1) 199(7): Tetrad, Spin Connection and Torsion for Any orbit.

Consider the orbit to be in the plane:

$$dZ = 0. \quad -(1)$$

In this plane the position vector is:

$$\underline{r} = X\,\underline{i} + Y\,\underline{j}. \quad -(2)$$

Now rotate this vector through the angle θ. The usual method is to keep the axes fixed:

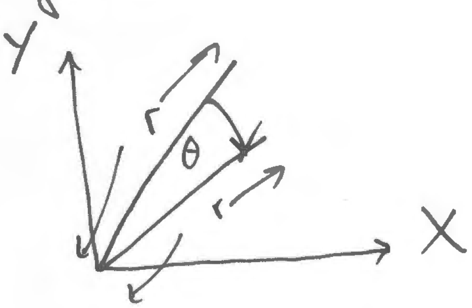

Fig (1)

In which case:

$$\underline{r}' = X'\,\underline{i} + Y'\,\underline{j} = (X\cos\theta + Y\sin\theta)\,\underline{i} + (-X\sin\theta + Y\cos\theta)\,\underline{j} \quad -(3)$$

with:

$$\begin{bmatrix} X' \\ Y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} \quad -(4)$$

In this case there is no connection, because the axes are fixed. The rotation tetrad is:

$$q^a_{\ \mu} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad -(5)$$

and a and μ must be indices of the same two dimensional space:

$$\left. \begin{array}{l} a = 1, 2 \\ \mu = 1, 2 \end{array} \right\} \quad -(6)$$

2) The torsion is:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu \quad -(7)$$

Note that the tetrad is the one that relates the cylindrical polar coordinate system to the Cartesian:

$$\begin{bmatrix} \underline{e}_r \\ \underline{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(8)$$

where $\underline{e}_r$ and $\underline{e}_\theta$ are the unit vectors of the cylindrical polar system in the plane.

Consider the coordinates of the cylindrical polar system:

$$(1, 2) = (r, \theta) \quad -(9)$$

then:

$$T^1_{21} = -T^1_{12} = \frac{\partial q^1_1}{\partial \theta} = -\sin\theta, \quad -(10)$$

$$T^2_{21} = -T^2_{12} = \frac{\partial q^2_1}{\partial \theta} = -\cos\theta \quad -(11)$$

So:

$$T^1_{\mu\nu} = \begin{bmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{bmatrix} \quad -(12)$$

$$T^2_{\mu\nu} = \begin{bmatrix} 0 & \cos\theta \\ -\cos\theta & 0 \end{bmatrix} \quad -(13)$$

Now consider the rotation as a movement of the axes wrt fixed vector:

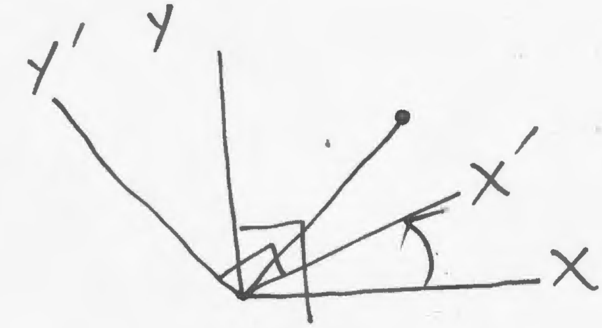

Fig (2)

Write eq. (3) as:

$$\underline{r}' = X\left(\underline{i}\cos\theta - \underline{j}\sin\theta\right) + Y\left(\underline{i}\sin\theta + \underline{j}\cos\theta\right) \quad -(14)$$

$$-(15)$$

so:

$$\underline{i}' = \underline{i}\cos\theta - \underline{j}\sin\theta \quad -(16)$$

$$\underline{j}' = \underline{i}\sin\theta + \underline{j}\cos\theta$$

$$\begin{bmatrix} \underline{i}' \\ \underline{j}' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{i} \\ \underline{j} \end{bmatrix} \quad -(17)$$

In this case X and Y are constant, so their derivatives are zero. Note that:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad -(18)$$

so

$$q^{\mu}_{a} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad -(19)$$

is the inverse rotation tetrad:

$$q^{a}_{\mu}\, q^{\mu}_{a} = 1, \quad -(20)$$

in Cartan's rotation

In this case there is a spin connection, t)

because the axes move (i.e. rotate counter clockwise)

The torsion is:

$$T^a_{\mu\nu} = \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(21)$$

$$= \omega^a_{\mu\nu} - \omega^a_{\nu\mu}.$$

However, the rotation is the same in both cases, so:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu \quad -(22)$$

$$= \omega^a_{\mu\nu} - \omega^a_{\nu\mu}$$

By antisymmetry:

$$* \quad \boxed{T^a_{\mu\nu} = \partial_\mu q^a_\nu = \omega^a_{\mu\nu}} \quad * \quad -(23)$$

and this is true for all planar functions ~~of~~:

$$r = r(\theta), \quad \theta = \theta(t) \quad -(24)$$

For an elliptical orbit:

$$r = \frac{d}{1 + \epsilon \cos\theta}, \quad -(25)$$

$$(\mu, \nu) = (1, 2) = (r, \theta) \quad -(26)$$

and so on.