207(2): Metric Compatibility and Equations of Motion

The metric compatibility equation is:

$$D_\rho g_{\mu\nu} = 0 \quad -(1)$$

where

$$g_{00} = 1, \; g_{11} = -\left(1 + \frac{r^2}{f^2}\right) -(2)$$

and

$$f = \frac{dr}{d\theta}. \quad -(3)$$

Let:

$$\rho = 0 \quad -(4)$$

then:

$$\partial_0 g_{\mu\nu} - \Gamma^\lambda_{0\mu} g_{\lambda\nu} - \Gamma^\lambda_{0\nu} g_{\mu\lambda} = 0 \quad -(5)$$

so

$$\partial_0 g_{00} - \Gamma^\lambda_{00} g_{\lambda 0} - \Gamma^\lambda_{00} g_{0\lambda} = 0. \quad -(6)$$

However:

$$\Gamma^\lambda_{00} = 0, \quad -(7)$$

so

$$\boxed{\partial_0 g_{00} = 0,} \quad -(8)$$

which is true because of eq. (2).

Also:

$$\partial_0 g_{11} - \Gamma^\lambda_{01} g_{\lambda 1} - \Gamma^\lambda_{01} g_{1\lambda} = 0 \quad -(9)$$

i.e.

$$\partial_0 g_{11} - \Gamma^1_{01} g_{11} - \Gamma^1_{01} g_{11} = 0 \quad -(10)$$

and

$$\boxed{\Gamma^1_{01} = \frac{1}{2g_{11}} \partial_0 g_{11}} \quad -(11)$$

Let

$$\rho = 1 \quad -(12)$$

2) Then:

$$\partial_1 g_{\mu\nu} - \Gamma^{\lambda}_{1\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{1\nu} g_{\mu\lambda} = 0 \quad -(13)$$

then

$$\partial_1 g_{00} - \Gamma^{\lambda}_{10} g_{\lambda 0} - \Gamma^{\lambda}_{10} g_{0\lambda} = 0 \quad -(14)$$

$$2\Gamma^{0}_{10} g_{00} = 0 \quad -(15)$$

$$\boxed{\Gamma^{0}_{10} = 0} \quad -(16)$$

Also:

$$\partial_1 g_{11} - \Gamma^{\lambda}_{11} g_{\lambda 1} - \Gamma^{\lambda}_{11} g_{1\lambda} = 0 \quad -(17)$$

i.e

$$\boxed{\partial_1 g_{11} = 0} \quad -(18)$$

Let $\rho = 2$, then

$$\partial_2 g_{\mu\nu} - \Gamma^{\lambda}_{2\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{2\nu} g_{\mu\lambda} = 0 \quad -(19)$$

For example:

$$\partial_2 g_{00} - \Gamma^{\lambda}_{20} g_{\lambda 0} - \Gamma^{\lambda}_{20} g_{0\lambda} = 0 \quad -(20)$$

i.e.

$$2\Gamma^{0}_{20} g_{00} = 0 \quad -(21)$$

$$\boxed{\Gamma^{0}_{20} = 0} \quad -(22)$$

Also

$$\partial_2 g_{11} - \Gamma^{1}_{21} g_{11} - \Gamma^{1}_{21} g_{11} = 0 \quad -(23)$$

and

$$\boxed{\Gamma^{1}_{21} = \frac{1}{2g_{11}} \partial_2 g_{11}} \quad -(24)$$

3)

The two equations of motion for the identity using output 3b are:

$$6(1+f)\frac{\partial^2 f}{\partial t^2} = 5\left(\frac{\partial f}{\partial t}\right)^2 \quad -(25)$$

and

$$\frac{\partial}{\partial t}\left(3\frac{\partial f}{\partial r} - (1+f)\frac{\partial f}{\partial r}\right) = 0$$

i.e.

$$\frac{\partial}{\partial t}\left(\left(3-(1+f)\right)\frac{\partial f}{\partial r}\right) = 0 \quad -(26)$$

For the precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(27)$$

$$\left.\begin{aligned} f = \frac{dr}{d\theta} &= \left(\frac{x\epsilon}{\alpha}\right) r^2 \sin(x\theta) \\ &= \left(\frac{x\epsilon}{\alpha}\right) r^2 \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \end{aligned}\right] \quad -(28)$$

Denote

$$g(r(t)) = 3-(1+f)\frac{\partial f}{\partial r} \quad -(29)$$

then eq. (26) is:

$$\boxed{\frac{\partial g(r(t))}{\partial t} = 0} \quad -(30)$$

4) From eq. (29):

$$g = g(r, \theta, t). \quad -(31)$$

so:

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial r}\frac{dr}{dt} + \frac{\partial g}{\partial \theta}\frac{d\theta}{dt} \quad -(32)$$

so eq. (30) is:

$$\frac{dg}{dt} = \frac{\partial g}{\partial r}\frac{dr}{dt} + \frac{\partial g}{\partial \theta}\frac{d\theta}{dt} \quad -(33)$$

because:

$$\frac{\partial g}{\partial r} = 0. \quad -(34)$$

However, if:

$$g = g(r, \theta) \quad -(35)$$

eq. (33) is ~~always true.~~

identity (26) is always true therefore the identity (26) is always true

if $f = f(r, \theta)$, or more accurately:

$$f = f(r(t), \theta(t)) \quad -(36)$$

Now use:

$$\frac{\partial f}{\partial t} = \frac{df}{dr}\frac{\partial r}{\partial t} \quad -(37)$$

and

$$\frac{\partial r}{\partial t} = \frac{dr}{d\theta}\frac{\partial \theta}{\partial r} \quad -(38)$$

From note 207(1):

$$\frac{\partial r}{\partial \theta} = 0 \quad -(39)$$

5) so the identity (25) is always true because
both sides go to infinity.

<u>This analysis</u> <u>shows yet again that the</u>
<u>Evans</u> <u>identity</u> <u>is always true.</u> It is

$$D_\mu T^{\kappa\mu\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(40)$$

and is an example of the Cartan identity.