Note 188(2) : Triple Cross Check

Consider a cyclic permutation of indices :

μ, ν, ρ
ρ, μ, ν
ν, ρ, μ

and permute as follows :

$$\mu \to \rho, \quad \nu \to \mu, \quad \rho \to \nu \qquad -(1)$$

Now permute twice on eq. (1) of note 188(1) :

$$2\left(\Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} + \Gamma^{\lambda}_{\rho\nu} g_{\lambda\mu}\right) = \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} \qquad -(2)$$

$$\downarrow$$

$$2\left(\Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu} + \Gamma^{\lambda}_{\nu\mu} g_{\lambda\rho}\right) = \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} \qquad -(3)$$

$$\downarrow$$

$$2\left(\Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} + \Gamma^{\lambda}_{\mu\rho} g_{\lambda\nu}\right) = \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \qquad -(4)$$

Add eqs. (2) and (4) to find again :

$$2\left(\Gamma^{\lambda}_{\rho\nu} g_{\lambda\mu} + \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho}\right) = -2\,\partial_\nu g_{\rho\mu} \qquad -(5)$$

i.e

$$\boxed{\partial_\nu g_{\rho\mu} = -\left(\Gamma^{\lambda}_{\rho\nu} g_{\lambda\mu} + \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho}\right)} \qquad -(6)$$

which is the same as eq. (5) of note 188(1) QED