208(4): Checking and Development of the Equation of Motion

First consider:

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(\omega\frac{df}{d\theta}\right) \quad -(1)$$

$$= \frac{1}{2}\left(\left(\frac{\partial \omega}{\partial t}\right)\left(\frac{df}{d\theta}\right) + \omega\frac{\partial}{\partial t}\left(\frac{df}{d\theta}\right)\right)$$

$$= 0.$$

Secondly consider:

$$F = \frac{\partial f}{\partial t}. \quad -(2)$$

It is seen that:

$$F = F\left(r(t), \theta(t)\right). \quad -(3)$$

For example, for a hyperbolic spiral:

$$f = \theta^2 \quad -(4)$$

and

$$\theta = \theta(r, t). \quad -(5)$$

From eq. (3), proceed as in note 207(8) to find that:

$$\frac{\partial F}{\partial t} = \frac{1}{2}\omega\frac{dF}{d\theta} \quad -(6)$$

so:

$$\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(\omega\frac{d}{d\theta}\left(\frac{\partial f}{\partial t}\right)\right)$$

$$= \frac{1}{2}\omega\frac{d}{d\theta}\left(\frac{1}{2}\omega\frac{df}{d\theta}\right) \quad -(7)$$

$$= \frac{1}{4}\omega\frac{d}{d\theta}\left(\omega\frac{df}{d\theta}\right) = 0$$

From eqs. (1) and (7):

$$\boxed{\left(\frac{\partial\omega}{\partial t}\right)\left(\frac{df}{d\theta}\right) + \omega\frac{\partial}{\partial t}\left(\frac{df}{d\theta}\right) = \frac{1}{2}\omega\frac{d}{d\theta}\left(\omega\frac{df}{d\theta}\right) = 0} \quad -(8)$$

The solutions of eq. (8) are:

$$\omega = 0 \quad -(9)$$

and

$$\frac{d}{d\theta}\left(\omega\frac{df}{d\theta}\right) = 0 \quad -(10)$$

so

$$\frac{d\omega}{d\theta}\frac{df}{d\theta} + \omega\frac{d^2f}{d\theta^2} = 0 \quad -(11)$$

$$\boxed{\frac{d\omega}{d\theta} = -\frac{\omega\, d^2f/d\theta^2}{df/d\theta}} \quad -(12)$$