1)

179(4): Derivation of Energy Equation from Metric

Consider the general metric:

$$ds^2 = c^2 d\tau^2 = n(r) c^2 dt^2 - m(r) dr^2 - r^2 d\phi^2 \quad -(1)$$

then:

$$\left(\frac{ds}{d\tau}\right)^2 = c^2 = c^2 \left(\frac{dt}{d\tau}\right)^2 n(r) - m(r) \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \quad -(2)$$

The Hamiltonian is:

$$H = \frac{1}{2} mc^2 = \frac{1}{2} mc^2 n(r) \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{2} m\, m(r) \left(\frac{dr}{d\tau}\right)^2 - \frac{1}{2} m r^2 \left(\frac{d\phi}{d\tau}\right)^2 \quad -(3)$$

The Euler Lagrange equation of motion is:

$$\frac{\partial}{\partial \tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \quad -(4)$$

where

$$\mathcal{L} = \frac{1}{2} mc^2 \quad -(5)$$

so

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \text{constant} \quad -(6)$$

where:

$$\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau} \quad -(7)$$

i.e.

$$\dot{x}^0 = \frac{dt}{d\tau}, \quad \dot{x}^1 = \frac{dr}{d\tau}, \quad \dot{x}^2 = \frac{d\phi}{d\tau} \quad -(8)$$

Therefore:

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$$\frac{\partial \mathcal{L}}{\partial \dot{x}^0} = n(r) m c^2 \frac{dt}{d\tau} \quad -(9)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^1} = -n(r) m \frac{dr}{d\tau} \quad -(10)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^2} = -m r^2 \frac{d\phi}{d\tau} \quad -(11)$$

These are constants and define the constants of motion:

$$E = n(r) m c^2 \frac{dt}{d\tau} \quad -(12)$$

$$p_e = n(r) m \frac{dr}{d\tau} \quad -(13)$$

$$L = m r^2 \frac{d\phi}{d\tau} \quad -(14)$$

Here E is the total energy, p_e is linear momentum and L is angular momentum. The total linear momentum is:

$$p^2 = p_e^2 + \frac{L^2}{r^2} \quad -(15)$$

With these definitions:

$$H = \frac{1}{2}\left(\frac{E^2}{mc^2} - \frac{p^2}{m}\right) \quad -(16)$$

$$= \frac{1}{2} mc^2$$

i.e.

$$\boxed{E^2 = c^2 p^2 + m^2 c^4} \quad -(17)$$

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So we obtain the important result that the format of the Einstein energy equation is the same in general relativity.

The ECE wave equation is:

$$(\square + R) \phi = 0. \quad -(18)$$

Using the Schroedinger postulate:

$$p^\mu = i \hbar \partial^\mu \quad -(19)$$

the d'Alembertian operator $\square$ becomes:

$$-\hbar^2 \square = -\hbar^2 p^\mu p_\mu = \frac{E^2}{c^2} - p^2 \quad -(20)$$

so the classical counterpart of eq. (18) is:

$$\boxed{E^2 = c^2 p^2 + c^2 \hbar^2 R} \quad -(21)$$

Comparing eqs. (17) and (21), it is seen that

$$R = \left(\frac{mc}{\hbar} \right)^2 \quad -(22)$$

and m in eq. (17) is the covariant mass.

It is defined by:

$$m^2 = \frac{1}{c^4} \left(E^2 - c^2 p^2 \right) \quad -(23)$$

i.e. by:

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$$m = \frac{1}{c^2}\left(E^2 - c^2 p^2\right)^{1/2} \quad -(24)$$

where:

$$E = n(r)\, m c^2 \frac{dt}{d\tau}, \quad -(25)$$

$$p^2 = p_r^2 + \frac{L^2}{r^2}, \quad -(26)$$

$$L = m r^2 \frac{d\phi}{d\tau}, \quad -(27)$$

$$p_r = m(r)\, m \frac{dr}{d\tau}. \quad -(28)$$

The measured mass m_0 is defined by:

$$m_0 = \frac{1}{c^2}\left(E_0^2 - c^2 p_0^2\right)^{1/2} \quad -(29)$$

where:

$$n(r) = m(r) = 1. \quad -(30)$$

So:

$$\boxed{\frac{m}{m_0} = \left(\frac{E^2 - c^2 p^2}{E_0^2 - c^2 p_0^2}\right)^{1/2}} \quad -(31)$$

The format of the fermion equation in general relativity is the same as in special relativity except that the mass m_0 is replaced by the covariant mass m, defined by eq. (31).