) \quad -(28)$$

where

$$x = \frac{r}{b} \quad -(29)$$

From eqs. (23) and (28):

$$\boxed{T^1{}_{01} = \frac{\omega_0}{c}\left(\frac{x}{1+x^2}\right)\exp\left(-x^2\right)} \quad -(30)$$

<u>Results</u>

For the Archimedes spiral the whirlpool galaxy is characterized by the torsion:

$$T^1{}_{01} = \frac{\omega_0}{c}\left(\frac{x}{1+x^2}\right)e^{-x^2} \quad -(31)$$

where

$$x = \frac{r}{b} \quad -(32)$$

and the angular velocity:

$$\omega = \omega_0\, e^{-x^2} \quad -(33)$$

So

$$T^1{}_{01} = \frac{\omega}{c}\left(\frac{x}{1+x^2}\right) \quad -(34)$$