1) 176(1): Quantum Hamilton Equations for Rotation in a Plane.

First Equation

In this case:

$$\hat{\phi}\psi = \phi\psi \quad -(1)$$

$$\hat{p}_\phi \psi = -\frac{i\hbar}{r}\frac{\partial\psi}{\partial\phi} \quad -(2)$$

and

$$[\hat{\phi}, \hat{p}_\phi]\psi = -\frac{i\hbar}{r}\psi \quad -(3)$$

where $\hat{p}_\phi$ is the angular momentum operator. Therefore:

$$\boxed{\frac{1}{r}\frac{d}{d\phi}\langle\hat{A}\rangle = -\frac{i}{\hbar}\langle[\hat{A}, \hat{p}_\phi]\rangle} \quad -(4)$$

When:

$$\hat{A} = \hat{H} \quad -(5)$$

then:

$$\frac{1}{r}\frac{d}{d\phi}\langle\hat{H}\rangle = -\frac{i}{\hbar}\langle[\hat{H}, \hat{p}_\phi]\rangle$$

$$= -\frac{d}{dt}\langle\hat{p}_\phi\rangle \quad -(6)$$

and the Hamilton equation is:

$$\frac{1}{r}\frac{dH}{d\phi} = -\frac{dp_\phi}{dt} \quad -(7)$$

The Hamiltonian is:

$$H = \frac{1}{2}mr^2\left(\frac{d\phi}{dt}\right)^2 \quad -(8)$$

and the angular momentum is:

$$p_\phi = mr^2\frac{d\phi}{dt} \quad -(9)$$

2) If the angular momentum commutes with the hamiltonian

then:
$$\langle [\hat{H}, \hat{p}_\phi] \rangle = 0 \quad - (10)$$

and
$$\frac{dp_\phi}{dt} = 0 \quad - (11)$$

together with:
$$\frac{1}{r}\frac{dH}{d\phi} = 0. \quad - (12)$$

<u>Check</u>
$$\frac{dH}{dp_\phi} = \frac{dH}{d\dot{\phi}}\frac{d\dot{\phi}}{dp_\phi} = \frac{mr^2}{mr^2}\frac{d\phi}{dt} = \frac{d\phi}{dt} \quad - (13)$$

Eq. (14) is obtained from the tautology:
$$\frac{d\langle\hat{\phi}\rangle}{d\phi} = \frac{1}{r} = -\frac{i}{\hbar}\langle [r\hat{\phi}, \hat{p}_\phi] \rangle \quad - (14)$$

<u>Second Equation</u>

This is derived from:
$$\frac{d\langle\hat{p}_\phi\rangle}{dp_\phi} = 1 = -\frac{i}{\hbar}\langle r\hat{\phi}, \hat{p}_\phi\rangle \quad - (15)$$

so
$$\frac{d\langle\hat{H}\rangle}{dp_\phi} = -\frac{i}{\hbar}\langle\hat{\phi}, \hat{H}\rangle \quad - (16)$$

$$= \frac{d\langle\hat{\phi}\rangle}{dt}$$

i.e
$$\frac{dH}{dp_\phi} = \frac{d\phi}{dt} \quad - (17)$$