193(3): Lagrangian Force Law for a Binary Pulsar

Let the orbit of the binary pulsar be described by the spiralling and precessing ellipse:

$$r = \frac{\alpha}{1+\epsilon\cos(x\theta)} e^{a\theta} \quad -(1)$$

so

$$\frac{1}{r} = \frac{1}{\alpha}\left(1+\epsilon\cos(x\theta)\right) e^{-a\theta} \quad -(2)$$

Therefore:

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\epsilon x}{\alpha}\sin(x\theta) e^{-a\theta} - \frac{a}{\alpha}\left(1+\epsilon\cos(x\theta)\right) e^{-a\theta}$$

$$= -\frac{e^{-a\theta}}{\alpha}\left(x\epsilon\sin(x\theta) + a\left(1+\epsilon\cos(x\theta)\right)\right) \quad -(3)$$

and:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{a}{\alpha}e^{-a\theta}\left(x\epsilon\sin(x\theta) + a\left(1+\epsilon\cos(x\theta)\right)\right)$$

$$- \frac{e^{-a\theta}}{\alpha}\left(x^2\epsilon\cos(x\theta) - a\epsilon x\sin(x\theta)\right) \quad -(4)$$

So:

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{e^{-a\theta}}{\alpha}\left[a x\epsilon\sin(x\theta) + a^2\left(1+\epsilon\cos(x\theta)\right) - x^2\epsilon\cos(x\theta) + a\epsilon x\sin(x\theta) + 1 + \epsilon\cos(x\theta)\right]$$

$$= \frac{e^{-a\theta}}{\alpha}\left[1+a^2+\epsilon\left(1-x^2+a^2\right)\cos(x\theta) + 2a\epsilon x\sin(x\theta)\right]$$

$$= -\frac{mr^2}{L^2} F(r) \quad -(5)$$

So:

$$F(r) = -\frac{L^2}{mr^2} \frac{e^{-a\theta}}{\alpha} \begin{bmatrix} 1 + a^2 + \epsilon(1 - x^2 + a^2)\cos(x\theta) \\ + 2a\epsilon x \sin(x\theta) \end{bmatrix} \quad -(6)$$

i.e.

$$\boxed{F(r) = -\frac{L^2}{mr^3}\left[\frac{1 + a^2 + \epsilon(1 - x^2 + a^2)\cos(x\theta) + 2a\epsilon \sin(x\theta)}{1 + \epsilon\cos(x\theta)}\right]} \quad -(7)$$

This is an inverse cube force law moderated by terms in $\cos(x\theta)$ and $\sin(x\theta)$. It is not a Hooke / Newton inverse square law and is not an Einsteinian force law of general relativity. However, it is the correct Lagrangian force law.

In the limit:

$$a \to 0 \quad -(8)$$

$$F(r) \to -\frac{L^2}{\alpha m r^2}\left[1 + \epsilon(1 - x^2)\cos(x\theta)\right] \quad -(9)$$

which is the same as eq. (15) of note 193(2). QED.