193(5): Light Deflection Due to Gravitation.

The orbit of any object of mass m around M is observed in the solar system to be a conic section:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} \quad -(1)$$

So

$$\frac{dr}{d\theta} = \frac{x\epsilon}{\alpha} r^2 \sin(x\theta) \quad -(2)$$

Here

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r} - 1\right) \quad -(3)$$

and

$$\cos^2(x\theta) + \sin^2(x\theta) = 1 \quad -(4)$$

so

$$\sin(x\theta) = \left(1 - \frac{1}{\epsilon^2}\left(\frac{\alpha}{r} - 1\right)^2\right)^{1/2} \quad -(5)$$

$$= \left(\frac{\epsilon^2 r^2 - (\alpha - r)^2}{\epsilon^2 r^2}\right)^{1/2}.$$

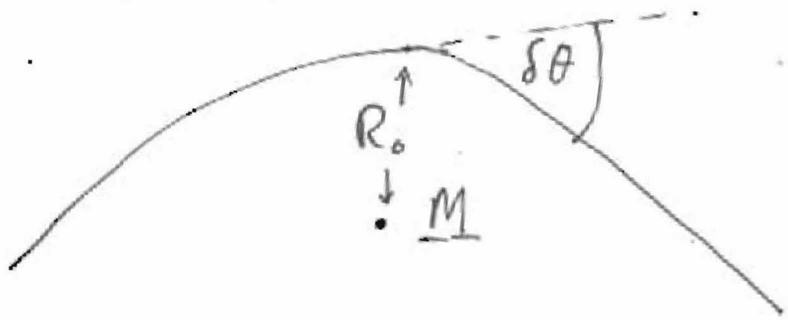
So:

$$\frac{dr}{d\theta} = \frac{xr}{\alpha}\left(\epsilon^2 r^2 - (\alpha - r)^2\right)^{1/2} \quad -(6)$$

and

$$\boxed{\frac{d\theta}{dr} = \frac{\alpha}{xr}\left(\epsilon^2 r^2 - (\alpha - r)^2\right)^{-1/2}} \quad -(7)$$

Fig (1)



2)

With reference to Fig (1):

$$\boxed{\Delta\theta = \frac{2\alpha}{x}\int_{R_0}^{\infty} \frac{1}{r}\left(\epsilon^2 r^2 - (\alpha - r)^2\right)^{-1/2} dr \quad - \pi} \quad -(8)$$

This is the correct calculation of the deflection by M_1 of m at a minimum distance of approach R_0. If the photon has mass m this must be true for the photon.

In general relativity:

$$\frac{d\theta}{dr} = \frac{1}{r^2}\left(\frac{1}{b^2} - m(r)\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\right)^{-1/2}$$

$$= \frac{1}{r^2}\left(\frac{1}{b^2} - m(r)\left(\frac{a^2 + r^2}{a^2 r^2}\right)\right)^{-1/2}$$

$$= \frac{1}{r^2}\left(\frac{a^2 r^2 - b^2 m(r)\left(a^2 + r^2\right)}{a^2 r^2 b^2}\right)^{-1/2} \quad -(9)$$

$$= \frac{ab}{r}\left(a^2 r^2 - b^2 m(r)\left(a^2 + r^2\right)\right)^{-1/2}$$

and

$$\Delta\theta = ab\int_{R_0}^{\infty} \frac{1}{r}\left(a^2 r^2 - b^2 m(r)\left(a^2 + r^2\right)\right)^{-1/2} dr - \pi \quad -(10)$$

) It can be seen that the two expressions (8) and (10) are similar, but eq. (8) uses only the experimental observation (1).

The most elegant approach is the lagrangian. For

$$x = 1 \quad -(11)$$

then

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \alpha, \quad -(12)$$

i.e

$$\frac{d^2 u}{d\theta^2} + u = \alpha \quad -(13)$$

with

$$u = 1/r. \quad -(14)$$

So

$$\frac{du}{dr} = -\frac{1}{r^2}, \quad \frac{d^2 u}{dr^2} = \frac{2}{r^3} \quad -(15)$$

and

$$u = \frac{1}{\alpha}\left(1 + \epsilon\cos(x\theta)\right) \quad -(16)$$

so

$$\frac{du}{d\theta} = -\frac{x\epsilon}{\alpha}\sin(x\theta)$$

$$= -\frac{x\epsilon}{\alpha}\left(1 - \frac{1}{\epsilon^2}\left(u - \frac{1}{\alpha}\right)^2\right)^{1/2}$$

$$= -\frac{x\epsilon}{\alpha}\left(\frac{\epsilon^2\alpha^2 - (\alpha u - 1)^2}{\epsilon^2\alpha^2}\right)^{1/2}$$

$$= -\frac{x}{\alpha^2}\left(\epsilon^2\alpha^2 - (\alpha u - 1)^2\right)^{1/2} \quad -(16)$$

So

$$\frac{d\theta}{du} = -\frac{\alpha^2}{x}\left(\epsilon^2\alpha^2-(\alpha u-1)^2\right)^{-1/2} \quad -(17)$$

and:

$$\boxed{\Delta\theta = -\frac{2\alpha^2}{x}\int_{1/R_0}^{0}\left(\epsilon^2\alpha^2-(\alpha u-1)^2\right)^{-1/2}du - \pi \quad -(18)}$$

If the path of the photon of mass m is assumed to be a parabola then

$$\epsilon = 1. \quad -(19)$$

In the solar system:

$$x = 1 \quad -(20)$$

to an excellent approximation, so:

$$\boxed{\Delta\theta = -2\alpha^2\int_{1/R_0}^{0}\left(\alpha^2-(\alpha u-1)^2\right)^{-1/2}du - \pi} \quad -(21)$$

Therefore α may be found from the experimental $\Delta\theta$.