199(6): Proof that Torsion and Curvature are Always Non Zero

Consider the Cartan structure equations:

$$T = d \wedge q + \omega \wedge q \quad -(1)$$

$$R = d \wedge \omega + \omega \wedge \omega \quad -(2)$$

and the Cartan identity:

$$D \wedge T = d \wedge T + \omega \wedge T := R \wedge q \quad -(3)$$

These are the basic equations of geometry used in the ECE theory of unified physics. Here, indexless notation has been used for clarity. $T$ is the torsion, $R$ the curvature, $\wedge$ the wedge product, $\omega$ the spin connection, $q$ the tetrad. These equations are equivalent to the Riemannian definitions given the tetrad postulate:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda = 0 \quad -(4)$$

So:

$$\Gamma^\lambda_{\mu\nu} q^a_\lambda = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu, \quad -(5)$$

where

$$\Gamma^\lambda_{\mu\nu} q^a_\lambda = \Gamma^a_{\mu\nu} \quad -(6)$$

$$\omega^a_{\mu b} q^b_\nu = \omega^a_{\mu\nu}. \quad -(7)$$

So

$$\Gamma^a_{\mu\nu} = \partial_\mu q^a_\nu + \omega^a_{\mu\nu}. \quad -(8)$$

The received dogma of the twentieth century relied completely on:

$$T \overset{?}{=} 0, \quad -(9)$$

[illegible] was an error introduced historically, the

2) reason being that torsion was unknown in the decade 1905 - 1915 when general relativity was developed. So the dogma incorrectly used:

$$T = d \wedge q + \omega \wedge q =? 0 \quad -(10)$$

$$R = d \wedge \omega + \omega \wedge \omega \quad -(11)$$

$$R \wedge q = ?0 \quad -(12)$$

Eq. (12) was known as "the first Bianchi identity":

$$R^{\lambda}{}_{\mu\nu\rho} + R^{\lambda}{}_{\rho\mu\nu} + R^{\lambda}{}_{\nu\rho\mu} =? 0. \quad -(13)$$

Unfortunately the whole of twentieth century general relativity was based on eq. (13), and all of it was incorrect.

The arbitrary assumption (9) arose because there was no way of defining the symmetry of the Christoffel connection $\Gamma^{\lambda}{}_{\mu\nu}$ in the decade 1905 - 1915. There is only one way of defining this symmetry:

$$[D_\mu, D_\nu] V^\rho = R^{\rho}{}_{\sigma\mu\nu} V^\sigma - T^{\lambda}{}_{\mu\nu} D_\lambda V^\rho \quad -(14)$$

$$[D_\mu, D_\nu] V^\rho := -[D_\nu, D_\mu] V^\rho. \quad -(15)$$

where: $[D_\mu, D_\nu]$ is the commutator of covariant derivatives.

Here $[D_\mu, D_\nu]$ is the commutator. It acts on any tensor, in this case the vector $V^\rho$, to produce two fundamental tensors, the torsion $T^{\kappa}{}_{\mu\nu}$ and the curvature $R^{\rho}{}_{\sigma\mu\nu}$:

$$T^{\lambda}{}_{\mu\nu} := \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}, \quad -(16)$$

$$R^{\rho} \ldots = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}$$

3) So as in UFT 137:

$$[D_\mu, D_\nu] V^\rho = -\Gamma^\lambda_{\mu\nu} D_\lambda V^\rho + \ldots \quad (18)$$

From eqs (15) and (18):

$$\boxed{\Gamma^\lambda_{\mu\nu} := -\Gamma^\lambda_{\nu\mu}} \quad -(19)$$

From the fundamental definitions of T and R, $\Gamma$ is also antisymmetric.

