1) 187(10): Calculations with the General Spherical Metric

From computer algebra:

$$\Gamma^{0}_{10} = \frac{\partial \alpha}{\partial r}, \quad \Gamma^{1}_{01} = \frac{\partial \beta}{\partial t},$$
$$\Gamma^{2}_{12} = \Gamma^{3}_{13} = \frac{1}{r}, \quad \Gamma^{3}_{23} = \frac{\cos\phi}{\sin\phi} \quad -(1)$$

$$R^{0}{}_{001} = \frac{\partial \alpha}{\partial r}\frac{\partial \beta}{\partial t} + \frac{\partial^2 \alpha}{\partial r \partial t} \quad -(2)$$

$$R^{0}{}_{101} = \frac{\partial^2 \alpha}{\partial r^2} + \left(\frac{\partial \alpha}{\partial r}\right)^2 \quad -(3)$$

$$R^{1}{}_{001} = -\left(\frac{\partial^2 \beta}{\partial t^2} + \left(\frac{\partial \beta}{\partial t}\right)^2\right) \quad -(4)$$

$$R^{1}{}_{101} = -\left(\frac{\partial \alpha}{\partial r}\frac{\partial \beta}{\partial t} + \frac{\partial^2 \beta}{\partial r \partial t}\right) \quad -(5)$$

$$R^{2}{}_{012} = -\frac{1}{r}\frac{\partial \beta}{\partial t} \quad -(6)$$

$$R^{3}{}_{013} = -\frac{1}{r}\frac{\partial \beta}{\partial t} \quad -(7)$$

$$R^{3}{}_{103} = \frac{1}{r}\frac{\partial \beta}{\partial t} \quad -(8)$$

$$R^{3}{}_{123} = -\frac{2}{r}\frac{\cos\phi}{\sin\phi} \quad -(9)$$

$$R^{3}{}_{223} = 1 \quad -(10)$$

with
$$g_{00} = -e^{2\alpha}, \quad g_{11} = e^{2\beta} \quad -(11)$$

Therefore:
$$R^{0}{}_{1}{}^{10} = -e^{\alpha+\beta} R^{0}{}_{110} \quad -(12)$$
$$= e^{\alpha+\beta}\left(\frac{\partial^2 \alpha}{\partial r^2} + \left(\frac{\partial \alpha}{\partial r}\right)^2\right)$$

Now consider:

$$D_1 T^{010} + D_2 T^{020} + D_3 T^{030} = R^{0}{}_{1}{}^{10} \quad -(13)$$

~~with~~ i.e.

$$2\left(D_1 \Gamma^{010} + D_2 \Gamma^{020} + D_3 \Gamma^{030}\right) = R^{0}{}_{1}{}^{10} \quad -(14)$$

i.e

$$2 D_1 \Gamma^{010} = R^{0}{}_{1}{}^{10} \quad -(15)$$

This equation is

$$2 \partial_1 \Gamma^{010} = R^{0}{}_{1}{}^{10} \quad -(16)$$

i.e.

$$-2 e^{\alpha+\beta} \partial_1 \Gamma^{0}{}_{10} = R^{0}{}_{1}{}^{10} \quad -(17)$$

which is

$$-2 \frac{\partial}{\partial r}\left(\frac{\partial \alpha}{\partial r}\right) = \frac{\partial^2 \alpha}{\partial r^2} + \left(\frac{\partial \alpha}{\partial r}\right)^2 \quad -(18)$$

or

$$\boxed{3 \frac{\partial^2 \alpha}{\partial r^2} + \left(\frac{\partial \alpha}{\partial r}\right)^2 = 0} \quad -(19)$$

This is a constraint on α.

If it is assumed that β and α are functions only of r:

$$R^{0}{}_{001} = R^{1}{}_{001} = R^{1}{}_{101} = R^{2}{}_{012} = R^{3}{}_{013}$$
$$= R^{3}{}_{103} = 0 \quad -(20)$$

and

$$\Gamma^{1}{}_{01} = 0 \quad -(21)$$

Therefore:

$$\boxed{\begin{aligned} \Gamma^{0}_{10} &= \frac{\partial \alpha}{\partial r} \\ 3\frac{\partial^2 \alpha}{\partial r^2} + \left(\frac{\partial \alpha}{\partial r}\right)^2 &= 0 \end{aligned}} \quad -(22)$$

with:

$$ds^2 = -e^{2\alpha} c^2 dt^2 + e^{2\beta} dr^2 + r^2 d\phi^2 \quad -(23)$$

in a plane.

There seems to be no basic inconsistency in this type of metric.