1) 196(9): Fully Relativistic Treatment of the Angular Momentum.

Consider the relativistic lagrangian:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) \left(1 + \omega c t_g \right) - U(r) \quad -(1)$$

in the Euler Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(2)$$

The total angular momentum L is a constant of motion and is:

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m r^2 \left(1 + \omega c t_g \right) \dot{\theta}. \quad -(3)$$

Define:

$$u = 1/r. \quad -(4)$$

Then

$$\frac{du}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta} = -\frac{1}{r^2} \frac{dr}{dt} \frac{dt}{d\theta}. \quad -(5)$$

From eq. (3):

$$\frac{dt}{d\theta} = \frac{m r^2}{L} \left(1 + \omega c t_g \right), \quad -(6)$$

so

$$\frac{du}{d\theta} = -\frac{m}{L} \left(1 + \omega c t_g \right) \frac{dr}{dt} \quad -(7)$$

i.e.

$$\boxed{\dot{r} = -\frac{L}{m \left(1 + \omega c t_g \right)} \frac{du}{d\theta}} \quad -(8)$$

Next evaluate:

$$\frac{d^2 u}{d\theta^2} = -\left(1 + \omega c t_g \right) \frac{m}{L} \frac{d}{d\theta} \left(\frac{dr}{dt} \right) \quad -(9)$$

2) where

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \quad -(10)$$

and

$$\frac{d}{d\theta} = \frac{dt}{d\theta}\frac{d}{dt}. \quad -(11)$$

So:
$$\frac{d^2u}{d\theta^2} = -\left(1+\omega ct_g\right)\frac{m}{L}\frac{dt}{d\theta}\frac{d^2r}{dt^2} \quad -(12)$$

From eq. (6):

$$\frac{d^2u}{d\theta^2} = -\left(1+\omega ct_g\right)^2\frac{m^2}{L^2}r^2\frac{d^2r}{dt^2} \quad -(13)$$

So:
$$\boxed{\ddot{r} = \frac{-L^2u^2}{m^2\left(1+\omega ct_g\right)^2}\frac{d^2u}{d\theta^2}} \quad -(14)$$

From eq. (3):
$$\boxed{r\dot{\theta}^2 = \frac{L^2u^3}{m^2\left(1+\omega ct_g\right)}} \quad -(15)$$

The rigorously relativistic force is therefore:

$$F(u) = \left(m\ddot{r} - mr\dot{\theta}^2\right)\left(1+\omega ct_g\right) - \frac{1}{2}mct_g\frac{\partial\omega}{\partial r}\left(\dot{\theta}^2r^2+\dot{r}^2\right) \quad -(16)$$

$$= -\frac{L^2u^2}{m\left(1+\omega ct_g\right)}\left(\frac{d^2u}{d\theta^2}+u\right) - \frac{ct_gL^2}{2\left(1+\omega ct_g\right)^2m}\frac{\partial\omega}{\partial r}\left(u^2+\left(\frac{du}{d\theta}\right)^2\right) \quad -(17)$$

3) So:

$$\boxed{-\frac{mF(u)}{L^2u^2} = \left(\frac{1}{1+\omega c t_f}\right)\left(\frac{d^2u}{d\theta^2}+u\right) + \frac{1}{2}ct_f\left(\frac{\partial\omega}{\partial r}\right)\left(\frac{1}{(1+\omega c t_f)^2}\right)\left(u^2+\left(\frac{du}{d\theta}\right)^2\right)} \quad -(18)$$

Therefore the force $F(u)$ can be found from an experimental knowledge of

$$u = \frac{1}{r} = f(\theta) \quad -(19)$$

in terms of the quantities ω, $\partial\omega/\partial r$, and the constant interval t_f. The function $f(\theta)$ is the orbit. If ω and $\partial\omega/\partial r$ approach zero then:

$$-\frac{mF(u)}{L^2u^2} = \frac{d^2u}{d\theta^2}+u. \quad -(20)$$

This equation shows that the ellipse:

$$r = \alpha/(1+\epsilon\cos\theta) \quad -(21)$$

gives:

$$F(r) = -mMG/r^2 \quad -(22)$$

which is the Newtonian limit.