201(5): Description of Orbit in the Most General Spherical Spacetime.

Consider the infinitesimal line element:

$$ds^2 = m(r,t)c^2 dt^2 - n(r,t)\,dr^2 - r^2 d\theta^2. \quad -(1)$$

In note 201(4) it was shown that:

$$m(r,t) = \frac{E^2}{mc^2(E+mc^2)} \quad -(2)$$

$$-(3)$$

and that:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{n(r,t)}\left(\frac{1}{m(r,t)b^2} - \frac{1}{a^2} + \frac{1}{r^2}\right)$$

where

$$b = \frac{Lc}{E}, \quad a = \frac{L}{mc} \quad -(4)$$

From eqs. (2) to (4):

$$\frac{1}{m(r,t)b^2} = \frac{mc^2(E+mc^2)}{E^2}\frac{E^2}{(Lc)^2} \quad -(5)$$

$$= \frac{m}{L^2}(E+mc^2)$$

so:

$$\frac{1}{m(r,t)b^2} - \frac{1}{a^2} = \frac{m}{L^2}(E+mc^2) - \frac{m^2c^2}{L^2}$$

$$= \frac{mE}{L^2} \quad -(6)$$

and

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^4}{n(r,t)}\left(\frac{mE}{L^2} + \frac{1}{r^2}\right) \quad -(7)$$

2) This is an equation which gives a description of any orbit in any spherically symmetric spacetime in terms of the function $n(r, t)$.

Note carefully that this description has not used the Einstein field equation.

Therefore:

$$\boxed{\frac{d\theta}{dr} = \frac{n(r,t)}{r^2}\left(\frac{mE}{L^2} + \frac{1}{r^2}\right)^{-1/2}} \quad -(8)$$

The correct description of light deflection due to gravitation is therefore:

$$\boxed{\Delta\theta = 2\int_{R_0}^{\infty} \frac{n(r,t)}{r^2}\left(\frac{mE}{L^2} + \frac{1}{r^2}\right)^{-1/2} dr - \pi} \quad -(9)$$

and this also gives the correct description of the time delay due to gravitation.

The correct description of a precessing elliptical orbit is given by:

$$r = \frac{\alpha}{1 + \epsilon \cos(x\theta)} \quad -(10)$$

so
$$\frac{dr}{d\theta} = \frac{\epsilon x \alpha \sin(x\theta)}{(1 + \epsilon\cos(x\theta))^2} = \frac{\epsilon x}{\alpha} r^2 \sin(x\theta)$$

and
$$\left(\frac{dr}{d\theta}\right)^2 = \left(\frac{\epsilon x}{\alpha}\right)^2 r^4 \sin^2(x\theta) \quad -(11)$$

3)

Therefore from eqs. (7) and (11):

$$\left(\frac{\epsilon x}{\alpha}\right)^2 \sin^2(x\theta) = \frac{1}{n(r,t)}\left(\frac{mE}{L^2} + \frac{1}{r^2}\right) \quad -(12)$$

and

$$n(r,t) = \left(\frac{\alpha}{\epsilon x}\right)^2 \frac{1}{\sin^2(x\theta)}\left(\frac{mE}{L^2} + \frac{1}{r^2}\right) \quad -(13)$$

The correct Newtonian description is given by:

$$x \rightarrow 1 \quad -(14)$$

Note θ is a function of time, so n depends both on r and t.

An elliptical orbit is therefore described by:

$$n(r,t) = \left(\frac{\alpha}{\epsilon x}\right)^2 \frac{1}{\sin^2\theta}\left(\frac{mE}{L^2} + \frac{1}{r^2}\right) \quad -(15)$$

$$n(r,t) = \frac{E^2}{mc^2(E+mc^2)} = \text{constant} \quad -(16)$$
