208(3): Arc Length Along the Hyperbolic Spiral.

The infinitesimal element of arc length is:

$$ds = \left(dr^2 + r^2 d\theta^2 \right)^{1/2} \quad -(1)$$

For the hyperbolic spiral:

$$ds^2 = \left(1 + f \right) dr^2 \quad -(2)$$

where

$$f = \theta^2 = \left(\frac{r_0}{r} \right)^2 . \quad -(3)$$

So

$$r = \int_{r_1}^{r_2} \left(1 + f \right)^{1/2} dr \quad -(4)$$

$$= \int_{r_1}^{r_2} \left(1 + \frac{r_0^2}{r^2} \right)^{1/2} dr$$

the integral can be evaluated analytically to give:

$$r = \left(r_2^2 + r_0^2 \right)^{1/2} - \left(r_1^2 + r_0^2 \right)^{1/2} - r_0 \left(\log_e \left(\frac{r_0 + \left(r_2^2 + r_0^2 \right)^{1/2}}{r_2} \right) - \log_e \left(\frac{r_0 + \left(r_1^2 + r_0^2 \right)^{1/2}}{r_1} \right) \right) \quad -(5)$$

$$= \int_{r_1}^{r_2} \left(1 + \theta^2 \right)^{1/2} dr$$

$$= - r_0 \int_{\theta_1}^{\theta_2} \left(\frac{1 + \theta^2}{\cancel{\theta^4}} \right)^{1/2} d\theta$$

2)

So:

$$r = \int_{r_1}^{r_2} \left(1 + \frac{r_0^2}{r^2}\right)^{1/2} dr = r_0 \int_{r_0/\theta_2}^{r_0/\theta_1} \left(\frac{1+\theta^2}{\theta^2}\right)^{1/2} d\theta \quad -(6)$$

If $\quad r_2 \to \infty, \quad r_1 \to 0 \quad -(7)$

then:

$$r \to \left(r_2^2 + r_0^2\right)^{1/2} - r_0 - \log_e\left(\frac{2r_0}{r_1}\right) \quad -(8)$$

$$\boxed{r \to r_2 - \log_e\left(\frac{2r_0}{r_1}\right)} \quad -(9)$$

Now use the standard integral:

$$\int \frac{(x^2+a^2)^{1/2}}{x^2} dx = -\frac{(x^2+a^2)^{1/2}}{x} + \log_e\left(x + (x^2+a^2)^{1/2}\right) \quad -(10)$$

so

$$\int \frac{(1+\theta^2)^{1/2}}{\theta^2} d\theta = -\frac{(1+\theta^2)^{1/2}}{\theta} + \log_e\left(\theta + (1+\theta^2)^{1/2}\right) \quad -(11)$$

From eqns. (6) and (10):

$$= r_0 \left(-\frac{(1+\theta^2)^{1/2}}{\theta} + \log_e\left(\theta + (1+\theta^2)^{1/2}\right) \Bigg|_{r_0/\theta_2}^{r_0/\theta_1} \right) \quad -(12)$$

So:

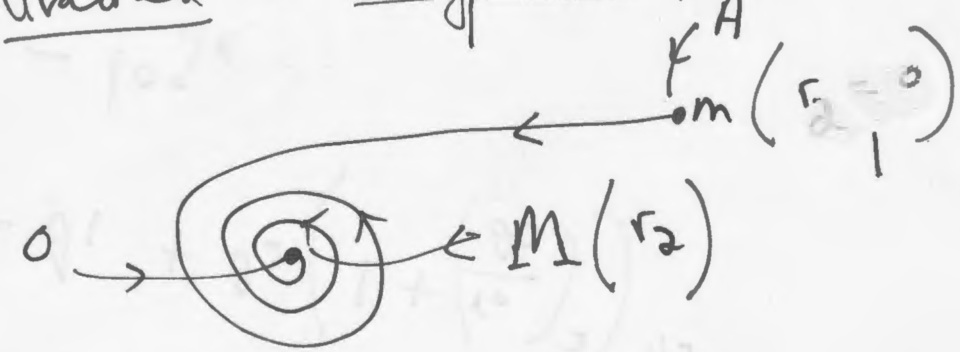
$$r = r_0\left(\frac{(1+\theta_2^2)^{1/2}}{\theta_2} - \frac{(1+\theta_1^2)^{1/2}}{\theta_1} + \log_e\left(\theta_1 + (1+\theta_1^2)^{1/2}\right) - \log_e\left(\theta_2 + (1+\theta_2^2)^{1/2}\right)\right) \quad -(13)$$

If $\theta_2 \to 0, \ \theta_1 \to \infty \quad -(14)$

then $$r \to r_0\left(\frac{1}{\theta_2} - 1 + \log_e(2\theta_1)\right) \quad -(15)$$

$$\to \infty$$

<u>Particle</u> <u>Attracted</u> on a <u>Hyperbolic Spiral</u>



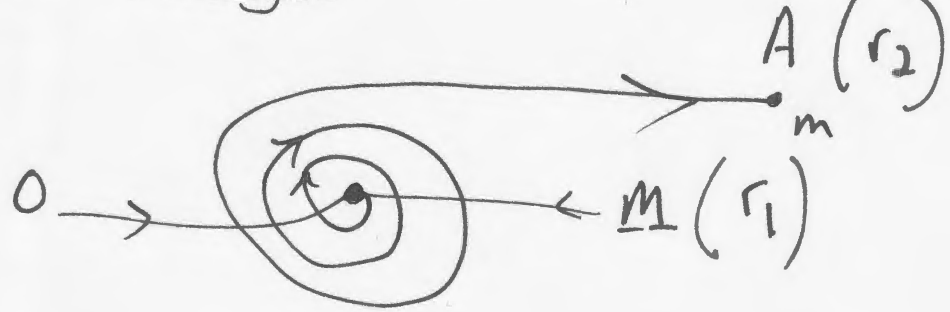
In this case: $r_2 > r_1 \quad -(16)$

The distance from A to O is given by eq. (5) or (13). In the limit (→) or (14) this distance becomes infinite. In the region near <u>M</u> the particle m acquires a very large angular velocity given by:

$$\frac{\omega}{\omega_0} = \frac{r}{r_0} = \frac{1}{\theta} \quad - (17)$$

The equation of motion (17) is derived from the constrained Minkowski method.

Whirlpool Galaxy



In this case:

$$r_2 > r_1 \quad - (18)$$

but the particle m (a star) is carried out of the central region of mass M by spacetime torsion:

$$T^{1}_{\ 01} = \frac{\partial f / \partial t}{c(1+f)}, \quad - (19)$$

$$f = \theta^2. \quad - (20)$$

Initially its ~~velocity~~ angular velocity is very large, and it gradually dissipates itself to zero, when the star travels on a straight line with constant linear velocity and no angular motion, as observed experimentally.