201(1): Some Transformation Properties of the New Relativity.

The covariant derivative is a derivative that is covariant under the general coordinate transformation:

$$D_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} D_{\mu} V^{\nu} \quad -(1)$$

This is a tensorial transformation law. It is defined by:

$$D_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda} \quad -(2)$$

where $\Gamma^{\nu}_{\mu\lambda}$ is the Christoffel connection. The covariant derivative is a tensor that reduces to the partial derivative of a vector in flat spacetime and transforms as a tensor in the arbitrary manifold.

In four dimensions $\Gamma^{\nu}_{\mu\lambda}$ has 64 components in general, and in general has no symmetry. In general $\Gamma^{\nu}_{\mu\lambda}$ has no symmetry in its lower two indices, i.e. is asymmetric in its lower two indices μ and λ.

The ordinary derivative $\partial_{\mu} V^{\nu}$ does not transform as a tensor, because:

$$\frac{\partial}{\partial x^{\mu'}}\left(V^{\nu'}\right) = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}} V^{\nu}\right) \quad -(3)$$

$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right) V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \frac{\partial V^{\nu}}{\partial x^{\mu}}$$

However, for flat spaces:

2)
$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right) = 0 \quad -(4)$$

so in flat space:
$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} \quad -(5)$$
and this is a tensor transformation.

The transformation law for the connection is derived from:
$$\partial_{\mu'} V^{\nu'} + \Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'}$$
$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \partial_{\mu} V^{\nu} + \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\nu} \frac{\partial}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} V^{\lambda} \quad -(6)$$
$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} D_{\mu} V^{\nu}$$
$$= \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \left(\partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda}\right)$$

so:
$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\nu} \frac{\partial}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} + \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^{\lambda}} V^{\lambda} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda} V^{\lambda} \quad -(7)$$

In the first term on the LHS change summation indices from ν to λ:
$$\frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\lambda} \frac{\partial}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\lambda}} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} V^{\nu} \frac{\partial}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \quad -(8)$$
Then V^{λ} cancels from eq. (7) to give:

$$\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\mu}\frac{\partial x^{\nu'}}{\partial x^\lambda}+\Gamma^{\nu'}_{\mu'\lambda'}\frac{\partial x^{\lambda'}}{\partial x^\lambda}=\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda}\quad-(9)$$

Finally multiply by $\partial x^\lambda/\partial x^{\lambda'}$ to give

$$\Gamma^{\nu'}_{\mu'\lambda'}=\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^\nu}\Gamma^\nu_{\mu\lambda}-\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^{\nu'}}{\partial x^\lambda}\right)\quad-(10)$$

As in the appended notes it follows that the difference of any two connections is a tensor:

$$S^\lambda_{\mu\nu}=\Gamma^\lambda_{\mu\nu}-\hat{\Gamma}^\lambda_{\mu\nu}\quad-(11)$$

In general relativity, the quantities of physics must be tensors. So the second, inhomogeneous, term in eq. (10) is unphysical.

Its symmetry is:

$$\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^\mu\partial x^\lambda}=\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^\lambda\partial x^\mu}\quad-(12)$$

...t this symmetry is irrelevant to physics.

It is well known that:

$$[D_\mu,D_\nu]V^\rho=R^\rho{}_{\sigma\mu\nu}V^\sigma-\left(\Gamma^\lambda_{\mu\nu}-\Gamma^\lambda_{\nu\mu}\right)D_\lambda V^\rho\quad-(13)$$

... definition:

$$[D_\mu,D_\nu]=-[D_\nu,D_\mu]\quad-(14)$$

:

4)

$$R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu} \quad -(15)$$

by definition. The commutator and curvature represent a round trip in the arbitrary manifold.

Therefore, by definition, the connection is anti-symmetric:

$$\Gamma^{\lambda}_{\mu\nu} = -\Gamma^{\lambda}_{\nu\mu}. \quad -(16)$$

When:

$$\mu = \nu \quad -(17)$$

$$[D_\mu, D_\nu] = [\partial_\mu, \partial_\nu] = 0 \quad -(18)$$

and

$$\Gamma^{\lambda}_{\mu\nu}(\mu = \nu) = 0. \quad -(19)$$

Similarly:

$$R^{\rho}{}_{\sigma\mu\nu}(\mu = \nu) = 0, \quad -(20)$$

and

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = -T^{\lambda}_{\nu\mu}, \quad -(21)$$

$$T^{\lambda}_{\mu\nu}(\mu = \nu) = 0. \quad -(22)$$

The connection is antisymmetric in all frames of reference. It follows that:

$$\boxed{\frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^{\mu} \partial x^{\lambda}} = 0} \quad -(23)$$

because it is symmetric, and the symmetric part of the connection is zero.

