1)

187(4): Self Consistent Solutions

The two simultaneous equations are:

$$\partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0 \quad -(1)$$

and

$$\partial_\mu T^{\kappa}_{\ \nu\sigma} = 0. \quad -(2)$$

Eq. (2) is:

$$\partial_\mu \Gamma^{\kappa}_{\ \nu\sigma} = 0 \quad -(3)$$

Eq (2) is derived from:

$$D_\mu T^{\kappa\mu\nu} = R^{\kappa\ \ \mu\nu}_{\ \mu} \quad -(4)$$

when

$$R^{\kappa\ \ \mu\nu}_{\ \mu} = 0. \quad -(5)$$

In eq. (4):

$$T^{\kappa\mu\nu} = g^{\mu\alpha} g^{\nu\beta} T^{\kappa}_{\ \alpha\beta} \quad -(6)$$

by definition. Therefore:

$$D_\mu T^{\kappa\mu\nu} = D_\mu \left(g^{\mu\alpha} g^{\nu\beta} T^{\kappa}_{\ \alpha\beta} \right) \quad -(7)$$

$$= g^{\mu\alpha} g^{\nu\beta} D_\mu T^{\kappa}_{\ \alpha\beta}$$

because by metric compatibility:

$$D_\mu \left(g^{\mu\alpha} g^{\nu\beta} \right) = 0 \quad -(8)$$

As in previous notes 187 eq (7) means:

$$D_\mu T^{\kappa}_{\ \alpha\beta} = \partial_\mu T^{\kappa}_{\ \alpha\beta} = 0 \quad -(9)$$

2)

So: $\partial_\mu \Gamma^{\kappa}_{\alpha\beta} = 0$ — (10)

which is eq. (3).

For a spherically symmetric spacetime:

$$\left. \begin{array}{l} g_{00} = e^{2\alpha}, \quad g_{11} = -e^{-2\beta}, \\ g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\phi \end{array} \right\} \quad \text{— (11)}$$

where:

$$\alpha = \alpha(r, t) \quad \text{— (12)}$$

$$\beta = \beta(r, t) \quad \text{— (13)}$$

In the solar system:

$$\exp(2\alpha) \rightarrow 1 - \frac{r_0}{r} \quad \text{— (14)}$$

$$\exp(-2\beta) \rightarrow \left(1 - \frac{r_0}{r}\right)^{-1} \quad \text{— (15)}$$

is a possibility, not the only possibility.

If in eq. (1):

$$\mu = \nu = 0 \quad \text{— (16)}$$

$$\Gamma^0_{10} = \frac{1}{2g_{00}} \frac{\partial}{\partial r} g_{00} \quad \text{— (17)}$$

if

$$\alpha = \alpha(r) \quad \text{— (18)}$$

i.e. the no time dependence.

So

$$\boxed{\Gamma^0_{10} = \frac{\partial\alpha}{\partial r}} \quad \text{— (19)}$$

2) The possibilities of eq. (10) are:

$$\partial_\mu \Gamma^0_{10} = 0 \quad -(20)$$

where:

$$\mu = 0, 1, 2, 3$$

The only possibility if α depends only on r is:

$$\frac{\partial}{\partial r} \Gamma^0_{10} = 0 \quad -(21)$$

i.e.

$$\boxed{\frac{\partial^2 \alpha}{\partial r^2} = 0} \quad -(22)$$

The Hooke / Newton force is:

$$\boxed{F = -mc^2 \frac{\partial \alpha}{\partial r}} \quad -(23)$$

constrained by $\frac{\partial^2 \alpha}{\partial r^2} = 0 \quad -(24)$

In the solar system, to a very good approximation:

$$g_{00} = 1 - \frac{2mG}{c^2 r} \quad -(25)$$

so $\alpha \sim \frac{1}{2} \log_e \left(1 - \frac{r_0}{r} \right) \quad -(26)$

If α depends both on r and t then eq.

4) (16) gives:

$$\partial_\rho g_{00} - \Gamma^{0}_{\rho 0} g_{00} - \Gamma^{0}_{\rho 0} g_{00} = 0 \quad -(27)$$

i.e. eq. (17) and an unphysical symmetric connection Γ^{0}_{00}.

For $\mu = \nu = 1 \quad -(28)$

the physically meaningful connection is:

$$\Gamma^{1}_{01} = \frac{1}{2 g_{11}} \frac{\partial}{\partial t} g_{11} \quad -(29)$$

$$= -\frac{\partial \beta}{\partial t}$$

i.e. $$\boxed{\Gamma^{1}_{01} = -\frac{\partial \beta}{\partial t}} \quad -(30)$$

castwardly $$\boxed{\frac{\partial^2 \beta}{\partial t^2} = 0.} \quad -(31)$$