196(7): Towards a New Relativity.

In previous notes for UFT 196 the force laws were derived for a precessing orbit in the limit when the connection goes to zero. The process consisted of:

$$\underset{\sim}{r} = r\,\underline{e}_r \quad -(1)$$

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(2)$$

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\underline{e}_\theta \quad -(3)$$

where:

$$\underline{v} = \frac{d\underline{r}}{dt}\,; \quad \underline{a} = \frac{d\underline{v}}{dt} \quad -(4)$$

in cylindrical polar coordinates (r, θ) in the plane of the orbit:

$$dZ = 0 \quad -(5)$$

In Cartesian coordinates:

$$\underset{\sim}{r} = r_x\,\underline{i} + r_y\,\underline{j} \quad -(6)$$

$$\underline{v} = \dot{\underline{r}} = \frac{d\underline{r}}{dt}\,, \quad \underline{a} = \dot{\underline{v}} = \frac{d\underline{v}}{dt} \quad -(7)$$

This is not a fully relativistic procedure however because the spin connection is missing. In UFT 143 the position tetrad was introduced:

$$R^a{}_\mu = R\,q^a{}_\mu \quad -(8)$$

from which the velocity is:

$$v^a = D \wedge R^a \quad -(9)$$

$$v^a{}_{\mu\nu} = c\left(\partial_\mu R^a{}_\nu - \partial_\nu R^a{}_\mu + \omega^a{}_{\mu b} R^b{}_\nu - \omega^a{}_{\nu b} R^b{}_\mu\right), \quad -(10)$$

$$\underline{v}^a = \partial \underline{r}^a + c\,\underline{\nabla} R^a_0 + c\,\omega^a{}_{0b}\,\underline{R}^b - c R^b_0\,\underline{\omega}^a{}_b$$

2) which is the general definition of velocity in general relativity based on Cartan geometry.

For the sake of illustration, apply antisymmetry as follows:

$$\frac{\partial \underline{R}^a}{\partial t} + c\,\omega^a{}_b\,\underline{R}^b = -c\,\underline{V}\,R^a_0 + c\,R^b_0\,\underline{\omega}^a{}_b \quad -(12)$$

Choosing a diagonal spin connection:

$$a = b \quad -(13)$$

then for each a:

$$\underline{v} = 2\left(\frac{\partial \underline{R}}{\partial t} + c\,\omega\,\underline{R}\right). \quad -(14)$$

The factor 2 can be eliminated by defining an initial position vector:

$$\underline{r} = \frac{1}{2}\underline{R} \quad -(15)$$

so

$$\boxed{\underline{v} = \left(\frac{\partial}{\partial t} + c\,\omega\right)\underline{r}}\,. \quad -(16)$$

It is convenient to define:

$$\frac{\partial}{\partial \tau} := \frac{\partial}{\partial t} + c\,\omega \quad -(17)$$

where τ is the proper time. It is seen that the derivative appearing in eq. (16) is a <u>covariant derivative</u>.

Using the definition (16), the effect of the

3) Spin connection or the kinetic energy can be worked out. In classical dynamics the work done is defined as:

$$W_{12} = T_2 - T_1 = \int_1^2 \underline{F} \cdot d\underline{r} \quad -(18)$$

where:

$$\underline{F} = m \frac{d\underline{v}}{dt} \quad -(19)$$

So:

$$\underline{F} \cdot d\underline{r} = m \frac{d\underline{v}}{dt} \cdot \frac{d\underline{r}}{dt} dt$$

$$= m \frac{d\underline{v}}{dt} \cdot \underline{v}\, dt \quad -(20)$$

$$= \frac{m}{2} \frac{d}{dt} \left(\underline{v} \cdot \underline{v} \right) dt$$

$$= d\left(\frac{1}{2} m v^2 \right)$$

So:

$$T_2 - T_1 = \int_1^2 d\left(\frac{1}{2} m v^2 \right) \quad -(21)$$

giving the familiar equation for classical kinetic energy:

$$\boxed{T = \frac{1}{2} m v^2} \quad -(22)$$

The first ~~theory~~ to change this definition was special relativity, where the momentum is replaced by the relativistic momentum:

$$\underline{p} = \gamma m \underline{v} \quad -(23)$$

4) Somewhat paradoxically, special relativity uses :

$$\underline{p} = m\frac{d\underline{r}}{d\tau} = \gamma m \frac{d\underline{r}}{dt} \quad -(24)$$

but defines force as:

$$\underline{F} = \frac{d\underline{p}}{dt}, \quad -(25)$$

i.e the proper time is used to define $\underline{p}$ but the time t of the frame of the observer is used to define $\underline{F}$.

It then follows in special relativity that:

$$W = T = \int \frac{d}{dt}\left(\gamma m \underline{v}\right)\cdot \underline{v}\, dt$$

$$= m\int_0^v v\, d(\gamma v) \quad -(26)$$

$$= \gamma m v^2 - m\int_0^v \frac{v\, dv}{(1-v^2/c^2)^{1/2}}$$

$$= \gamma m v^2 + mc^2(1-v^2/c^2)^{1/2} - mc^2$$

i.e.

$$\boxed{T = \gamma mc^2 - mc^2} \quad -(27)$$

and this is the well known relativistic kinetic energy.

The total energy is :

$$E = \gamma mc^2 \quad -(28)$$

$$\boxed{E = T + mc^2} \quad -(29)$$

The first step towards a new relativity is to define the kinetic energy from eqs. (18), (25) and (16). So as a first step the method of special relativity is used, i.e. that the force is defined by eq. (25), using the time derivative in the frame of the observer, but the momentum is defined using the covariant derivative, eqs. (16) and (17).

It is not known why the latter definition eq. (25) of special relativity appears to work experimentally. This is especially so when one considers the existence of the relativistic force equation of special relativity as used for example with the Lorentz force.

However, accepting eq. (25) for the sake of argument the kinetic energy is:

$$T = \frac{1}{2} m v^2 + mc \int \omega \, \underline{v} \cdot d\underline{r} \quad -(30)$$

$$= \frac{1}{2} m v^2 + mc \int \omega \frac{d\underline{v}}{dt} \cdot d\underline{r} \, dt$$

$$= \frac{1}{2} m v^2 + mc \iint \omega \frac{d\underline{v}}{dt} \cdot \frac{d\underline{r}}{dt} \, dt \, dt$$

$$= \frac{1}{2} m v^2 + c \iint \omega \, d\left(\frac{1}{2} m v^2\right) dt$$

$$T = \frac{1}{2} m v^2 \left(1 + c \int \omega \, dt\right) \quad -(31)$$

Assume that the integral in eq. (31) is carried out from an initial point $t = 0$ to an instant of time t_f, then:

$$T = \frac{1}{2}mv^2\left(1 + c\int_0^{t_f}\omega\,dt\right) \quad -(32)$$

$$\boxed{T = \frac{1}{2}mv^2\left(1 + c\omega t_f\right)} \quad -(33)$$

The lagrangian is therefore:

$$\boxed{\begin{aligned} \mathcal{L} &= T - V \\ &= \frac{1}{2}mv^2\left(1 + \omega c t_f\right) - V \end{aligned}} \quad -(34)$$

and the hamiltonian is:

$$\boxed{\begin{aligned} H &= T + V \quad -(35) \\ &= \frac{1}{2}mv^2\left(1 + \omega c t_f\right) + V \end{aligned}}$$

<u>Units</u>

The units of ω are m^{-1}, so the units of eqs. (33) to (35) are correct.

The next step is to work out an equation for the force law relevant to any orbit using the relativistic lagrangian (34).