187(2): Proof of the Antisymmetry of the Connection

The basic equation is:

$$[D_\mu, D_\nu] V^\rho = -\left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) D_\lambda V^\rho + R^\rho{}_{\sigma\mu\nu} V^\sigma \quad -(1)$$

If the connection is symmetric:

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}. \quad -(2)$$

For example:

$$\mu = \nu = 1, \quad -(3)$$

in which case:

$$[D_1, D_1] V^\rho = R^\rho{}_{\sigma 11} V^\sigma = 0. \quad -(4)$$

The assumption (2) means that both the torsion and curvature vanish. This means that the spacetime is flat:

$$[D_\mu, D_\nu] = [\partial_\mu, \partial_\nu] = 0 \quad -(5)$$

i.e.

$$D_\mu V^\sigma = \partial_\mu V^\sigma, \quad -(6)$$

and the symmetric connection is zero, because:

$$D_\mu V^\sigma = \partial_\mu V^\sigma + \Gamma^\sigma_{\mu\lambda} V^\lambda. \quad -(7)$$

From eqs. (6) and (7):

$$\Gamma^\sigma_{\mu\lambda} = 0 \quad \text{if } \mu = \lambda \quad -(8)$$

Q.E.D. The connection is antisymmetric in μ and ν for the same reason as to why $R^\sigma{}_{\sigma\mu\nu}$ is anti-symmetric in μ and ν, transport around a loop.