5)

The antisymmetric connection is a tensor:

$$\boxed{\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\lambda}}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} \Gamma^{\nu}_{\mu\lambda}} \quad -(4)$$

and the antisymmetric connection is physical.

The transformation properties of the equivalence theorem of UFT199 now follow. The equivalence theorem is:

$$\partial_\mu q^a_\nu = \omega^a_{\mu\nu} \quad -(5)$$

and

$$\partial_\mu V^a = \omega^a_{\mu b} V^b \quad -(6)$$

where

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b. \quad -(7)$$

Eq. (6) implies:

$$\boxed{\partial_\mu V^\nu = \Gamma^\nu_{\mu\lambda} V^\lambda} \quad -(8)$$

static frame, vector moving, flat space

static vector, general manifold, ordinary derivative zero

Both sides of eq. (8) are tensorial because:

$$\partial_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \partial_\mu V^\nu \quad -(9)$$

$$\Gamma^{\nu'}_{\mu'\lambda'} V^{\lambda'} = \Gamma^{\nu'}_{\mu'\lambda'} \frac{\partial x^{\lambda'}}{\partial x^\lambda} V^\lambda \quad -(10)$$

Appendix to Note 201(1)

Transformation of the Connection

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\Gamma^{\nu}_{\mu\lambda} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}} \quad -(1)$$

Symmetries

In general, $\Gamma^{\nu'}_{\mu'\lambda'}$ is asymmetric in its lower two indices, i.e. in isolation of any other argument it has no symmetry in μ' and λ' or μ and λ. So the torsion is:

$$\begin{aligned} T^{\nu'}_{\mu'\lambda'} &= \Gamma^{\nu'}_{\mu'\lambda'} - \Gamma^{\nu'}_{\lambda'\mu'} \\ &= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\left(\Gamma^{\nu}_{\mu\lambda} - \Gamma^{\nu}_{\lambda\mu}\right) \\ &= \left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\nu'}}{\partial x^{\nu}}\right)T^{\nu}_{\mu\lambda} \end{aligned} \quad -(2)$$

and is a tensor which is generally covariant. This is very well known.

For any symmetry:

$$\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\mu}\partial x^{\lambda}} = \frac{\partial x^{\lambda}}{\partial x^{\lambda'}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial^2 x^{\nu'}}{\partial x^{\lambda}\partial x^{\mu}} \quad -(3)$$

3 is very well known.

THE ARGUMENTS OF RODRIGUES ARE INCORRECT

The details of eq. (3) are as follows:

1) $$\frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \quad -(4)$$

This is an algebraic result:

$$ab = ba \quad -(5)$$

2) $$\frac{\partial^2 x^{\nu'}}{\partial x^\lambda \partial x^\mu} = \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} \quad -(6)$$

is a property of the ordinary derivative.

In general, a large number of connections can be defined on a manifold. For any two connections, their difference is a (1, 2) tensor (S. P. Carroll "Spacetime and Geometry: an Introduction to General Relativity" (Addison Wesley, NY, 2004). So in general:

$$S^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \hat{\Gamma}^\lambda_{\mu\nu} \quad -(7)$$

is a tensor which transforms as:

$$S^{\lambda'}_{\mu'\nu'} = \Gamma^{\lambda'}_{\mu'\nu'} - \hat{\Gamma}^{\lambda'}_{\mu'\nu'}$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^\lambda} \Gamma^\lambda_{\mu\nu} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\mu \partial x^\nu}$$

$$- \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^\lambda} \hat{\Gamma}^\lambda_{\mu\nu} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial^2 x^{\lambda'}}{\partial x^\mu \partial x^\nu}$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^\lambda} \left(\Gamma^\lambda_{\mu\nu} - \hat{\Gamma}^\lambda_{\mu\nu} \right)$$

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \frac{\partial x^{\lambda'}}{\partial x^\lambda} S^\lambda_{\mu\nu} \quad -(8)$$

3) which is the tensor transformation law (Q.E.D.)

Note carefully that the difference of any two connections is a tensor. This result does not depend on the symmetry of the lower two indices of the connection.

Now consider the special case:

$$\hat{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}. \quad -(9)$$

It follows that:

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \quad -(10)$$

is a tensor. It is known as the torsion tensor. From consideration of the commutator:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^{\lambda}_{\mu\nu} D_\lambda V^\rho \quad -(11)$$

where

$$[D_\mu, D_\nu] = -[D_\nu, D_\mu] \quad -(12)$$

Eq. (11) is:

$$[D_\mu, D_\nu] V^\rho = -\left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right) D_\lambda V^\rho + R^\rho{}_{\sigma\mu\nu} V^\sigma \quad -(13)$$

As is well known: $R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu} \quad -(14)$

so

$$\boxed{\Gamma^{\lambda}_{\mu\nu} = -\Gamma^{\lambda}_{\nu\mu}} \quad -(15)$$