3.4(?) : Further Comments

a) If there is a metric defined by:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -(1+f) \end{bmatrix} \quad -(1)$$

the line element is:

$$ds^2 = g_{00}\, dx^0 dx^0 + g_{11}\, dx^1 dx^1 \quad -(2)$$

So the metric compatibility equation is

$$D_\rho\, g_{\mu\nu} = 0 \quad -(3)$$

i.e

$$D_\rho\, g_{00} = 0 \quad -(4)$$

$$D_\rho\, g_{11} = 0 \quad -(5)$$

The indices ρ, μ and ν refer to the same mathematical space with a defined dimensionality. We have:

$$\mu, \nu = 0 \text{ and } 1 \quad -(6)$$

So:

$$\rho = 0 \text{ and } \rho = 1 \quad -(7)$$

b) Once f is defined then it can be worked out but this is not a trivial problem. However, astronomers have made progress in defining various orbital quantities.

c) We may write:

2)

$$f = f\left(r(t), \theta(t)\right) \quad — (8)$$

so $$\frac{df}{dt} = \frac{\partial f}{\partial r}\frac{dr}{dt} + \frac{\partial f}{\partial \theta}\frac{d\theta}{dt}. \quad — (9)$$

Also: $$f = f\left(t(\theta), r(\theta)\right) \quad — (10)$$

so $$\frac{df}{d\theta} = \frac{\partial f}{\partial r}\frac{dr}{d\theta} + \frac{\partial f}{\partial t}\frac{dt}{d\theta}. \quad — (11)$$

Also: $$f = f\left(\theta(t), t(r)\right) \quad — (12)$$

so $$\frac{df}{dr} = \frac{\partial f}{\partial \theta}\frac{d\theta}{dr} + \frac{\partial f}{\partial t}\frac{dt}{dr} \quad — (13)$$

Although t is a function of r, it is also a function of f. Although t is a function of θ it is also a function of f. For example if:

$$f = \sin x(t) + \cos y(t); \quad x(t) = 2t, \quad y(t) = t^2, \quad — (14)$$

then $$f = \sin(2t) + \cos(t^2)$$

$$\frac{df}{dt} = 2\cos 2t - 2t\sin(t^2) \neq 0$$

So $$\boxed{\frac{df}{dt} \neq 0} \quad — (15)$$

3) The total derivative does not vanish.

If we write:

$$f = \sin x(t) + \cos y(t) \quad -(16)$$

$$= \sin(2t) + \cos y$$

then

$$\left(\frac{\partial f}{\partial t}\right)_y = 2\cos(2t) \quad -(17)$$

If we write:

$$f = \sin x + \cos t^2 \quad -(18)$$

then

$$\left(\frac{\partial f}{\partial t}\right)_x = -2t \sin t^2 \quad -(19).$$

In general neither the total nor the partial derivatives vanish. The general relation between df/dt and $\partial f/\partial t$ can be obtained from eqs. (9), (11) and (13). For example:

$$\frac{\partial f}{\partial \theta} = \left(\frac{df}{dr} - \frac{\partial f}{\partial t}\frac{dt}{dr}\right)\frac{dr}{d\theta} \quad -(20)$$

so

$$\frac{df}{dt} = \frac{\partial f}{\partial r}\frac{dr}{dt} + \frac{d\theta}{dt}\frac{dr}{d\theta}\left(\frac{df}{\partial r} - \frac{\partial f}{\partial t}\frac{dt}{dr}\right)$$

$$= \frac{\partial f}{\partial r}\frac{dr}{dt} + \frac{df}{dt} - \frac{\partial f}{\partial t} \quad -(21)$$

4)

So:

$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial r}\frac{dr}{dt}} \quad -(22)$$

This is the result obtained from:

$$f = f(u), \quad u = u(x, y) \quad -(23)$$

with:

$$x = t, \quad u = r, \quad r = r(t, \theta) \quad -(24)$$

So in general:

$$\boxed{\frac{\partial f}{\partial t} \neq 0} \quad -(25)$$

The only relevant connection is:

$$\Gamma^1_{01} = \frac{1}{2c\, g_{11}}\frac{\partial g_{11}}{\partial t} \quad -(26)$$

because the indices ρ, μ and ν in the metric compatibility equation (3) ~~can~~ must refer to the same space of dimensionality two. So the only possibility is:

$$\boxed{D_0\, g_{11} = 0} \quad -(27)$$