193(4): Lagrangian Dynamics of Any Planar Orbit.

The lagrangian is:

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \quad -(1)$$

where μ is the reduced mass:

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad -(2)$$

and $V(r)$ the potential of attraction. If:

$$m_2 \gg m_1 \quad -(3)$$

then

$$\mu \doteq m_1 \quad -(4)$$

to a good approximation. We denote by M the mass of the attracting object (e.g. the sun) and by m the mass of the attracted object (e.g. a planet). So:

$$\mu \sim m \quad -(5)$$

The lagrangian is cyclically symmetric in θ so the angular momentum L conjugate to the coordinate θ is conserved:

$$\dot{L} = \frac{\partial \mathcal{L}}{\partial \theta} = 0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \quad -(6)$$

i.e.

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad -(7)$$

$$= \text{constant}$$

The angular momentum L is the first integral of the motion, and is constant. It is the total angular momentum.

) The Lagrange equation for r is:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \quad -(8)$$

From eqs. (1) and (8):

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{\partial V}{\partial r} = F(r) \quad -(9)$$

Here $F(r)$ is the force of attraction between m and M.

Make the change of variable:

$$u = 1/r \quad -(10)$$

Then

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} \quad -(11).$$

From eq. (7):

$$\frac{d\theta}{dt} = \frac{L}{mr^2} \quad -(12)$$

so

$$\frac{du}{d\theta} = -\frac{m}{L}\frac{dr}{dt} = -\frac{m}{L}\dot{r} \quad -(13)$$

and

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{m}{L}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{m}{L}\dot{r}\right) = \frac{-m}{L\dot{\theta}}\ddot{r} \quad -(14)$$

Also:

$$\frac{d^2u}{d\theta^2} = -\frac{m^2}{L^2}r^2\ddot{r} \quad -(15).$$

Therefore:

$$\ddot{r} = -\frac{L^2}{m^2} u^2 \frac{d^2 u}{d\theta^2} \quad -(16)$$

$$r\dot{\theta}^2 = \frac{L^2}{m^2} u^3 \quad -(17)$$

So

$$\frac{d^2 u}{d\theta^2} + u = -\frac{m}{L^2} \frac{1}{u^2} F(u) \quad -(18)$$

and the general force law for any planar orbit is:

$$\boxed{F(r) = -\frac{L^2}{mr^2}\left[\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r}\right]} \quad -(19)$$

This is true for any potential of attraction defined by:

$$F(r) = -\frac{\partial V(r)}{\partial r} \quad -(20)$$

The Static Conic Sections

These are all defined by:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos\theta\right) \quad -(21)$$

where α is the half right latitude and ϵ is the eccentricity. The ellipse is defined by $0 < \epsilon < 1$, the hyperbola by $\epsilon > 1$, the parabola by $\epsilon = 1$, and the circle by $\epsilon = 0$.

4) For all the conic sections the force law is:

$$F(r) = -\frac{L^2}{m\alpha r^2} \quad -(22)$$

and the eccentricity ϵ does not appear in the force law.

For the circle for example:

$$\alpha = r \quad -(23)$$

so for a circular orbit:

$$F(r) = -\frac{L^2}{mr^3} \quad -(24)$$

which is an inverse cube force law. From eq. (22):

$$F(u) = -\left(\frac{L^2}{m\alpha}\right)u^2 \quad -(25)$$

so from eqs. (18) and (25):

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} \quad -(26)$$

for all conic sections, i.e.

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}} \quad -(27)$$

The conic sections are sketched as follows:

