1)

187(3): Correction to Note 186(3)

Note 186(3) produced the correct general formula:

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda}$$
$$+ \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} + \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} + \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} + \Gamma^\lambda_{\nu\mu} g_{\rho\lambda}$$
$$= 0 \quad -(1)$$

where

$$g_{\mu\nu} = g_{\nu\mu} \quad -(2)$$
$$\Gamma^\lambda_{\mu\nu} = -\Gamma^\lambda_{\nu\mu} \quad -(3)$$

Therefore:

$$\Gamma^\lambda_{\mu\nu} g_{\lambda\rho} + \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0. \quad -(4)$$

Diagonal Metric

In this case:

$$\mu = \nu \quad -(5)$$

so

$$\Gamma^\lambda_{\mu\rho} g_{\nu\lambda} + \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} = 0 \quad -(6)$$

Therefore eq (1) simplifies to:

$$\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} = \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} + \Gamma^\lambda_{\rho\nu} g_{\mu\lambda},$$
$$\mu = \nu \quad -(7)$$

2) Let $\mu = \nu = \alpha$ — (8)

Then:

$$\boxed{\begin{aligned}&\partial_\rho g_{\alpha\alpha} - \partial_\alpha g_{\alpha\rho} - \partial_\alpha g_{\rho\alpha} \\ &= 2\Gamma^\lambda_{\rho\alpha} g_{\lambda\alpha}\end{aligned}} \quad \text{— (9)}$$

multiply both sides by $g^{\lambda\alpha}$:

$$g^{\lambda\alpha}\left(\partial_\rho g_{\alpha\alpha} - \partial_\alpha g_{\alpha\rho} - \partial_\alpha g_{\rho\alpha}\right) \quad \text{— (10)}$$

$$= 2 g^{\lambda\alpha} \Gamma^\lambda_{\rho\alpha} g_{\lambda\alpha}$$

$$= 2\Gamma^\lambda_{\rho\alpha} \quad \text{— (11)}$$

so $\Gamma^\lambda_{\rho\alpha} = \frac{1}{2} g^{\lambda\alpha}\left(\partial_\rho g_{\alpha\alpha} - \partial_\alpha g_{\alpha\rho} - \partial_\alpha g_{\rho\alpha}\right)$

the metric must be diagonal so: — (12)

$$\Gamma^\alpha_{\rho\alpha} = \frac{1}{2} g^{\alpha\alpha}\left(\partial_\rho g_{\alpha\alpha} - \partial_\alpha g_{\alpha\rho} - \partial_\alpha g_{\rho\alpha}\right)$$

because $\alpha = \lambda$. — (13)

the connection must be antisymmetric so

$$\rho \neq \alpha \quad \text{— (14)}$$

3) in which case:

$$g_{\alpha\rho} = g_{\rho\alpha} = 0 \quad -(15)$$

and

$$\boxed{\Gamma^{\alpha}_{\rho\alpha} = \frac{1}{2} g^{\alpha\alpha} \partial_\rho g_{\alpha\alpha}} \quad -(16)$$

Cross Checks

If $\alpha = 0$, $\rho = 1$: $\partial_1 g_{00} = 2\Gamma^0_{10} g_{00}$ $-(17)$

If $\alpha = 2, \rho = 1$: $\partial_1 g_{22} = 2\Gamma^2_{12} g_{22}$ $-(18)$

If $\alpha = 3, \rho = 1$: $\partial_1 g_{33} = 2\Gamma^3_{13} g_{33}$ $-(19)$

If $\alpha = 3, \rho = 2$: $\partial_2 g_{33} = 2\Gamma^3_{23} g_{33}$ $-(20)$

These are the same results as from the metric compatibility equation, Q.E.D