1) 199(8): The Equivalence of Active and Passive Rotations

The active rotation is the rotation of a vector with axes fixed. This is equivalent to the passive rotation keeping the vector fixed and rotating the frame. The rotations of the frame and vector must be in opposite senses.

Consider the covariant derivative of Cartan:

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b \quad -(1)$$

where $\omega^a_{\mu b}$ is the spin connection. The quantity $D_\mu V^a$ retains its format under the general coordinate transform. In this sense it is generally relativistic in any mathematical space in any dimension.

Consider now the rotation of a vector in two dimensions as in note 199(7). The active rotation is defined by:

$$D_\mu V^a = \partial_\mu V^a \quad -(2)$$

because the axes do not move, meaning that the spin connection is zero. The two dimensional plane XY is a flat two dimensional space. The passive rotation is defined by:

$$D_\mu V^a = \omega^a_{\mu b} V^b \quad -(3)$$

because the vector V^a is static and the frame moves. If the frame moves there exists a non-zero spin connection by definition. So we obtain a new theorem of general relativity:

$$\boxed{\partial_\mu V^a = \omega^a_{\mu b} V^b} \quad -(4)$$

2) for rotation.

This theorem can be generalized for any motion.

Example

Consider the position vector in two dimensions:

$$\underline{r} = X(t)\,\underline{i} + Y(t)\,\underline{j} \quad -(5)$$

where X and Y depend on t. This is the Cartesian representation of an orbit in a plane. Here:

$$V^1 = X(t), \quad V^2 = Y(t). \quad -(6)$$

So:

$$\partial_0 X = \omega^1{}_{0b} V^b = \omega^1{}_{01} X + \omega^1{}_{02} Y \quad -(7)$$

$$\partial_0 Y = \omega^2{}_{0b} V^b = \omega^2{}_{01} X + \omega^2{}_{02} Y \quad -(8)$$

where

$$\partial_0 = \frac{1}{c}\frac{\partial}{\partial t} \quad -(9)$$

so:

$$\frac{\partial X}{\partial t} = c\left(\omega^1{}_{01} X + \omega^1{}_{02} Y\right) \quad -(10)$$

$$\frac{\partial Y}{\partial t} = c\left(\omega^2{}_{01} X + \omega^2{}_{02} Y\right) \quad -(11)$$

where:

$$X = r\cos\theta \quad -(12)$$

$$Y = r\sin\theta. \quad -(13)$$

Here the cylindrical polar coordinates are:

$$(1, 2) = (r, \theta) \quad -(14)$$

Therefore:

$$\frac{\partial}{\partial t}\left(r\cos\theta\right) = \frac{\partial r}{\partial t}\cos\theta + r\,\frac{\partial \cos\theta(t)}{\partial t} \quad -(15)$$

Use the chain rule:

$$\frac{\partial f(\theta)}{\partial t} = \frac{\partial f(\theta)}{\partial \theta}\frac{\partial \theta}{\partial t} \quad - (16)$$

So
$$\frac{\partial \cos\theta(t)}{\partial t} = -\frac{\partial\theta}{\partial t}\sin\theta(t) \quad - (17)$$

and
$$\frac{\partial}{\partial t}\left(r\cos\theta\right) = \frac{\partial r}{\partial t}\cos\theta - \frac{\partial\theta}{\partial t} r\sin\theta(t) \quad - (18)$$

Similarly:
$$\frac{\partial}{\partial t}\left(r\sin\theta\right) = \frac{\partial r}{\partial t}\sin\theta + \frac{\partial\theta}{\partial t} r\cos\theta(t) \quad - (19)$$

Comparing eqs. (10) and (18):

$$c\omega^1{}_{01}\, r\cos\theta + c\omega^1{}_{02}\, r\sin\theta = \frac{\partial r}{\partial t}\cos\theta - \frac{\partial\theta}{\partial t} r\sin\theta(t) \quad - (20)$$

So
$$\boxed{\begin{aligned} \omega^1{}_{01} &= \frac{1}{cr}\frac{\partial r}{\partial t} \quad &- (21)\\ \omega^1{}_{02} &= -\frac{1}{c}\frac{\partial\theta}{\partial t} \quad &- (22)\end{aligned}}$$

Comparing eqs (11) and (19):

$$c\omega^2{}_{01}\, r\cos\theta + c\omega^2{}_{02}\, r\sin\theta = \frac{\partial r}{\partial t}\sin\theta + r\frac{\partial\theta}{\partial t}\cos\theta(t) \quad - (23)$$

$$\boxed{\begin{aligned} \omega^2{}_{01} &= \frac{1}{c}\frac{\partial\theta}{\partial t} \quad &- (24)\\ \omega^2{}_{02} &= \frac{1}{cr}\frac{\partial r}{\partial t} \quad &- (25)\end{aligned}}$$

So:
$$\omega^1{}_{01} = \omega^2{}_{02} \quad - (26)$$
$$\omega^1{}_{02} = -\omega^2{}_{01} \quad - (27)$$

4) The spin connection is therefore:

$$\omega^a{}_{0b} = \begin{bmatrix} \omega^1{}_{01} & \omega^1{}_{02} \\ \omega^2{}_{01} & \omega^2{}_{02} \end{bmatrix} \quad -(28)$$

$$= \frac{1}{c}\begin{bmatrix} \frac{1}{r}\frac{\partial r}{\partial t} & -\frac{\partial \theta}{\partial t} \\ \frac{\partial \theta}{\partial t} & \frac{1}{r}\frac{\partial r}{\partial t} \end{bmatrix}$$

Therefore:

$$r c \, \omega^a{}_{0b} = \begin{bmatrix} \frac{\partial r}{\partial t} & -r\frac{\partial \theta}{\partial t} \\ r\frac{\partial \theta}{\partial t} & \frac{\partial r}{\partial t} \end{bmatrix} \quad -(29)$$

The orbital velocity is:

$$\underline{v} = \frac{dr}{dt}\underline{e}_r + r\frac{d\theta}{dt}\underline{e}_\theta \quad -(30)$$

where $\underline{e}_r$ and $\underline{e}_\theta$ are the unit vectors of the cylindrical polar coordinate system in a plane.

It is seen that the components of $\underline{v}$ are spin connection components:

$$\frac{\partial r}{\partial t} = r c \, \omega^1{}_{01} = r c \, \omega^2{}_{02}, \quad -(31)$$

$$r\frac{\partial \theta}{\partial t} = - r c \, \omega^1{}_{02} = r c \, \omega^2{}_{01}, \quad -(32)$$

and

$$\boxed{\begin{aligned} \underline{v} &= r c \left(\omega^1{}_{01}\underline{e}_r - \omega^1{}_{02}\underline{e}_\theta \right) \\ &= r c \left(\omega^2{}_{02}\underline{e}_r + \omega^2{}_{01}\underline{e}_\theta \right) \end{aligned}} \quad -(33)$$

5) The rotational kinetic energy and lagrangian are:

$$\mathcal{L} = T = \frac{1}{2} m v^2 \quad -(34)$$

where:

$$\left.\begin{aligned} v^2 &= r^2 c^2 \left((\omega^1{}_{01})^2 + (\omega^1{}_{02})^2 \right) \\ &= r^2 c^2 \left((\omega^2{}_{02})^2 + (\omega^2{}_{01})^2 \right) \\ &= \dot{r}^2 + r^2 \dot{\theta}^2 . \end{aligned}\right\} \quad -(35)$$

The equivalence of active and passive rotation means that the lagrangian is defined in terms of spin connection components, and this is a fully relativistic theory of all orbits.