Unfortunately, the Einstein field equation had been inferred (1915) and "tested" (early twenties) before eq. (14) was known. Torsion was never incorporated correctly. The curvature was inferred originally around the turn of the twentieth century, circa 1900 to 1905, by Levi-Civita, Bianchi and Ricci. Torsion was not inferred until twenty years later, by the Cartan school in Paris. The latter defined the connection as eq. (8) with:

$$\Gamma^\lambda_{\mu\nu} := \Gamma^a_{\mu\nu} q^\lambda_a \quad -(20)$$

where

$$q^a_\lambda q^\lambda_a := 1. \quad -(21)$$

It follows from eqs. (5) and (19) that:

$$\partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu = -\left(\partial_\nu q^a_\mu + \omega^a_{\nu b} q^b_\mu\right) \quad -(22)$$

Therefore:

$$\partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu + \partial_\nu q^a_\mu + \omega^a_{\nu b} q^b_\mu = 0 \quad -(23)$$

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4) If the erroneous assumption is made of zero torsion then:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu =? 0 \quad -(24)$$

Adding and subtracting eqns. (23) and (24) give:

$$\Gamma^a_{\mu\nu} = \partial_\mu q^a_\nu + \omega^a_{\mu\nu} =? 0, \quad -(25)$$

$$\Gamma^a_{\nu\mu} = \partial_\nu q^a_\mu + \omega^a_{\nu\mu} =? 0, \quad -(26)$$

so

$$\Gamma^\lambda_{\mu\nu} =? 0 \quad -(27)$$

and

$$\boxed{R^\rho{}_{\sigma\mu\nu} =? 0} \quad -(28)$$

Q.E.D.

Conclusions

Torsion and curvature are always both non-zero in any spacetime characterized by a connection. The torsion, curvature and connection are all anti-symmetric in the indices of the commutator $\mu$ and $\nu$. The obsolete argument was that the connection, as defined by Christoffel in the eighteen sixties, could take any symmetry in $\mu$ and $\nu$. Circa 1900 - 1905, Levi Civita defined the curvature without use of the commutator, and erroneously omitted torsion. At best, Levi-Civita subjectively ignored torsion, at worst he was simply wrong. The correct method was finally developed by Cartan nearly twenty years later.

5) Conversely, the use of an erroneously symmetric connection:

$$\Gamma^{\lambda}_{\mu\nu} \overset{?}{=} \Gamma^{\lambda}_{\nu\mu} \quad -(29)$$

means that:

$$[D_\mu, D_\nu] V^\rho = 0, \quad -(30)$$

which is true if and only if:

$$D_\mu = \partial_\mu \quad -(31)$$

and

$$\Gamma^{\lambda}_{\mu\nu} = 0. \quad -(32)$$

The symmetric connection is zero. It is obvious that the symmetric curvature and torsion are also zero, because:

$$T^{\lambda}_{\mu\nu} := -T^{\lambda}_{\nu\mu} \quad -(33)$$

$$R^{\rho}{}_{\sigma\mu\nu} := -R^{\rho}{}_{\sigma\nu\mu}. \quad -(34)$$

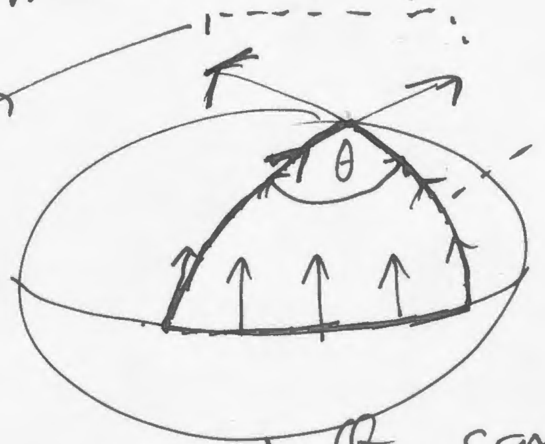
Similarly the symmetric electromagnetic field tensor is zero because:

$$F^{a}_{\mu\nu} := -F^{a}_{\nu\mu}. \quad -(35)$$

The commutator is fundamental because it represents parallel transport around a loop. Parallel transport of a vector or any tensor around a closed loop in any space with non-zero connection leads to a transformation of the vector. The transformation depends both on the curvature and the torsion.

The usual example of this idea is to take a vector on the earth's equator, point it

b) along a line of constant longitude and transport it to the north pole. Then take the same vector, transport it along the equator (i.e parallel transport it through an angle $\theta$), then move it again to the north pole.




The vector arrives at the same north pole with two different values – it has been transformed by parallel transport around a closed loop. The commutator describes this.