1)

Note 188(1): Unique Solution for Antisymmetric Metric.

Starting eq. (12) of note 186(3):

$$2\left(\Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} + \Gamma^{\lambda}_{\rho\nu} g_{\lambda\mu}\right) = \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} \quad -(1)$$

Now let:

$$\rho \rightarrow \mu, \ \mu \rightarrow \rho. \quad -(2)$$

This relabels indices and leaves eq. (1) the same. So:

$$2\left(\Gamma^{\lambda}_{\mu\rho} g_{\lambda\nu} + \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho}\right) = \partial_\mu g_{\rho\nu} - \partial_\rho g_{\nu\mu} - \partial_\nu g_{\mu\rho} \quad -(3)$$

Add (1) and (3) using the symmetry of the metric

$$2\left(\Gamma^{\lambda}_{\rho\nu} g_{\lambda\mu} + \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho}\right) = -2\, \partial_\nu g_{\rho\mu} \quad -(4)$$

i.e.

$$\boxed{\partial_\nu g_{\rho\mu} = -\left(\Gamma^{\lambda}_{\rho\nu} g_{\lambda\mu} + \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho}\right)} \quad -(5)$$

This is a general result for any metric.

For a diagonal metric:

$$\rho = \mu \quad -(6)$$

2)

So :

$$\boxed{\partial_\nu g_{\mu\mu} = 2\Gamma^{\lambda}_{\nu\mu}\, g_{\mu\lambda}} \quad -(7)$$

with

$$\boxed{\nu \neq \mu} \quad -(8)$$

<u>by definition.</u>

1) If $\mu = 0$ $\quad -(9)$

$$\partial_\nu g_{00} = 2\Gamma^{0}_{\nu 0}\, g_{00} \quad -(10)$$

2) If $\mu = 1$ $\quad -(11)$

$$\partial_\nu g_{11} = 2\Gamma^{1}_{\nu 1}\, g_{11} \quad -(12)$$

3) If $\mu = 2$ $\quad -(13)$

$$\partial_\nu g_{22} = 2\Gamma^{2}_{\nu 2}\, g_{22} \quad -(14)$$

4) If $\mu = 3$ $\quad -(15)$

$$\partial_\nu g_{33} = 2\Gamma^{3}_{\nu 3}\, g_{33}. \quad -(16)$$