193(2): The Newtonian Force Law for a Precessing Ellipse.

From the Euler Lagrange equation of any planar orbit:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{L^2} F(r) \quad -(1)$$

where

$$F(r) = -\frac{\partial V}{\partial r}, \quad -(2)$$

in cylindrical polar coordinates (r, θ). Here m is the mass of the attracted object, L is the constant angular momentum, $V(r)$ the potential and $F(r)$ the force of attraction between m and M.

For an ellipse:

$$\frac{1}{r} = \frac{1}{\alpha}\left(1 + \epsilon \cos\theta\right) \quad -(3)$$

so

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha} \quad -(4)$$

and

$$\boxed{F(r) = -\left(\frac{L^2}{m\alpha}\right)\frac{1}{r^2}} \quad -(5)$$

This is the precise form of the Hooke / Newton inverse square law.

In modern notation Hooke and Newton wrote it as:

2)
$$F(r) = -\frac{mMG}{r^2} \quad -(6)$$

where G is Newton's constant. In order for eq. (5) and (6) to be the same:

$$\alpha = \frac{L^2}{m^2 MG} \quad -(7)$$

Eq. (6) leads to eq. (3) if and only if eq. (7) is true.

The angular momentum is:

$$L = mr^2 \frac{d\theta}{dt} = \text{constant} \quad -(8)$$

The total energy E is:

$$E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + V(r) \quad -(9)$$

where

$$V(r) = -\frac{mMG}{r} \quad -(10)$$

so

$$\theta(r) = \int \frac{1}{r^2} \left(2m \left(E - V - \frac{L^2}{2mr^2} \right) \right)^{-1/2} dr \quad -(11)$$

Eq. (11) leads to eq. (7).

However, for a precessing ellipse:

$$\frac{1}{r} = \frac{1}{\alpha} \left(1 + \epsilon \cos(x\theta) \right) \quad -(12)$$

and

3)

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{x^2\epsilon}{\alpha}\cos(x\theta) \quad -(13)$$

so:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\left(1-x^2\right)\cos(x\theta)\right)$$

$$= -\frac{mr^2}{L^2}F(r) \quad -(14)$$

so:

$$F(r) = -\frac{L^2}{m\alpha r^2}\left(1+\epsilon\left(1-x^2\right)\cos(x\theta)\right) \quad -(15)$$

in which:

$$\cos(x\theta) = \frac{1}{\epsilon}\left(\frac{\alpha}{r}-1\right) \quad -(16)$$

so:

$$F(r) = -\frac{L^2}{m\alpha r^2}\left(1+\left(1-x^2\right)\left(\frac{\alpha}{r}-1\right)\right) \quad -(17)$$

$$\boxed{F(r) = -\left(\frac{L^2x^2}{m\alpha}\right)\frac{1}{r^2} - \frac{L^2}{mr^3}\left(1-x^2\right)}$$

$-(18)$

This result means that the precessing ellipse is described by a combination of an inverse square and inverse cube law.

This result is one of Lagrangian dynamics at the classical level, there is no need for

4) the theory of general relativity to describe the precessing ellipse. The force can be rewritten as:

$$F(r) = -\frac{L^2}{mr^2}\left(\frac{x^2}{\alpha} + \frac{1}{r}\left(1-x^2\right)\right) \quad -(19)$$

The corresponding result claimed from Einsteinian GR is

$$F(r) = ? -\frac{mMG}{r^2} - \frac{3MGL^2}{mc^2r^4} \quad -(20)$$

It can be seen that eqns. (19) and (20) are not the same.

Conclusion

Einsteinian GR does not give a precessing ellipse as claimed.