196(5): General Derivation of the Inverse Square Law from an Elliptical Orbit.

Given the elliptical orbit:

$$r(t) = \frac{\alpha}{1 + \epsilon \cos \theta(t)} \quad -(1)$$

This note derives the inverse square law using a general method that is fully relativistic and which does not use a non-relativistic lagrangian method.

Consider the unit vectors $\underline{e}_r$ and $\underline{e}_\theta$ of the cylindrical polar system. As in Marion and Thornton, "Classical Dynamics", pp 31 ff, third ed., 1988,

$$\dot{\underline{e}}_r = \dot{\theta}\, \underline{e}_\theta, \quad \dot{\underline{e}}_\theta = -\dot{\theta}\, \underline{e}_r \quad -(2)$$

The linear velocity is:

$$\dot{\underline{r}} = \underline{v} = \frac{d\underline{r}}{dt} = \frac{d}{dt}\left(r\underline{e}_r\right) = \dot{r}\,\underline{e}_r + r\,\dot{\underline{e}}_r \quad -(3)$$

The acceleration is:

$$\underline{a} = \frac{d\underline{v}}{dt} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta \quad -(4)$$

The calculation takes place in the frame of the observer, so the time t is used. In this frame the elliptical orbit (1) is observed. The force is:

$$\underline{F} = m\,\underline{a} \quad -(5)$$

for ... the mass of an object in orbit attracted by

) an object M, the Sun.

The unit vectors are related to the Cartesian unit vectors by:

$$\underline{e}_r = \underline{i}\cos\theta + \underline{j}\sin\theta \quad -(6)$$

$$\underline{e}_\theta = -\underline{i}\sin\theta + \underline{j}\cos\theta \quad -(7)$$

As in eq. (4.16) of UFT 126 this analysis gives:

$$L = mr^2\dot{\theta} = mr^2\frac{d\theta}{dt} \quad -(8)$$

as a constant of the motion in the plane of the orbit. Here L is the magnitude of the total angular momentum.

From eq. (1):

$$\frac{dr}{d\theta} = \frac{\epsilon}{\alpha}r^2\sin\theta \quad -(9)$$

so

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \left(\frac{L\epsilon}{m\alpha}\right)\sin\theta \quad -(10)$$

i.e.

$$\dot{r} = \left(\frac{L\epsilon}{m\alpha}\right)\sin\theta \quad -(11)$$

$$\dot{\theta} = \frac{L}{mr^2} \quad -(12)$$

Therefore

$$\ddot{r} = \left(\frac{L\epsilon}{m\alpha}\right)\frac{d}{dt}(\sin\theta) \quad -(13)$$

$$\ddot{\theta} = \frac{L}{m}\frac{d}{dt}\left(\frac{1}{r^2}\right) \quad -(14)$$

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Now use:

$$\frac{df(r)}{d\theta} = \frac{df(r)}{dr}\frac{dr}{d\theta} \quad -(15)$$

$$\frac{df(\theta)}{dr} = \frac{df(\theta)}{d\theta}\frac{d\theta}{dr} \quad -(16)$$

So:

$$\frac{d}{dt}\left(\frac{1}{r^2}\right) = -\frac{2}{r^3}\frac{dr}{dt} \quad -(17)$$

$$\frac{d}{dt}\left(\sin\theta\right) = \cos\theta\,\frac{d\theta}{dt} \quad -(18)$$

Therefore:

$$\ddot{r} = \left(\frac{\epsilon L^2}{m^2 \alpha}\right)\frac{1}{r^2}\cos\theta \quad -(19)$$

$$\ddot{\theta} = -\left(\frac{2L^2\epsilon}{m^2\alpha}\right)\frac{\sin\theta}{r^3}. \quad -(20)$$

It follows that:

$$\ddot{r} - r\dot{\theta}^2 = \frac{\epsilon MG}{r^2}\cos\theta - \frac{L^2}{m^2 r^3} \quad -(21)$$

where G is Newton's constant. We have used:

$$\alpha = \frac{L^2}{2m^2 MG} \quad -(22)$$

Similarly:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \quad -(23)$$

So the force is :

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$$\underline{F} = m\left(\frac{\epsilon \underline{M} G}{r^2}\cos\theta - \frac{L^2}{m^2 r^3}\right)\underline{e}_r \quad -(24)$$

and is radially directed.

Now use:

$$\epsilon \cos\theta = \frac{\alpha}{r} - 1 \quad -(25)$$

from eq. (1), to find that:

$$\underline{F} = \left(-\frac{m\underline{M}G}{r^2} + \frac{L^2}{mr^3} - \frac{L^2}{mr^3}\right)\underline{e}_r \quad -(26)$$

i.e.

$$\underline{F} = -\frac{m\underline{M}G}{r^2}\underline{e}_r \quad -(27)$$

The only thing that has been used in this calculation is the definition of the cylindrical polar coordinates in a plane, and time derivatives of $\underline{r}$. As in Marion and Thornton eq. (7.32) the centrifugal force is defined as:

$$\underline{F}_c = \frac{L^2}{mr^3}\underline{e}_r \quad -(28)$$

So:

$$\underline{F} = -\frac{m\underline{M}G}{r^2}\underline{e}_r + \left(\underline{F}_c - \underline{F}_c\right). \quad -(29)$$

= circular orbit :

5)

$$f(r_0, \theta) = 0 \quad -(30)$$

and

$$r = \alpha . \quad -(31)$$

In this case the force is:

$$\underline{F} = -\frac{L^2}{mr^3}\, \underline{e}_r \quad -(32)$$

and agrees with the result obtained by the lagrangian method. The force for a circular orbit is the negative of that due to the centrifugal force.

In an orbit the net force on m is zero, so this suggests that $\underline{F}$ in eq. (27) comes from the potential energy U in the hamiltonian:

$$H = T + U \quad -(33)$$

i.e.

$$F = -\frac{\partial U}{\partial r} \quad -(34)$$

and

$$U = -\frac{mMG}{r} = m\,\Phi \quad -(35)$$

where Φ is the gravitational potential. The complete potential therefore is:

$$U = -\frac{mMG}{r} + \frac{L^2}{2mr^2} - \frac{L^2}{2mr^2} \quad -(36)$$

In the next note, this procedure will be repeated for the precessing ellipse.

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From eqs. (2) and (3):

$$\underline{v} = \dot{r}\,\underline{e}_r + r\dot{\theta}\,\underline{e}_\theta \quad -(37)$$

so

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2 r^2} \quad -(38)$$

So:

$$\underline{F} = m\frac{d\underline{v}}{dt} = -\frac{mMG}{r^2}\,\underline{e}_r \quad -(39)$$

This is usually referred to as the equivalence principle, and using this method it is seen that the elliptical orbit implies the equivalence principle.

Usually, $\underline{F}$ is described as the radially directed force between m and M. However, its complete definition is, from eq. (4):

$$\underline{F} = m\frac{d\underline{v}}{dt} = \left(\ddot{r} - r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\underline{e}_\theta \quad -(30)$$

$$= -\frac{mMG}{r^2}\,\underline{e}_r$$

given that:

$$r = \frac{\alpha}{1+\epsilon\cos\theta}. \quad -(31)$$

In other words the acceleration of m in the orbit (31) is:

$$\underline{a} = -\frac{MG}{r^2}\,\underline{e}_r. \quad -(32)$$

Usually, this is known as the acceleration due to gravity, the result of the curve (31).