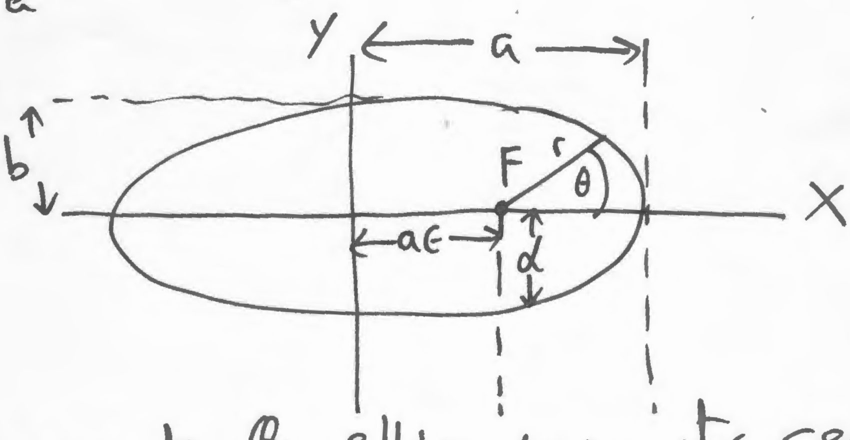
199(2): The Ellipse as a Tetrad.

First note that the ellipse is:

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=1 \qquad -(1)$$



Eq. (1) represents the ellipse using its centre. In orbital theory the ellipse is represented by:

$$r=\frac{\alpha}{1+\epsilon\cos\theta} \qquad -(2)$$

where r and θ are measured from a focus F.

Use:

$$X = c + r\cos\theta \qquad -(3)$$

$$Y = r\sin\theta \qquad -(4)$$

and

$$\epsilon = \left(1-\frac{b^2}{a^2}\right)^{1/2} \qquad -(5)$$

so

$$\left.\begin{aligned} b &= a\left(1-\epsilon^2\right) = \left(a^2-c^2\right)^{1/2}, \\ c &= \left(a^2-b^2\right)^{1/2} = a\epsilon, \quad \epsilon = \frac{c}{a} \end{aligned}\right\} \qquad -(6)$$

Here ϵ is the eccentricity.

So:

$$\frac{(c+r\cos\theta)^2}{a^2}+\frac{r^2\sin^2\theta}{b^2}=1 \qquad -(7)$$

where

$$\sin^2\theta = 1-\cos^2\theta. \qquad -(8)$$

So:

$$b^2c^2 + 2rcb^2\cos^2\theta + b^2r^2\cos^2\theta + a^2r^2 - a^2r^2\cos^2\theta = a^2b^2. \quad -(9)$$

i.e.

$$a^2(1-\epsilon^2)a^2\epsilon^2 + 2a\epsilon a^2(1-\epsilon^2)r\cos\theta + a^2(1-\epsilon^2)r^2\cos^2\theta + a^2r^2 - a^2r^2\cos^2\theta = a^2(a^2(1-\epsilon^2)),$$

or

$$r^2 = (\epsilon r\cos\theta - a(1-\epsilon^2))^2. \quad -(10)$$

When:

$$\epsilon = 0, \quad -(11)$$

then

$$r = \pm a \quad -(12)$$

but a is always positive so:

$$r = a(1-\epsilon^2) - \epsilon r\cos\theta, \quad -(13)$$

i.e.

$$r = \frac{\alpha}{1+\epsilon\cos\theta} \quad -(14)$$

where

$$\alpha = a(1-\epsilon^2) \quad -(15)$$

QED. Here 2α is the latus rectum or right magnitude of the orbit.

The orbit can also be represented as eq. (1), which is:

$$\left(\frac{b}{a^2+b^2}\right)X^2 + \left(\frac{a}{a^2+b^2}\right)Y^2 = 1 \quad -(16)$$

which is similar to the circle:

$$X^2 + Y^2 = 1. \quad -(17)$$

Now define:

$$A = \left(\frac{b}{a^2+b^2}\right)^{1/2}, \quad B = \left(\frac{a}{a^2+b^2}\right)^{1/2} \quad -(18)$$

and the tetrad vector:

$$\boxed{\underline{q}^{(1)} = \frac{1}{\sqrt{2}}\left(AX\,\underline{i} - iBY\,\underline{j}\right)} \quad -(19)$$

This is the required representation of the ellipse as a tetrad. Note that:

$$\underline{q}^{(1)} \cdot \underline{q}^{(2)} = A^2X^2 + B^2Y^2 \quad -(20)$$

$$= 1.$$

Therefore $\underline{q}^{(1)}$ is a unit vector of elliptical polarization in optics. The components of the tetrad are:

$$q_X^{(1)} = \frac{AX}{\sqrt{2}} \;; \quad q_Y^{(1)} = -\frac{iBY}{\sqrt{2}} \quad -(21)$$

Here:

$$\boxed{\underline{q}^{(2)} = \frac{1}{\sqrt{2}}\left(AX\,\underline{i} + iBY\,\underline{j}\right)} \quad -(22)$$

and is the complex conjugate of $\underline{q}^{(1)}$.

Note that:

$$\underline{q}^{(1)} \times \underline{q}^{(2)} = \frac{1}{2}\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ AX & -iBY & 0 \\ AX & iBY & 0 \end{vmatrix} \quad -(23)$$

$$= i\,ABXY\underline{k}$$

4) By definition:

$$V^a = q^a_\mu V^\mu \quad -(24)$$

where

$$a = (1), (2), (3) \quad -(25)$$

$$\mu = 1, 2, 3. \quad -(26)$$

These are two representations of three dimensional space. For a planar orbit it is sufficient to consider:

$$a = (1), (2) \quad -(27)$$

$$\mu = 1, 2 \quad -(28)$$

Thus:

$$\boxed{\begin{bmatrix} \underline{q}^{(1)} \\ \underline{q}^{(2)} \end{bmatrix} = \begin{bmatrix} q^{(1)}_x & q^{(1)}_y \\ q^{(2)}_x & q^{(2)}_y \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix}} \quad -(29)$$

where:

$$q^{(1)}_x = \frac{AX}{\sqrt{2}}, \quad q^{(1)}_y = -\frac{iBY}{\sqrt{2}} \quad -(30)$$

$$q^{(2)}_x = \frac{AX}{\sqrt{2}}, \quad q^{(2)}_y = \frac{iBY}{\sqrt{2}} \quad -(31)$$

For a particle m orbiting M in an ellipse,

$$X = X(t) \; ; \quad Y = Y(t), \quad -(32)$$

i.e. X and Y are functions of time.

The next note will generalize this analysis to the position tetrad. r^a_μ.