1)

188(6): Theorem of the Antisymmetric Connection

Consider three metric compatibility conditions in cyclic permutation:

$$\partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0 \quad -(1)$$

$$\partial_\mu g_{\nu\rho} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda} = 0 \quad -(2)$$

$$\partial_\nu g_{\rho\mu} - \Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} = 0 \quad -(3)$$

Subtract (2) and (3) from (1):

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} + \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} + \Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda} + \Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu} + \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} = 0 \quad -(4)$$

Subtract (1) and (2) from (3):

$$\partial_\nu g_{\rho\mu} - \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} + \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} + \Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda} + \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} + \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0 \quad -(5)$$

Subtract (1) and (3) from (2):

$$\partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu} - \partial_\nu g_{\rho\mu} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda} + \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} + \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} + \Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu} + \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} = 0 \quad -(6)$$

Now apply antisymmetry:

$$\Gamma^{\lambda}_{\mu\nu} = -\Gamma^{\lambda}_{\nu\mu} \quad -(7)$$

2) to obtain:

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} = 2\left(\Gamma^\lambda_{\rho\mu} g_{\lambda\nu} + \Gamma^\lambda_{\rho\nu} g_{\lambda\mu}\right) \quad -(8)$$

$$\partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} = 2\left(\Gamma^\lambda_{\nu\rho} g_{\lambda\mu} + \Gamma^\lambda_{\nu\mu} g_{\lambda\rho}\right) \quad -(9)$$

$$\partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} = 2\left(\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} + \Gamma^\lambda_{\mu\rho} g_{\lambda\nu}\right) \quad -(10)$$

Add eq. (8) and (10):

$$\boxed{\partial_\nu g_{\rho\mu} = -\left(\Gamma^\lambda_{\rho\nu} g_{\lambda\mu} + \Gamma^\lambda_{\mu\nu} g_{\lambda\rho}\right)} \quad -(11)$$

This equation relates the general metric to the antisymmetric connection.

For a diagonal metric:

$$\rho = \mu \quad -(12)$$

so

$$\partial_\nu g_{\mu\mu} = 2\Gamma^\mu_{\nu\mu} g_{\mu\mu} \quad -(13)$$

$$\boxed{\Gamma^\mu_{\nu\mu} = \frac{1}{2 g_{\mu\mu}} \partial_\nu g_{\mu\mu} \qquad \nu \neq \mu} \quad -(14)$$

This is the criterion of the antisymmetric connection:

$$\Gamma^\mu_{\nu\mu} = -\Gamma^\mu_{\mu\nu}. \quad -(15)$$