1) 176(5): The Higher Order Quantum Hamilton Equations and Reduction to Poisson Brackets.

Consider the quantum Hamilton equation in operator format:

$$i\hbar \frac{d\hat{A}}{dx} = [\hat{A}, \hat{p}] \quad -(1)$$

$$i\hbar \frac{d\hat{A}}{dp} = -[\hat{A}, \hat{x}] \quad -(2)$$

in Cartesian coordinates in one dimension. Here $\hat{A}$ is any valid, hermitian operator of quantum mechanics. The higher order quantum Hamilton equations are built up with:

$$[\hat{A}, \hat{p}\hat{p}] = [\hat{A}, \hat{p}]\hat{p} + \hat{p}[\hat{A}, \hat{p}] \quad -(3)$$

and higher order commutator algebra. For example:

$$[\hat{A}, \hat{p}\hat{p}]\psi = i\hbar\left(\frac{d\hat{A}}{dx}\hat{p} + \hat{p}\frac{d\hat{A}}{dx}\right)\psi \quad -(4)$$

If:

$$\hat{A} = \hat{x}\hat{x} \quad -(5)$$

then:

$$[\hat{x}\hat{x}, \hat{p}\hat{p}]\psi = i\hbar\left(\frac{d\hat{x}\hat{x}}{dx}\hat{p} + \hat{p}\frac{d\hat{x}\hat{x}}{dx}\right)\psi \quad -(6)$$

where

$$\hat{x}\psi = x\psi \quad -(7)$$

so

$$\hat{x}\hat{x}\psi = x^2\psi \quad -(8)$$

and

$$\left(\frac{d\hat{x}\hat{x}}{dx}\right)\psi = 2x\psi \quad -(9)$$

Therefore

2)

$$[\hat{x}\hat{x}, \hat{p}\hat{p}]\phi = 2i\hbar(\hat{x}\hat{p} + \hat{p}\hat{x})\phi \quad -(10)$$

$$= 2i\hbar\{\hat{x}, \hat{p}\}\phi \quad -(11)$$

Q.E.D as in previous work.

In the position representation:

$$\hat{p}\phi = -i\hbar\frac{\partial\phi}{\partial x} \quad -(12)$$

and $$\{\hat{x}, \hat{p}\}\phi = \hat{x}(\hat{p}\phi) + \hat{p}(\hat{x}\phi) \quad -(13)$$

$$= -i\hbar\left(x\frac{\partial\phi}{\partial x} + \frac{\partial}{\partial x}(x\phi)\right)$$

$$= -i\hbar\left(\phi + 2x\frac{\partial\phi}{\partial x}\right)$$

$$\boxed{\{\hat{x}, \hat{p}\}\phi = -i\hbar\phi + 2x\hat{p}\phi} \quad -(14)$$

Therefore:

$$\boxed{[\hat{x}\hat{x}, \hat{p}\hat{p}]\phi = 2\hbar^2\phi + 4ix\hat{p}\phi} \quad -(15)$$

Q.E.D as in previous work.

If $\hat{G}$ and $\hat{F}$ are two operators of quantum mechanics then <u>the</u> <u>Poisson</u> <u>bracket</u> of G and F is defined by:

3)

$$\frac{\langle [\hat{G}, \hat{F}] \rangle}{i\hbar} \xrightarrow[\hbar \to 0]{} (G, F) \quad -(16)$$

where

$$(G, F) = \frac{\partial G}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p} \quad -(17)$$

Eq. (16) was discovered by Dirac for his Ph.D. thesis. Using this equation the new quantum Hamilton equations become:

$$\boxed{\begin{aligned} \frac{d}{dx}\langle \hat{A} \rangle &\xrightarrow[\hbar \to 0]{} (A, p) \quad -(18) \\ \frac{d}{dp}\langle \hat{A} \rangle &\xrightarrow[\hbar \to 0]{} -(A, x) \quad -(19) \end{aligned}}$$

Eqs. (18) and (19) generalize the Hamilton equations because

$$\frac{dA}{dx} = \frac{\partial A}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial p}{\partial x} = \frac{\partial A}{\partial x} \quad -(20)$$

$$\frac{dA}{dp} = -\frac{\partial A}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial A}{\partial p}\frac{\partial x}{\partial x} = \frac{\partial A}{\partial p} \quad -(21)$$

and we obtain the classical equations

$$\boxed{\frac{dA}{dx} = (A, p), \quad \frac{dA}{dp} = -(A, x)} \quad -(22)$$

from the quantum Hamilton equations (1) and (2).

4) Finally use the equation:

$$\frac{dH}{dx} = (H, p) \;,\; \frac{dH}{dp} = -(H, x) \quad -(23)$$

to find the Hamilton equations:

$$\frac{dH}{dp} = \frac{dx}{dt} \;,\; \frac{dH}{dx} = -\frac{dp}{dt} \quad -(24)$$

Quantum	Classical.
$i\hbar \frac{d\hat{A}}{dx} \phi = [\hat{A}, \hat{p}] \phi$	$\frac{dA}{dx} = (A, p)$
$i\hbar \frac{d\hat{A}}{dp} \phi = -[\hat{A}, \hat{x}] \phi$	$\frac{dA}{dp} = -(A, x)$
$i\hbar \frac{d\hat{H}}{dx} \phi = [\hat{H}, \hat{p}] \phi$	$\frac{dH}{dx} = (H, p)$
$i\hbar \frac{d\hat{H}}{dp} \phi = -[\hat{H}, \hat{x}] \phi$	$\frac{dH}{dp} = -(H, x)$

These equations of the Hamiltonian formulation of quantum dynamics are novel equations. The older Newtonian formulation is:

Quantum	Classical
$i\hbar \frac{d\hat{A}}{dt} \phi = [\hat{A}, \hat{H}] \phi$	$\frac{dA}{dt} = (A, H)$

5) Discussion

As in previous work the commutator (15) is zero for the harmonic oscillator and non-zero for the H wavefunctions. So eq. (15) cannot be interpreted by the Copenhagen philosophy.

We have:

$$\frac{\langle[\hat{x}\hat{x},\hat{p}\hat{p}]\rangle}{i\hbar} \xrightarrow[\hbar\to 0]{} (x^2, p^2) = 4xp \quad -(25)$$

so this is an example of a commutator with a non-zero classical limit. The usual commutator is:

$$\frac{\langle[\hat{x},\hat{p}]\rangle}{i\hbar} \xrightarrow[\hbar\to 0]{} (x,p) = 1 \quad -(26).$$

which also has a non-zero classical limit of 1 if the Poisson bracket limit is used. In both cases the general rule is:

$$\frac{\langle\cdot[\hat{A},\hat{B}]\rangle}{i\hbar} \xrightarrow[\hbar\to 0]{} (A,B) \quad -(27)$$

This is a very clear refutation of the Copenhagen philosophy because the latter is wholly concentrated on one special case of the general result (27), the special case being:

$$(A,B) = 1 \quad -(28)$$

$$\delta A\,\delta B = \hbar/2 = \text{constant}.$$

i.e

In general, (A,B) is not constant.