177(6) : Force Eigenvalue for Planar Rotation

In this case the quantum force equation is:

$$\left(\hat{H} - E\right)\left(\frac{d\phi}{dx} + \frac{d\phi}{dy}\right) = F\phi \quad -(1)$$

where:

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \quad -(2)$$

In cylindrical coordinates:

$$\frac{\partial f}{\partial x} = \cos\phi \frac{\partial f}{\partial r} - \frac{\sin\phi}{r}\frac{\partial f}{\partial \phi} \quad -(3)$$

$$\frac{\partial f}{\partial y} = \sin\phi \frac{\partial f}{\partial r} + \frac{\cos\phi}{r}\frac{\partial f}{\partial \phi} \quad -(4)$$

and if r is constant:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} \quad -(5)$$

Therefore:

$$\frac{\partial^3 f}{\partial x^3} + \frac{\partial^3 f}{\partial y^3} = \frac{1}{r^3}\left(\cos\phi - \sin\phi\right)\frac{\partial^3 f}{\partial \phi^3} \quad -(6)$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} = \frac{1}{r}\left(\cos\phi - \sin\phi\right)\frac{\partial f}{\partial \phi} \quad -(7)$$

Thus:

$$\hat{H}\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right) = -\frac{\hbar^2}{2mr^3}\left(\cos\phi - \sin\phi\right)\frac{\partial^3\phi}{\partial\phi^3} \quad -(8)$$

2)

and $$-E\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\right)=-\frac{E}{r}\left(\cos\phi-\sin\phi\right)\frac{\partial\phi}{\partial\phi}\quad-(9)$$

In general:

$$\phi=A\exp\left(im_J\phi\right)+B\exp\left(-im_J\phi\right)\quad-(10)$$

$$E=\frac{m_J^2\hbar^2}{2mr^2}\quad-(11)$$

Therefore:

$$\frac{\partial\phi}{\partial\phi}=im_J\left(Ae^{im_J\phi}-Be^{-im_J\phi}\right)\quad-(12)$$

$$\frac{\partial^2\phi}{\partial\phi^2}=-m_J^2\left(Ae^{im_J\phi}+Be^{-im_J\phi}\right)$$

$$=-m_J^2\phi\quad-(13)$$

and $$\frac{\partial^3\phi}{\partial\phi^3}=-im_J^3\left(Ae^{im_J\phi}-Be^{-im_J\phi}\right)\quad-(14)$$

so:

$$\hat{H}\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\right)-E\left(\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\right)=F\phi$$

$$=\frac{i\hbar^2m_J^3}{2mr^3}\left(\cos\phi-\sin\phi\right)\left(Ae^{im_J\phi}-Be^{-im_J\phi}\right)$$

$$-\frac{i\hbar^2m_J^3}{2mr^3}\left(\cos\phi-\sin\phi\right)\left(Ae^{im_J\phi}-Be^{-im_J\phi}\right)\quad-(15)$$

$$=0$$

3) Therefore for all m_J:

$$\boxed{F = 0.} \quad -(16)$$

This means that for angular or rotational motion in a plane there is no net force for all m_J.

This result is consistent with the assumption that r is constant, and checks that eq. (1) is correct.

The next check is rotation on a sphere, where the wavefunctions are the spherical harmonics.