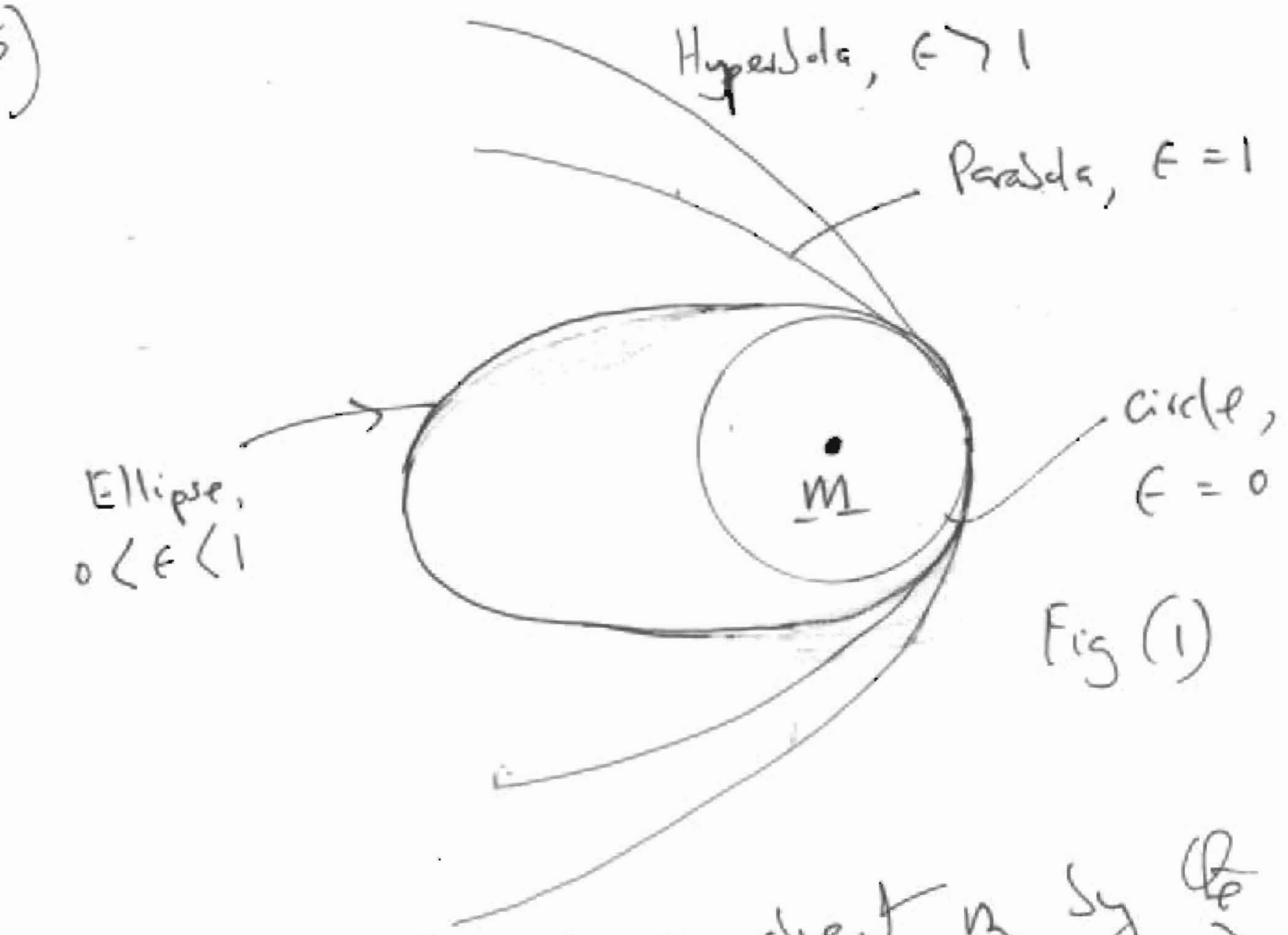

Fig (1)

So the deflection of an object m by the object $\underline{M}$ can be calculated from eq. (27) or eq. (26). The deflection does not depend on ϵ.

The Precessing Conic Sections

In this case:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos(x\theta)\right) \quad -(28)$$

and

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{x^2}{\alpha} + \frac{1}{r}\left(1 - x^2\right)\right) \quad -(29)$$

so

$$F(u) = -\frac{L^2 x^2}{m\alpha} u^2 - \frac{L^2\left(1-x^2\right)}{m} u^3 \quad -(30)$$

and

$$\boxed{\frac{d^2 u}{d\theta^2} + u = \frac{x^2}{\alpha} + \left(1 - x^2\right) u} \quad -(31)$$