176(3): The Commutator $[\hat{x}\hat{x}, \hat{p}\hat{p}]\phi$ — (1)

This is a special case of:

$$[\hat{A}\hat{B}, \hat{C}\hat{D}]\phi = \left([\hat{A}\hat{B}, \hat{C}]\hat{D} + \hat{C}[\hat{A}\hat{B}, \hat{D}]\right)\phi$$

where:

$$[\hat{A}, \hat{B}\hat{C}]\phi = \left([\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]\right)\phi. \quad -(2)$$

So for eq. (2) in (1):

$$[\hat{C}\hat{D}, \hat{A}\hat{B}]\phi = \left([\hat{C}, \hat{A}]\hat{B}\hat{D} + \hat{A}[\hat{C}, \hat{B}]\hat{D} + \hat{C}[\hat{D}, \hat{A}]\hat{B} + \hat{C}\hat{A}[\hat{D}, \hat{B}]\right)\phi. \quad -(3)$$

This is true for all representations as is the case for all commutator equations.

Thus:

$$[\hat{x}\hat{x}, \hat{p}\hat{p}]\phi = \left([\hat{x}, \hat{p}]\hat{p}\hat{x} + \hat{p}[\hat{x}, \hat{p}]\hat{x} + \hat{x}[\hat{x}, \hat{p}]\hat{p} + \hat{x}\hat{p}[\hat{x}, \hat{p}]\right)\phi \quad -(4)$$

Now use: $[\hat{x}, \hat{p}]\phi = i\hbar\phi$, — (5)

so

$$[\hat{x}\hat{x}, \hat{p}\hat{p}]\phi = \left(i\hbar\hat{p}\hat{x} + i\hbar\hat{p}\hat{x} + i\hbar\hat{x}\hat{p} + i\hbar\hat{x}\hat{p}\right)\phi$$

$$= 2i\hbar\left(\hat{p}\hat{x} + \hat{x}\hat{p}\right)\phi \quad -(6)$$

$$= 2i\hbar\,[\hat{p}, \hat{x}]\phi$$

as in UFT 175, Q.E.D.

; Therefore:

$$\boxed{[\hat{x}\hat{x}, \hat{p}\hat{p}]\phi = 2i\hbar[\hat{p}, \hat{x}]\phi} \quad -(7)$$

in which:

$$[\hat{p}, \hat{x}]\phi = \hat{p}\hat{x}\phi - x\hat{p}\phi. \quad -(8)$$

In the position representation:

$$[\hat{p}, \hat{x}]\phi = -i\hbar\left(\frac{\partial}{\partial x}(x\phi) - x\frac{\partial\phi}{\partial x}\right)$$

$$= -i\hbar\phi + 2i\hbar x\frac{\partial\phi}{\partial x} \quad -(9)$$

$$= -i\hbar\phi - 2x\hat{p}\phi$$

So $$\boxed{[\hat{x}\hat{x}, \hat{p}\hat{p}]\phi = (2\hbar^2 + 4i\hbar x\hat{p})\phi} \quad -(10)$$

This result is given in E. Merzbacher, "Quantum [me]chanics" (Wiley, 1970), p. 340.

From eq. (4) it is seen that $[\hat{x}\hat{x}, \hat{p}\hat{p}]\phi$ [is] made up of terms containing $[\hat{x}, \hat{p}]\phi$. In [UF]T 175 it was shown that $[\hat{x}\hat{x}, \hat{p}\hat{p}]\phi$ is [some]times zero and sometimes non-zero. It is [ze]ro for harmonic oscillator wavefunction and non-

3) zero for H wavefunctions. This is despite the fact that

$$[\hat{x}, \hat{p}]\,\phi = i\hbar\phi \quad -(11)$$

is always true simply because:

$$\hat{p}\phi = -i\hbar\frac{\partial\phi}{\partial x}. \quad -(12)$$

Therefore despite the fact that

$$\delta x\,\delta p \geq \frac{\hbar}{2}, \quad -(13)$$

the so called Heisenberg uncertainty principle, we have:

$$[\hat{x}\hat{x}, \hat{p}\hat{p}]\,\phi = 0 \quad (\text{harmonic oscillator}) \quad -(14)$$

$$[\hat{x}\hat{x}, \hat{p}\hat{p}]\,\phi \neq 0 \quad (\text{H atom}) \quad -(15)$$

from the Schroedinger equation. According to Copenhagen, eq. (14) means that $\hat{x}\hat{x}$ and $\hat{p}\hat{p}$ are both known precisely, while eq. (15) means that either $\hat{x}\hat{x}$ or $\hat{p}\hat{p}$ is "unknowable". The Copenhagen interpretation ~~can be~~ is untenable.

In the classical limit:

$$\frac{1}{i\hbar}[\hat{x}\hat{x} - \hat{p}\hat{p}] \xrightarrow[\hbar \to 0]{} 4\hat{x}\hat{p} \quad -(16)$$

and the expectation value of $4\hat{x}\hat{p}$ is the Poisson bracket:

$$\langle 4\hat{x}\hat{p}\rangle = (x^2, p^2)$$

$$= \frac{\partial x^2}{\partial x}\frac{\partial p^2}{\partial p} - \frac{\partial p^2}{\partial x}\frac{\partial x^2}{\partial p} = 4xp$$