1) 199(5): Tetrad of the Ellipse in Cylindrical Polar Representation.

First consider the vector:

$$\underset{\sim}{r} = X\,\underline{i} + Y\,\underline{j} \quad -(1)$$

where

$$X = r\cos\theta, \quad Y = r\sin\theta, \quad -(2)$$

so

$$\underline{r} = r\cos\theta\,\underline{i} + r\sin\theta\,\underline{j}. \quad -(3)$$

The unit vectors of the cylindrical polar representation are defined by

$$\underline{e}_r = \frac{\partial \underline{r}/\partial r}{|\partial \underline{r}/\partial r|} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} \quad -(4)$$

$$\underline{e}_\theta = \frac{\partial \underline{r}/\partial\theta}{|\partial \underline{r}/\partial\theta|} = -\sin\theta\,\underline{i} + \cos\theta\,\underline{j}. \quad -(5)$$

Therefore

$$\underline{r} = r\,\underline{e}_r. \quad -(6)$$

In an orbit, both r and θ depend on t.

The ellipse is defined by:

$$X = \epsilon a + r\cos\theta \quad -(7)$$

$$Y = r\sin\theta \quad -(8)$$

where ϵ and a do not depend on t. So the unit vectors (4) and (5) are ~~the~~ the same, but:

$$\underline{r} = (\epsilon a + r\cos\theta)\,\underline{i} + r\sin\theta\,\underline{j} \quad -(9)$$

The tetrad is therefore defined by:

$$\begin{bmatrix} \epsilon a + r\cos\theta \\ r\sin\theta \end{bmatrix} = \begin{bmatrix} \epsilon a + r\cos\theta & 0 \\ 0 & r\sin\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad -(10)$$

) and is

$$\boxed{r^{a}_{\mu} = \begin{bmatrix} \epsilon a + r\cos\theta & 0 \\ 0 & r\sin\theta \end{bmatrix}} \quad -(11)$$

If

$$a = (1), (2), \quad -(12)$$

$$\mu = 1, 2 \quad -(13)$$

then

$$r^{(1)}_{1} = \epsilon a + r\cos\theta \quad -(14)$$

$$r^{(1)}_{2} = 0 \quad -(15)$$

$$r^{(2)}_{1} = 0 \quad -(16)$$

$$r^{(2)}_{2} = r\sin\theta \quad -(17)$$

and

$$\underline{r} = \underline{r}^{(1)} + \underline{r}^{(2)} \quad -(18)$$

where

$$\underline{r}^{(1)} = r^{(1)}_{1}\,\underline{i} \quad -(19)$$

$$\underline{r}^{(2)} = r^{(2)}_{2}\,\underline{j}. \quad -(20)$$

Therefore:

$$\frac{\partial r^{(1)}_{1}}{\partial t} = -r\sin\theta + \dot{r}\cos\theta \quad -(21)$$

$$\frac{\partial r^{(2)}_{2}}{\partial t} = r\cos\theta + \dot{r}\sin\theta \quad -(22)$$

<u>because both r and</u> θ <u>depend on t.</u>

the ellipse is then:

$$r = \frac{\alpha}{1 + \epsilon\cos\theta} \quad -(23)$$

where

$$\alpha = a\left(1 - \epsilon^{2}\right) \quad -(24)$$

3) Double Cartesian Tetrad

Consider the position vector:

$$\underline{r}(t) = X(t)\,\underline{i} + Y(t)\,\underline{j} \quad -(25)$$

which depends on time t. The tetrad in this case is defined by:

$$\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} X(t) & 0 \\ 0 & Y(t) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad -(26)$$

i.e.

$$r^a_{\ \mu} = \begin{bmatrix} X(t) & 0 \\ 0 & Y(t) \end{bmatrix} \quad -(27)$$

The movement represented by eq. (25) can be thought of as a dynamic unit vector, i.e. in terms of a frame of reference that evolves with time. The tetrad is the 2×2 matrix that links this dynamic frame to the static frame:

$$\underline{e}(t) = \underline{i} + \underline{j}\,. \quad -(28)$$

Eq. (26) is an example of the definition:

$$V^a = q^a_{\ \mu} V^\mu \quad -(29)$$

where the components of the complete vector field V are V^a and V^μ in two different frames. These can be a dynamic frame and a static frame using the same coordinates.

4) Complete Proof of the Elliptical Equations.

To Prove

That $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be written as

$r = d/(1 + \epsilon\cos\theta)$

Proof use $X = c + r\cos\theta$, $Y = r\sin\theta$ — (1)

then:

$b^2(c^2 + 2rc\cos\theta + r^2\cos^2\theta) + a^2 r^2(1-\cos^2\theta) = (ab)^2$ — (2)

Here $c = \epsilon a$, — (3)

$b = (1-\epsilon^2)^{1/2} a$ — (4)

so:

$a^2(1-\epsilon^2)\epsilon^2 a^2 + 2ra^2(1-\epsilon^2)\epsilon a\cos\theta$
$+ a^2(1-\epsilon^2)r^2\cos^2\theta + a^2 r^2(1-\cos^2\theta) = a^4(1-\epsilon^2)$.

i.e.

$a^2(1-\epsilon^2)\epsilon^2 + r^2(1-\cos^2\theta + (1-\epsilon^2)\cos^2\theta) + 2r(1-\epsilon^2)a\epsilon\cos\theta$
$= a^2 r^2(1-\epsilon^2)$ — (5)

i.e. $r^2 = a^2(1-\epsilon^2) - a^2\epsilon^2(1-\epsilon^2) - 2r(1-\epsilon^2)a\epsilon\cos\theta$
$+ r^2\epsilon^2\cos^2\theta$

$= a^2(1-\epsilon^2)^2 - 2r(1-\epsilon^2)a\epsilon\cos\theta + r^2\epsilon^2\cos^2\theta$

$= (\epsilon r\cos\theta - a(1-\epsilon^2))^2$ — (6)

$r^2 = (\epsilon r\cos\theta - a(1-\epsilon^2))^2$ — (7)

$r = \pm(\epsilon r\cos\theta - a(1-\epsilon^2))$

The negative root is needed because

$r = a$ when $\epsilon = 0$ — (8)

$r = a(1-\epsilon^2) - \epsilon r\cos\theta$ — (9)

So $r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} = \frac{d}{1+\epsilon\cos\theta}$ — (10)

QED