207(6): Summary of Results

Consider the precessing elliptical orbit:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)}, \quad -(1)$$

then:

$$\frac{dr}{d\theta} = \left(\frac{x\epsilon}{\alpha}\right) r^2 \sin(x\theta). \quad -(2)$$

In an orbit:

$$r = r(t), \quad \theta = \theta(t) \quad -(3)$$

because r and θ are functions of time. Eq. (2) is the total derivative of r with respect to θ.

Consider the function:

$$f = r^2 \left(\frac{d\theta}{dr}\right)^2 \quad -(4)$$

where eq. (3) applies. Then:

$$f = f(r(t), \theta(t)). \quad -(5)$$

This means that f is a function both of r and θ, and r is a function of t and θ is a function of t. It follows that:

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial r}\frac{dr}{dt} + \frac{\partial f}{\partial \theta}\frac{d\theta}{dt}} \quad -(6)$$

This is the chain rule.

2)

Therefore:

$$\boxed{\frac{\partial f}{\partial \theta} = \frac{dt}{d\theta}\left(\frac{df}{dt} - \frac{\partial f}{\partial r}\frac{dr}{dt}\right)} \quad -(7)$$

Similarly:

$$\frac{df}{dr} = \frac{\partial f}{\partial \theta}\frac{d\theta}{dr} + \frac{\partial f}{\partial t}\frac{dt}{dr} \quad -(8)$$

so:

$$\boxed{\frac{\partial f}{\partial t} = \frac{dr}{dt}\left(\frac{df}{dr} - \frac{\partial f}{\partial \theta}\frac{d\theta}{dr}\right)} \quad -(9)$$

The connections are:

$$\Gamma^1_{01} = \frac{1}{2c\, g_{11}}\frac{\partial g_{11}}{\partial t} \quad -(10)$$

$$\Gamma^1_{21} = \frac{1}{2 g_{11}}\frac{\partial g_{11}}{\partial \theta} \quad -(11)$$

where

$$g_{11} = -(1 + f) \quad -(12)$$

Therefore:

$$\Gamma^{1}{}_{01} = -\frac{1}{2c(1+f)} \cdot \frac{\partial f}{\partial t} \quad -(13)$$

$$\Gamma^{1}{}_{21} = -\frac{1}{2r(1+f)} \frac{\partial f}{\partial \theta} \quad -(14)$$

i.e.

$$\boxed{\Gamma^{1}{}_{01} = -\frac{1}{2c(1+f)} \frac{dr}{dt} \left(\frac{df}{dr} - \frac{\partial f}{\partial \theta} \frac{d\theta}{dr} \right)} \quad -(15)$$

$$\boxed{\Gamma^{1}{}_{21} = -\frac{1}{2r(1+f)} \frac{dt}{d\theta} \left(\frac{df}{dr} - \frac{\partial f}{\partial r} \frac{dr}{dt} \right)} \quad -(16)$$

1) Note carefully that the partial derivatives $\partial f / \partial \theta$ and $\partial f / \partial t$ are defined by the chain rule, and contain time derivatives.

2) The total derivatives $df / d\theta$ and df / dt are different in structure. Only the total derivatives can be evaluated from eq. (4), i.e only df / dr and $df / d\theta$.

4) The code is formally correct in terms of f, but to go any further it is necessary to know the various quantities in eqs. (15) and (16).

There are three chain rules:

$$\frac{df}{dt} = \frac{\partial f}{\partial r}\frac{dr}{dt} + \frac{\partial f}{\partial \theta}\frac{d\theta}{dt} \quad -(17)$$

$$\frac{df}{dr} = \frac{\partial f}{\partial \theta}\frac{d\theta}{dr} + \frac{\partial f}{\partial t}\frac{dt}{dr} \quad -(18)$$

$$\frac{df}{d\theta} = \frac{\partial f}{\partial r}\frac{dr}{d\theta} + \frac{\partial f}{\partial t}\frac{dt}{dr} \quad -(19)$$

The metric definition is:

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu = c^2 dt^2 - dr^2 - r^2 d\theta^2 \quad -(20)$$

with $g_{00} = 1$, $g_{11} = -1$, $g_{22} = -1$

$dx^0 = c\,dt$, $dx^1 = dr$, $dx^2 = r\,d\theta$ $\quad -(21)$

The constraint is:

$$dx^2 = \left(r\frac{d\theta}{dr}\right)dr \quad -(22)$$

The ~~metric~~ line element is:

5)

$$ds^2 = g_{00}\, dx^0 dx^0 + g_{11}\, dx^1 dx^1 + g_{22}\, dx^2 dx^2$$

$$= c^2 dt^2 - dr^2 - r^2 \left(\frac{d\theta}{dr}\right)^2 dr^2 \qquad -(23)$$

i.e.

$$dx^2 = \frac{d\theta}{dr}\, r\, dr = r\, d\theta \qquad -(24)$$

$$dx^1 = dr \qquad -(25)$$

so

$$\boxed{dx^2 = \left(r \frac{d\theta}{dr}\right) dx^1} \qquad -(26)$$

Therefore:

$$\boxed{\begin{aligned} ds^2 = {} & g_{00}\, dx^0 dx^0 + g_{11}\, dx^1 dx^1 \\ & + g_{22} \left(r \frac{d\theta}{dr}\right)^2 dx^1 dx^1 \end{aligned}} \qquad -(27)$$

<u>The number of coordinates has been reduced from three to two.</u>

This can be interpreted in two ways.

1) The spacetime remains flat, i.e.

$$g_{00} = 1, \; g_{11} = -1, \; g_{22} = -1 \qquad -(28)$$

6)

2) The spacetime becomes non-Minkowski in the sense that:

$$ds^2 = g_{00}\, dx^0 dx^0 + g_{11}' \, dx^1 dx^1 \quad -(29)$$

where:

$$\boxed{g_{11}' = g_{11} + \left(r \frac{d\theta}{dr}\right)^2 g_{22}} \quad -(30)$$

The metric for case (2) is:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -(1+f) \end{bmatrix} \quad -(31)$$

The metric for case (1) is:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad -(32)$$

So there is only one convention, eq. (15).