208(a): Development of the Equation of Motion for Orbits in the Solar System.

The solar system orbit is the precessing ellipse.

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad - (1)$$

and the new equation of motion is:

$$\frac{d\omega}{d\theta} = - F\omega \quad - (2)$$

where

$$F = \frac{d^2 f / d\theta^2}{df / d\theta} \quad - (3)$$

The equation (3) can be expressed as:

$$\frac{d\omega}{d\theta}\frac{df}{d\theta} + \omega \frac{d^2 f}{d\theta^2} = 0. \quad - (4)$$

For a circular orbit:

$$\frac{d\omega}{d\theta} = 0, \quad \omega = vr = \text{constant} \quad - (5)$$

so

$$\frac{d^2 f}{d\theta^2} = 0 \quad - (6)$$

where

$$f = r^2 \left(\frac{d\theta}{dr}\right)^2 \quad - (7)$$

and

$$\frac{d\theta}{dr} = \lim_{\delta r \to 0} \left(\frac{f(r + \delta r) - f(r)}{\delta r} \right). \quad - (8)$$

by definition.

2)

For a circular orbit θ does not depend on r and θ does not change with r, i.e. θ and r are independent and

$$d\theta / dr = 0 \quad -(9)$$

So eq. (6) is true.

In the solar system:

$$\epsilon << 1 \quad -(10)$$

by observation.

So:

$$r = \alpha \left(1 + \epsilon \cos x\theta \right)^{-1}$$

$$\sim \alpha \left(1 - \epsilon \cos x\theta \right). \quad -(11)$$

It is also observed that:

$$x = 1 \quad -(12)$$

to an excellent approximation. So:

$$\boxed{r \sim \alpha \left(1 - \epsilon \cos \theta \right)} \quad -(13)$$

Therefore:

$$\frac{dr}{d\theta} = \epsilon \alpha \sin\theta \quad -(14)$$

and

$$f = \left(\frac{r}{\epsilon \alpha \sin\theta} \right)^2 \quad -(15)$$

$$= \left(\frac{1 - \epsilon \cos\theta}{\epsilon \sin\theta} \right)^2$$

3) Using eq. (10):

$$f \sim \left(\frac{1}{\epsilon \sin\theta}\right)^2 \quad -(16)$$

So

$$\frac{df}{d\theta} = \frac{1}{\epsilon^2}\frac{d}{d\theta}\left(\frac{1}{\sin^2\theta}\right)$$

$$= -\frac{1}{\epsilon^2}\frac{\cos\theta}{\sin^3\theta}$$

and:

$$\frac{d^2 f}{d\theta^2} = -\frac{1}{\epsilon^2}\left(\frac{-\sin^4\theta - 3\sin^2\theta\cos^2\theta}{\sin^6\theta}\right)$$

$$= \frac{1}{\epsilon^2}\left(\frac{\sin^2\theta + 3\cos^2\theta}{\sin^4\theta}\right)$$

$$= \frac{1}{\epsilon^2}\left(\frac{1+2\cos^2\theta}{\sin^4\theta}\right). \quad -(17)$$

So

$$F = -\left(\frac{1+2\cos^2\theta}{\sin^4\theta}\right)\frac{\sin^3\theta}{\cos\theta}$$

$$\boxed{F = -\left(\frac{1+2\cos^2\theta}{\sin\theta\cos\theta}\right) \text{ for } \epsilon \ll 1} \quad -(18)$$

and

$$\frac{d\omega}{d\theta} = \left(\frac{1+2\cos^2\theta}{\sin\theta\cos\theta}\right)\omega \quad -(19)$$

4)

Resolution is :

$$\int \frac{d\omega}{\omega} = \int \left(\frac{1 + 2\cos^2\theta}{\sin\theta\cos\theta} \right) d\theta = g(\theta) \quad -(20)$$

where $g(\theta)$ can be found by computer or integral tables. So

$$\omega = \omega_0 \exp(g(\theta)). \quad -(21)$$

However, in the limit (10), the orbital angular frequency is almost constant. Therefore eq. (21) is a small perturbation on the complete solution:

$$\Omega = \omega_1 + \omega \quad -(22)$$

where ω_1 is a constant such that:

$$\frac{d\Omega}{d\theta} = \frac{d\omega}{d\theta}. \quad -(23)$$

So

$$\boxed{\Omega = \omega_1 + \omega_0 \exp(g(\theta))} \quad -(24)$$

<u>Conclusion</u>

The theory predicts small perturbations in the orbit - given by the second term in eq. (24). These are not present in Newtonian orbits