158(7): The Most General Metric of a Spherically Symmetric Spacetime

Consider an n dimensional manifold foliated by m dimensional manifolds. Consider a set of m coordinate functions u^i on the submanifold and a set of $n-m$ coordinate functions v^I to define the submanifold. Therefore:

$$i = 1 \text{ to } m, \quad I = 1 \text{ to } n-m \quad -(1)$$

If the submanifolds are maximally symmetric spaces such as two spheres then:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{IJ}(v) dv^I dv^J + f(v) \gamma_{ij}(u) du^i du^j \quad -(2)$$

where $\gamma_{ij}(u)$ is the metric on the submanifold.

If the submanifolds are two-spheres then:

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2. \quad -(3)$$

Consider a four dimensional spacetime. Eq. (2) implies

$$ds^2 = g_{aa}(a,b) da^2 + g_{ab}(a,b)(da\,db + db\,da) + g_{bb} db^2 + r^2(a,b) d\Omega^2 \quad -(4)$$

here a and b are coordinates, and where $r(a,b)$ is to be determined.

Let

$$(a,b) \rightarrow (a,r) \quad -(5)$$

$$\boxed{ds^2 = g_{aa}(a,r) da^2 + g_{ar}(a,r)(da\,dr + dr\,da) + g_{rr}(a,r) dr^2 + r^2 d\Omega^2 \quad -(6)}$$

This is the most general metric of a spherically

2)
symmetric spacetime.

In order to find a diagonal metric:

$$g_{aa}\, da^2 + g_{ab}\,(da\,db + db\,da) + g_{bb}\, db^2$$

$$= g_{aa}\, da^2 + g_{ab}\,(da\,dr + dr\,da) + g_{rr}\, dr^2 \quad -(7)$$

$$= c^2 m\, dt^2 + n\, dr^2$$

This is possible with the constraints:

$$g_{aa} = c^2 m \left(\frac{\partial t}{\partial a}\right)^2 \quad -(8)$$

$$g_{rr} = n + m \left(\frac{\partial t}{\partial r}\right)^2 \quad -(9)$$

$$g_{ar} = -m \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right) \quad -(10)$$

If t is not a function of r, then:

$$g_{aa} = c^2 m \left(\frac{\partial t}{\partial a}\right)^2 \quad -(11)$$

$$g_{rr} = n \quad -(12)$$

$$g_{ar} = 0 \quad -(13)$$

This analysis shows that a spherically symmetric spacetime is diagonal unless:

$$\frac{\partial r}{\partial t} \neq 0. \quad -(14)$$

Therefore:

$$\boxed{ds^2 = c^2 m(r,t)\, dt^2 + n(r,t)\, dr^2 + r^2 d\Omega^2 \quad -(15)}$$

for all coordinate systems in which r and t are

5)

independent. For a flat Minkowski spacetime, a Lorentzian spacetime, then:

$$ds^2 = -c^2dt^2 + dr^2 + r^2 d\Omega^2 \quad -(16)$$

and the Minkowski metric is spherically symmetric, so in general:

$$\boxed{ds^2 = -m(r,t)\,c^2dt^2 + n(r,t)\,dr^2 + r^2 d\Omega^2} \quad -(17)$$

This is the most general metric that is likely to be relevant to physics, and it is diagonal:

$$g_{\mu\nu} = \begin{bmatrix} -m(r,t) & 0 & 0 & 0 \\ 0 & n(r,t) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\phi \end{bmatrix} \quad -(18.)$$

in cylindrical polar coordinates.

As in note 188(4) it is governed by the metric compatibility equation:

$$\partial_\nu g_{\mu\mu} = 2\Gamma^{\lambda}_{\nu\mu}\, g_{\mu\lambda} \quad -(19)$$

$$\mu \neq \nu \quad -(20)$$

Therefore:

$$\Gamma^{0}_{10} = \frac{1}{2g_{00}} \frac{\partial g_{00}}{\partial r}, \quad -(21)$$

i.e.

$$\boxed{\Gamma^{0}_{10} = -\Gamma^{0}_{01} = \frac{1}{2m(r,t)} \frac{\partial m(r,t)}{\partial r}} \quad -(22)$$

$$\partial_0 g_{11} = 2\Gamma^1_{01} g_{11} \quad -(23)$$

$$\frac{1}{c}\frac{\partial}{\partial t} n(r,t) = 2\Gamma^1_{01} n(r,t) \quad -(24)$$

$$\boxed{\Gamma^1_{01} = -\Gamma^1_{10} = \frac{1}{2\,n(r,t)} \cdot \frac{1}{c}\frac{\partial n(r,t)}{\partial t}} \quad -(25)$$

Also:

$$\partial_1 g_{22} = 2\Gamma^2_{12} g_{22} \quad -(26)$$

i.e.

$$\frac{\partial}{\partial r}\left(r^2\right) = 2\Gamma^2_{12} r^2 \quad -(27)$$

$$\boxed{\Gamma^2_{12} = -\Gamma^2_{21} = \frac{1}{r}} \quad -(28)$$

$$\partial_1 g_{33} = 2\Gamma^3_{13} g_{33} \quad -(29)$$

i.e.

$$\boxed{\Gamma^3_{13} = -\Gamma^3_{31} = \frac{1}{r}} \quad -(30)$$

Finally:

$$\partial_2 g_{33} = 2\Gamma^3_{23} g_{33} \quad -(31)$$

$$\frac{\partial}{\partial \phi}\left(r^2 \sin^2\phi\right) = 2\Gamma^3_{23} r^2 \sin^2\phi \quad -(32)$$

$$\boxed{\Gamma^3_{23} = -\Gamma^3_{32} = \frac{\cos\phi}{\sin\phi}} \quad -(33)$$

These are five independent antisymmetric connections of a spherically symmetric spacetime:

5)

$$\boxed{\begin{aligned}
\Gamma^{0}_{10} &= -\Gamma^{0}_{01} = \frac{1}{2m(r,t)}\,\frac{\partial m(r,t)}{\partial r}, \\
\Gamma^{1}_{01} &= -\Gamma^{1}_{10} = \frac{1}{2n(r,t)}\,\frac{1}{c}\,\frac{\partial n(r,t)}{\partial t}, \\
\Gamma^{2}_{12} &= -\Gamma^{2}_{21} = \frac{1}{r} \\
\Gamma^{3}_{13} &= -\Gamma^{3}_{31} = \frac{1}{r} \\
\Gamma^{3}_{23} &= -\Gamma^{3}_{32} = \frac{\cos\phi}{\sin\phi}
\end{aligned}} \quad -(34)$$

The functions $m(r,t)$ and $n(r,t)$ are found from the Evans identity:

$$\boxed{D_\mu T^{\kappa\mu\nu} := R^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad -(35)$$

The equation of orbits is found from eq. (17). The overall aim of cosmology is to find functions $m(r,t)$ and $n(r,t)$ so that all known orbits are described.