1)

196(8): Relativistic Calculation of Force Law

The force law is calculated from the lagrangian:

$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\theta}^2\right)\left(1+\omega c t_g\right) - U(r) \qquad -(1)$$

and the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}, \quad \frac{\partial \mathcal{L}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \qquad -(2)$$

Assume that the spin connection is a function of r, but not of θ. Then:

$$\frac{\partial \mathcal{L}}{\partial \omega} = m r \dot{\theta}^2\left(1+\omega c t_g\right) + \frac{1}{2} m c t_g \left(\dot{r}^2 \frac{\partial \omega}{\partial r} + \dot{\theta}^2 r^2 \frac{\partial \omega}{\partial r}\right) - \partial U / \partial r \qquad -(3)$$

$$= m r \dot{\theta} + \frac{1}{2} m c t_g \left(\dot{r}^2 \frac{\partial \omega}{\partial r} + \dot{\theta}^2 \frac{\partial}{\partial r}\left(\omega r^2\right)\right) - \frac{\partial U}{\partial r}$$

with

$$\frac{\partial \omega}{\partial r} \neq 0. \qquad -(4)$$

From eqs. (2):

$$L = m r^2 \dot{\theta} = \text{constant} \qquad -(5)$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m \ddot{r}\left(1+\omega c t_g\right) \qquad -(6)$$

So:

$$\boxed{F(r) = \left(m\ddot{r} - m r \dot{\theta}^2\right)\left(1+\omega c t_g\right) - \frac{1}{2} m c t_g \frac{\partial \omega}{\partial r}\left(\dot{\theta}^2 r^2 + \dot{r}^2\right)} \qquad -(7)$$

2) To put this equation into orbital format use:

$$\ddot{r} = -\frac{L^2}{m^2} u^2 \frac{d^2 u}{d\theta^2}, \quad r\dot{\theta}^2 = \frac{L^2}{m^2} u^3, \quad \dot{r} = -\frac{L}{m}\frac{du}{d\theta} \qquad (8)$$

where $$u = 1/r. \qquad (9)$$

So:

$$F(u) = -\frac{L^2}{m}\left(u^2 \frac{d^2 u}{d\theta^2} + u^3\right)\left(1 + \omega c t_f\right) - \frac{1}{2} c t_f \left(\frac{\partial \omega}{\partial r}\right) \frac{L^2}{m}\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) \qquad (10)$$

(11)

i.e.:

$$\left(\frac{d^2 u}{d\theta^2} + u\right)\left(1 + \omega c t_f\right) + \frac{1}{2} c t_f \left(\frac{\partial \omega}{\partial r}\right)\left(1 + \frac{1}{u^2}\left(\frac{du}{d\theta}\right)^2\right) = -\frac{m F(u)}{L^2 u^2}$$

or:

$$\boxed{\left(\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right)\left(1 + \omega c t_f\right) + \frac{1}{2} c t_f \left(\frac{\partial \omega}{\partial r}\right)\left(1 + r^2\left(\frac{d}{d\theta}\left(\frac{1}{r}\right)\right)^2\right) = -\frac{m r^2}{L^2} F(r)} \qquad (12)$$

<u>Newtonian Limit</u>

$$\omega \to 0, \quad d\omega / dr \to 0 \qquad (13)$$

so

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{m r^2}{L^2} F(r) \qquad (14)$$