207(1): Calculations based on the 2 x 2 metric

Consider the metric:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -\left(1+\frac{r^2}{f^2}\right) \end{bmatrix} \quad -(1)$$

where

$$f = \frac{dr}{d\theta}. \quad -(2)$$

Then:

$$g_{00} = 1, \; g_{11} = -\left(1+\frac{r^2}{f^2}\right). \quad -(3)$$

Here

$$f = f(r(\theta)). \quad -(4)$$

As in note 206(3) the two connections are:

$$\Gamma^1_{01} = \frac{1}{2c\, g_{11}} \frac{\partial g_{11}}{\partial t} \quad -(5)$$

and

$$\Gamma^1_{21} = \frac{1}{2 g_{11}} \frac{\partial g_{11}}{\partial \theta}. \quad -(6)$$

Here:

$$g_{11} = g_{11}(r(\theta)) \quad -(7)$$

so from eq. (3) of UFT206:

$$\frac{\partial g_{11}}{\partial \theta} = \frac{d g_{11}}{dr} \frac{dr}{d\theta} \quad -(8)$$

In order to compute dg_{11}/dr express r^2/f^2 completely in terms of r.

2) For example, for the precessing ellipse:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad - (9)$$

$$f = \frac{dr}{d\theta} = \left(\frac{x\epsilon}{\alpha}\right) r^2 \sin(x\theta) \quad - (10)$$

and express $\sin(x\theta)$ in terms of r using:

$$1 + \epsilon \cos(x\theta) = \frac{\alpha}{r} \quad - (11)$$

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad - (12)$$

$$\cos^2(x\theta) + \sin^2(x\theta) = 1 \quad - (13)$$

$$\sin(x\theta) = \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad - (14)$$

so

$$f = \left(\frac{x\epsilon}{\alpha}\right) r^2 \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad - (15)$$

Therefore the code can find dg_{11}/dr from eqns (3) and (15).

So:

$$\frac{dg_{11}}{d\theta} \neq 0. \quad - (16)$$

However [It is possible to go further using eq. (7) of UFT 206, so:

3)

$$\frac{dr}{d\theta} = \frac{\partial r}{\partial \theta} + \frac{\partial r}{\partial t}\frac{dt}{d\theta} \quad -(17)$$

$$\frac{dr}{dt} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial \theta}\frac{d\theta}{dt} \quad -(18)$$

So:
$$\frac{\partial r}{\partial \theta} = \frac{dr}{d\theta} - \frac{1}{\omega}\frac{\partial r}{\partial t} \quad -(19)$$

where
$$\omega = \frac{d\theta}{dt}. \quad -(20)$$

Therefore:
$$\frac{\partial r}{\partial \theta} = \frac{dr}{d\theta} - \frac{1}{\omega}\left(\frac{dr}{dt} - \omega\frac{\partial r}{\partial \theta}\right)$$
$$= \frac{dr}{d\theta} + \frac{\partial r}{\partial \theta} - \frac{1}{\omega}\frac{dr}{dt} \quad -(21)$$

So
$$\boxed{\frac{dr}{dt} = \omega\frac{dr}{d\theta} = \frac{d\theta}{dt}\frac{dr}{d\theta}} \quad -(22)$$

which is the rule for
$$r = r(\theta), \quad \theta = \theta(t) \quad -(23)$$

QED.

In general from eqs. (8) and (19):
$$\frac{\partial g_{11}}{\partial \theta} = \frac{dg_{11}}{dr}\left(\frac{dr}{d\theta} - \frac{1}{\omega}\frac{\partial r}{\partial t}\right) \quad -(24)$$
$$\neq 0$$