1917(5): Elliptical Orbit from Spacetime Torsion.

Define angular velocity and angular momentum in general by:

$$\boxed{\omega^a_{\mu\nu} = \frac{1}{mr^2} J^a_{\mu\nu} = c T^a_{\mu\nu},} \quad -(1)$$

i.e as vector valued two-forms of Cartan geometry:

$$\omega^a = \frac{1}{mr^2} J^a = c T^a \quad -(2)$$

In terms of magnitude:

$$\omega = \dot{\theta} = \frac{J}{mr^2} = cT. \quad -(3)$$

Therefore:

$$\boxed{T = \frac{J}{mcr^2}} \quad -(4)$$

If the angular momentum is assumed to be constant this is an inverse square law for torsion.

Assume now that the force form is given by:

$$\boxed{F^a_{\mu\nu} = -\left(\frac{Jc}{\alpha}\right) T^a_{\mu\nu}} \quad -(5)$$

where α is defined by:

$$\alpha = \frac{J^2}{m^2 M_2 G} \quad -(6)$$

then eq. (4) becomes:

$$F = -\frac{mM_2 G}{r^2}, \quad -(7)$$

$$= -\left(\frac{Jc}{\alpha}\right) T. \quad -(8)$$

In vector notation:

$$\underline{F} = -\left(\frac{Jc}{d}\right)\underset{\sim}{T} \quad -(9)$$

and is due to the orbital torsion.

Now assume that $\underset{\sim}{F}$ is radially directed:

$$\underline{F} = -\frac{mMG}{r^2}\underline{e}_r = m\,\underline{a} \quad -(10)$$

where the acceleration $\underline{a}$ is:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta \quad -(11)$$

It is known that the result (10) is obtained from the ellipse:

$$r = \frac{d}{1 + \epsilon\cos\theta} \quad -(12)$$

Therefore the definitions (1) and (5) lead to the elliptical orbit (12). The force law from eq. (9) shows that the origin of $\underline{F}$ is the orbital torsion. The idea of orbital torsion replaces the seventeenth century idea of force of gravitation.

In terms of magnitude the angular momentum is:

$$J = mcr^2 T = mr^2\omega \quad -(13)$$

and

$$\omega = cT. \quad -(14)$$

3) In this theory the angular momentum is assumed to be a constant, but the torsion form is not constant. It is governed by the identities:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(15)$$

and

$$D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b. \quad -(16)$$

The torsion form is defined by:

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad -(17)$$

This theory can be tied in to the classical definition of angular momentum:

$$\underline{J} = \underline{r} \times \underline{p} \quad -(18)$$

by using the argument leading to eq. (9) of note 197(4):

$$d \wedge q^a = \omega^a{}_b \wedge q^b \quad -(19)$$

i.e. angular momentum can be generated either by rotating a vector with fixed axes or keeping the vector fixed and rotating the axes in a different sense. Therefore the classical lagrangian can be used:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) \quad -(20)$$

with

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta} = 0, \quad -(21)$$

i.e.

$$J = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \dot{\theta} = \text{constant}.$$