177(3): Calculation of the F_1 Eigenvalue of the Harmonic Oscillator.

The force equation of quantum mechanics is:

$$(\hat{H} - E)\frac{d\phi}{dx} = F\phi \quad -(1)$$

The $n = 1$ eigenfunction of the harmonic oscillator is:

$$\phi_1 = \sqrt{2}\left(\frac{\alpha}{\pi}\right)^{1/4} y \exp\left(-\frac{y^2}{2}\right) \quad -(2)$$

$$= A x \exp\left(-\frac{\alpha x^2}{2}\right) \quad -(3)$$

where

$$A = \left(\frac{4\alpha}{\pi}\right)^{1/4} \alpha^{1/2} \quad -(4)$$

and

$$\alpha = m\omega / \hbar. \quad -(5)$$

The energy for $n = 1$ is:

$$E_1 = \left(n + \frac{1}{2}\right)\hbar\omega = \frac{3}{2}\hbar\omega. \quad -(6)$$

So:

$$\frac{\partial \phi_1}{\partial x} = A \exp\left(-\frac{\alpha x^2}{2}\right) - A x^2 \alpha \exp\left(-\frac{\alpha x^2}{2}\right) \quad -(7)$$

$$\frac{\partial^2 \phi_1}{\partial x^2} = -3\alpha x A \exp\left(-\frac{\alpha x^2}{2}\right) + A x^3 \alpha^2 \exp\left(-\frac{\alpha x^2}{2}\right) \quad -(8)$$

and

$$\frac{\partial^3 \phi_1}{\partial x^3} = \left(6\alpha^2 x - \frac{3\alpha}{x} - x^3 \alpha^3\right)\phi_1 \quad -(9)$$

So:

2)

$$\hat{H}\frac{\partial \phi_1}{\partial x} = -\frac{\hbar^2}{2m}\frac{\partial^3 \phi_1}{\partial x^3} = -\frac{\hbar^2}{2m}\left(6\alpha^2 x - \frac{3\alpha}{x} - x^3\alpha^3\right)\phi_1 \quad -(10)$$

and

$$E\frac{\partial \phi_1}{\partial x} = \frac{3}{2}\hbar\omega\left(\frac{1}{x} - x\alpha\right)\phi_1 \quad -(11)$$

Therefore:

$$F_1 = -\frac{3\hbar^2}{m}\frac{m^2\omega^2 x}{\hbar^2} + \frac{3\hbar^2\alpha}{2mx} + \frac{\hbar^2 x^3 m^3\omega^3}{\hbar^3 2m} - \frac{3\hbar\omega}{2x} + \frac{3\hbar\omega x}{2}\frac{m\omega}{\hbar}$$

$$\boxed{F_1 = -\frac{m\omega^2 x}{2}\left(3 - \frac{m\omega x^2}{\hbar}\right)} \quad -(12)$$

From note 172(2):

$$F_0 = -m\omega^2 x = -kx \quad -(13)$$

so the results so far are:

$$\boxed{\begin{aligned} F_0 &= -kx \\ F_1 &= -\frac{kx}{2}\left(3 - \frac{kx^2}{\hbar\omega}\right)\end{aligned}} \quad -(14)$$

where:

$$k = m\omega^2$$

is the constant of Hooke's law.

This is the first time that the force eigenvalues of the harmonic oscillator have been calculated, and the first time that force eigenvalues have been calculated in quantum mechanics.