1)

176(4): Applications of the Quantum Hamilton Equations, Part 2.

In the new Hamiltonian formulation of quantum dynamics the two fundamental equations are:

$$i\hbar \frac{d\langle \hat{A} \rangle}{dx} = \langle [\hat{A}, \hat{p}] \rangle \quad -(1)$$

and

$$i\hbar \frac{d\langle \hat{A} \rangle}{dp} = -\langle [\hat{A}, \hat{x}] \rangle \quad -(2)$$

for any operator $\hat{A}$. In the older Newtonian formulation of quantum mechanics there was one equation of motion:

$$i\hbar \frac{d\langle \hat{A} \rangle}{dt} = \langle [\hat{A}, \hat{H}] \rangle . \quad -(3)$$

The operator format of eq. (3) was obtained by assuming:

$$\left\langle \frac{d\hat{A}}{dt} \right\rangle = \frac{d\langle \hat{A} \rangle}{dt} \quad -(4)$$

for each state. So

$$i\hbar \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}] \quad -(5)$$

and

$$i\hbar \frac{d\hat{A}}{dt} \psi = [\hat{A}, \hat{H}] \psi . \quad -(6)$$

Similarly, if it is assumed that:

$$\left\langle \frac{d\hat{A}}{dx} \right\rangle = \frac{d\langle \hat{A} \rangle}{dx}, \quad \left\langle \frac{d\hat{A}}{dp} \right\rangle = \frac{d\langle \hat{A} \rangle}{dp} \quad -(7)$$

then:

$$\boxed{i\hbar \frac{d\hat{A}}{dx} \psi = [\hat{A}, \hat{p}] \psi} \quad -(8)$$

and:

$$\boxed{i\hbar \frac{d\hat{A}}{dp}\phi = -[\hat{A}, \hat{x}]\phi} \quad -(9)$$

These are the operator equations of Hamiltonian quantum dynamics. The Cartesian representation is used for eqs. (8) and (9), but they can be generalized to the canonical representation:

$$i\hbar \frac{d\hat{A}}{dq}\phi = [\hat{A}, \hat{p}]\phi \quad -(10)$$

$$i\hbar \frac{d\hat{A}}{dp}\phi = -[\hat{A}, \hat{q}]\phi \quad -(11)$$

From eq. (3):

$$i\hbar \frac{d\hat{p}}{dt} = [\hat{p}, \hat{H}], \quad i\hbar \frac{d\hat{q}}{dt} = [\hat{q}, \hat{H}] \quad -(12)$$

so, for

$$\hat{A} = \hat{H} \quad -(13)$$

eqs (10) to (13) give:

$$\frac{d\hat{H}}{dq}\phi = -\frac{d\hat{p}}{dt}\phi \quad -(14)$$

and

$$\frac{d\hat{H}}{dp}\phi = \frac{d\hat{q}}{dt}\phi. \quad -(15)$$

In the classical limit these are the Hamilton equations of motion:

3)

$$\frac{dH}{dq} = -\frac{dp}{dt}, \quad \frac{dH}{dp} = \frac{dq}{dt} \quad -(16)$$

using:

$$\frac{dH}{dq} = \left\langle \frac{d\hat{H}}{dq} \right\rangle ; \quad \frac{dH}{dp} = \left\langle \frac{d\hat{H}}{dp} \right\rangle \quad -(17)$$

The canonical equations (10) and (11) are very powerful with wide range of applicability, because p and q are <u>generalized</u> momentum and position, and <u>$\hat{A}$ is any operator of quantum mechanics</u>

In the older quantum dynamics, eq. (12), <u>only the $\hat{H}$ operator can be used</u>. <u>The reduction of eq. (12) to Newtonian dynamics must involve an extra assumption</u>:

$$\hbar \rightarrow 0 \quad -(18)$$

i.e.

$$\operatorname{Lim} \frac{\langle [\hat{x}, \hat{H}] \rangle}{i\hbar} \xrightarrow[\hbar \to 0]{} \frac{\partial H}{\partial p} \quad -(19)$$

$$\operatorname{Lim} \frac{\langle [\hat{p}, \hat{H}] \rangle}{i\hbar} \xrightarrow[\hbar \to 0]{} -\frac{\partial H}{\partial x} \quad -(20)$$

and these are <u>the correspondence principles</u>.

<u>The new equations (10) and (11) give the Hamilton equations without assuming $\hbar \rightarrow 0$. This is a major advance because $\hbar$ is a universal constant</u>.

Consider the harmonic oscillator as an example:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 q^2 \quad -(21)$$

From eq. (10), with $\hat{A} = \hat{H}$:

$$i\hbar \frac{d\hat{H}}{dx}\phi = [\hat{H}, \hat{p}]\phi = i\hbar m\omega^2 x \phi \quad -(22)$$

so

$$\frac{dp}{dt} = -\frac{dH}{dx} = -m\omega^2 x \quad -(23)$$

or

$$\frac{d^2 x}{dt^2} + \omega^2 x = 0 \quad -(24)$$

which is the equation of motion Q.E.D.

Check

$$[\hat{H}, \hat{p}]\phi = \hat{H}(\hat{p}\phi) - \hat{p}(\hat{H}\phi) \quad -(25)$$

$$= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2\right)\left(-i\hbar\frac{\partial \phi}{\partial x}\right) + i\hbar\frac{\partial}{\partial x}\left(\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2} m\omega^2 x^2\right)\phi\right)$$

$$= -\frac{i\hbar}{2m}\left(\frac{1}{2} m\omega^2 x^2\right)\frac{\partial \phi}{\partial x} + \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left(\frac{1}{2} m\omega^2 x^2 \phi\right)$$

$$= i\hbar m\omega^2 x \phi \qquad \text{Q.E.D.}$$

Eqs. (10) and (11) can be applied to rotational motion, angular momentum, relativistic quantum mechanics, quantum field theory and quantum optics, and generalize the Hamilton equations of classical mechanics.