1) 175(14): Tautological Nature of the Heisenberg Uncertainty Principle

Consider the equation:

$$[\hat{p}, x]\phi = -i\hbar\phi \quad -(1)$$

then:

$$\langle[\hat{p}, x]\rangle = \int \phi^* [\hat{p}, x]\phi \, d\tau \quad -(2)$$

$$= -i\hbar$$

Therefore: $$\int \phi^* (\hat{p}x - x\hat{p})\phi \, d\tau = -i\hbar. \quad -(3)$$

The observables in quantum mechanics correspond to real eigen-values and Hermitian operators with the property:

$$\int \phi_m^* \Omega \phi_n \, d\tau = \int (\Omega \phi_m)^* \phi_n \, d\tau. \quad -(4)$$

So eq. (3) is:

$$\int \left((\hat{p}\phi)^* x \phi - \phi^* x \hat{p} \phi \right) d\tau = -i\hbar \quad -(5)$$

with:

$$\hat{p}\phi = -i\hbar \frac{\partial \phi}{\partial x}, \quad (\hat{p}\phi)^* = i\hbar \frac{\partial \phi^*}{\partial x}. \quad -(6)$$

Therefore:

$$i\hbar \int \left(\frac{\partial \phi^*}{\partial x} x \phi + \phi^* x \frac{\partial \phi}{\partial x} \right) d\tau = -i\hbar. \quad -(7)$$

Now make a comparative analysis as follows with the well known derivation of the equation of motion in quantum mechanics.

$$\frac{d}{dt}\langle\hat{\Omega}\rangle = \frac{d}{dt}\int \psi^*(t)\,\hat{\Omega}\,\psi(t)\,d\tau$$

$$= \int \frac{\partial\psi^*}{\partial t}\hat{\Omega}\psi\,d\tau + \int \psi^*\hat{\Omega}\frac{\partial\psi}{\partial t}\,d\tau$$

$$= \frac{1}{i\hbar}\left(-\int(\hat{H}\psi)^*\hat{\Omega}\psi\,d\tau + \int\psi^*\hat{\Omega}\hat{H}\psi\,d\tau\right)$$

$$= \frac{1}{i\hbar}\left(-\int\psi^*\hat{H}\hat{\Omega}\psi\,d\tau + \int\psi^*\hat{\Omega}\hat{H}\psi\,d\tau\right)$$

$$= \frac{i}{\hbar}\int\psi^*\left(\hat{H}\hat{\Omega} - \hat{\Omega}\hat{H}\right)\psi\,d\tau$$

$$= \frac{i}{\hbar}\langle[\hat{H},\hat{\Omega}]\rangle$$

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t},\quad \hat{H}\psi^* = -i\hbar\frac{\partial\psi^*}{\partial t}$$

$$\frac{d}{dx}\langle\hat{x}\rangle = \frac{d}{dt}\int\psi^*\hat{x}\psi\,d\tau$$

$$= \int\frac{\partial\psi^*}{\partial x}\hat{x}\psi\,d\tau + \int\psi^*\hat{x}\frac{\partial\psi}{\partial x}\,d\tau$$

$$= -\frac{1}{i\hbar}\left(\int(\hat{p}\psi)^*\hat{x}\psi\,d\tau - \int\psi^*\hat{x}\hat{p}\psi\,d\tau\right)$$

$$= -\frac{1}{i\hbar}\left(\int\left(\psi^*\hat{p}\hat{x}\psi - \psi^*\hat{x}\hat{p}\psi\right)d\tau\right)$$

$$= -\frac{1}{i\hbar}\int\psi^*\left(\hat{p}\hat{x} - \hat{x}\hat{p}\right)\psi\,d\tau$$

$$= \frac{i}{\hbar}\langle[\hat{p},\hat{x}]\rangle$$

$$\hat{p}\psi = -i\hbar\frac{\partial\psi}{\partial x},\quad \hat{p}\psi^* = i\hbar\frac{\partial\psi^*}{\partial x}$$

Therefore: $$\frac{d\langle\hat{x}\rangle}{dx} = \frac{i}{\hbar}\langle[\hat{p},\hat{x}]\rangle. \quad -(8)$$

In general: $$\boxed{\frac{d\langle\hat{A}\rangle}{dx} = \frac{i}{\hbar}\langle[\hat{p},\hat{A}]\rangle} \quad -(9)$$

here $\hat{A}$ is any operator. Also:

$$\boxed{\frac{d\langle\hat{A}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle} \quad -(10)$$

Eq. (10) is the well known quantum mechanical equivalent Newton's first and second laws, while eq. (9) is new equation of motion of quantum mechanics

3)

It can be seen that when $\hat{A} = \hat{x}$:

$$\boxed{\frac{d\langle \hat{x} \rangle}{dx} = 1} \quad -(11)$$

so the Schroedinger postulate:

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \quad -(12)$$

is equivalent to the tautology (11). The Heisenberg uncertainty principle is obtained from:

$$[\hat{p}, \hat{x}]\phi = -i\hbar\phi \quad -(13)$$

and is

$$\delta p \, \delta x \geq \frac{\hbar}{2} = \frac{\langle \hbar \rangle}{2} . \quad -(14)$$

so the so-called "principle" is a tautology, eqn. (11), in which:

$$\langle \hat{x} \rangle = x . \quad -(15)$$

When $\hat{A} = \hat{p}$ then:

$$\boxed{\frac{d\langle \hat{p} \rangle}{dx} = 0} \quad -(16)$$

because:

$$[\hat{p}, \hat{p}]\phi = 0 \quad -(17)$$

When $\hat{A} = x^2$, then:

$$[\hat{p}, x^2] = -i\hbar \frac{dx^2}{dx} = -2i\hbar x \quad -(18)$$

which is Atkins' eq. on page 93 of "Molecular Quantum Mechanics" (OUP, 1983, 2nd. ed.). This result checks that the method is correct.

4) In the interacting case:

$$\hat{A} = \hat{H} \quad -(19)$$

where $\hat{H}$ is the Hamiltonian operator, then:

$$\frac{d}{dx}\langle \hat{H} \rangle = \frac{i}{\hbar}\langle [\hat{p}, \hat{H}] \rangle \quad -(20)$$

and from eq. (10):

$$\frac{d}{dt}\langle \hat{p} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{p}] \rangle \quad -(21)$$

Therefore:

$$\boxed{\frac{d}{dx}\langle \hat{H} \rangle = -\frac{d}{dt}\langle \hat{p} \rangle} \quad -(22)$$

which is one of Hamilton's equations of motion:

$$\dot{q} = \frac{\partial H}{\partial p} \; ; \; \dot{p} = -\frac{\partial H}{\partial q} \quad -(23)$$

Conclusion

The tautology: $\frac{d}{dx}\langle \hat{x} \rangle = 1 \quad -(24)$

implies that: $[\hat{p}, \hat{x}]\phi = -i\hbar\phi \quad -(25)$

and that $\hat{p}\phi = -i\frac{\partial \phi}{\partial x} \quad -(26)$

which is Schroedinger's postulate.