1)

175(4): Counter Example to the Heisenberg Principle

It is well known that:

$$[\hat{x}, \hat{p}]\phi = i\hbar\phi \quad -(1)$$

where

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \quad -(2)$$

$$\hat{x} = x. \quad -(3)$$

Eq. (1) follows from:

$$\left[x, -i\hbar\frac{\partial}{\partial x}\right]\phi = -i\hbar x\frac{\partial\phi}{\partial x} + i\hbar\frac{\partial}{\partial x}(x\phi)$$

$$= -i\hbar x\frac{\partial\phi}{\partial x} + i\hbar x\frac{\partial\phi}{\partial x} + i\hbar\phi\frac{\partial x}{\partial x}$$

$$= i\hbar\phi \quad -(4)$$

Now consider:

$$[\hat{x}^2, \hat{p}^2]\phi = \left[x^2, -\hbar^2\frac{\partial^2}{\partial x^2}\right]\phi$$

$$= -\hbar^2 x^2\frac{\partial^2\phi}{\partial x^2} + \hbar^2\left(2\phi + 4x\frac{\partial\phi}{\partial x} + x^2\frac{\partial^2\phi}{\partial x^2}\right)$$

$$= 2\hbar^2\phi + 4x\hbar^2\frac{\partial\phi}{\partial x} \quad -(5)$$

where from eq. (2):

$$\frac{\partial\phi}{\partial x} = i\frac{\hat{p}}{\hbar}\phi \quad -(6)$$

2)

So:

$$\boxed{[\hat{x}^2, \hat{p}^2]\phi = 2\hbar^2\phi + 4i\hbar\hat{x}\hat{p}\phi} \quad -(7)$$

The expectation value of eqn. (1) is:

$$\boxed{\langle[\hat{x}, \hat{p}]\rangle = i\hbar} \quad -(8)$$

Similarly:

$$\boxed{\langle[\hat{x}^2, \hat{p}^2]\rangle = 2\hbar^2 + 4i\hbar\langle\hat{x}\hat{p}\rangle \quad -(9)}$$

In the usual Copenhagen interpretation, if the commutator is zero then $\hat{x}$ and $\hat{p}$ cannot be simultaneously measurable. If the commutator is zero then the expectation value of the commutator is zero. From eq. (9), the commutator on the left hand side is in general non-zero. If it is zero then $\hat{x}^2$ and $\hat{p}^2$ are simultaneously measurable. However, this violates the basic idea of the Heisenberg uncertainty principle, which is

$$[A, B] = iC \quad -(10)$$

3)

The Harmonic Oscillator

The zeroith wave function is:

$$\phi_0 = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp\left(-\frac{\alpha}{2}x^2\right) \quad -(11)$$

Therefore,

$$\langle \hat{x}\hat{p} \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha x^2/2}\, x \hat{p}\, e^{-\alpha x^2/2}\, dx \quad -(12)$$

$$= -i\hbar \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha x^2/2}\, x \frac{\partial}{\partial x} e^{-\alpha x^2/2}\, dx$$

$$= i\hbar \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} \alpha x^2 e^{-\alpha x^2}\, dx \quad -(13)$$

Now use:

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}\, dx = \frac{1}{2}\left(\frac{\pi}{\alpha^3}\right)^{1/2} \quad -(14)$$

So

$$\langle \hat{x}\hat{p} \rangle = \frac{i\hbar}{2} \quad -(15)$$

Similarly:

$$\langle \hat{p}\hat{x} \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha x^2/2}\, \hat{p} \left(x e^{-\alpha x^2/2}\right) dx \quad -(16)$$

where:

$$\hat{p}\left(x e^{-\alpha x^2/2}\right) = -i\hbar \frac{\partial}{\partial x}\left(x e^{-\alpha x^2/2}\right)$$

$$= -i\hbar \left(1 - \alpha x^2\right) e^{-\alpha x^2/2} \quad -(17)$$

So:

$$\langle \hat{p} x \rangle = -i\hbar \left(\frac{\alpha}{\pi}\right)^{1/2} \left(\int_{-\infty}^{\infty} e^{-\alpha x^2} dx - \int_{-\infty}^{\infty} \alpha x^2 e^{-\alpha x^2} dx \right) \quad -(18)$$

Use:

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \left(\frac{\pi}{\alpha}\right)^{1/2} \quad -(19)$$

So:

$$\langle \hat{p} x \rangle = -\frac{i\hbar}{2} \quad -(20)$$

and

$$\langle [\hat{x}, \hat{p}] \rangle = i\hbar \quad -(21)$$

Q.E.D.

From eq. (15) & eq. (9):

$$\boxed{\langle [\hat{x}^2, \hat{p}^2] \rangle = 0} \quad -(22)$$

This means that $\hat{x}^2$ and $\hat{p}^2$ are measurable simultaneously, a counter example to Copenhagen.

We also have the results:

$$[\hat{p}^2, \hat{x}]\phi = -2\hbar i \hat{p} \phi \quad -(23)$$

and

$$[\hat{p}, \hat{x}^2]\phi = -2i\hbar x \phi \quad -(24)$$

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So:

$$\langle[\hat{p}^2, \hat{x}]\rangle = -2\hbar i\langle\hat{p}\rangle \quad -(25)$$

$$\langle[\hat{p}, \hat{x}^2]\rangle = -2i\hbar\langle\hat{x}\rangle \quad -(26)$$

For the zeroth wave function of the harmonic oscillator:

$$\langle\hat{p}\rangle = 0 \quad -(27)$$

$$\langle\hat{x}\rangle = 0. \quad -(28)$$

So

$$\langle[\hat{p}^2, \hat{x}]\rangle = 0 \quad -(29)$$

$$\langle[\hat{p}, \hat{x}^2]\rangle = 0. \quad -(30)$$

These are two further counter examples.

For the particle in a box wave function:

$$\psi = \left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{n\pi x}{L}\right) \quad -(31)$$

so

$$\langle\hat{x}\rangle = \frac{2}{L}\int_0^L x\sin^2\left(\frac{n\pi x}{L}\right)dx \quad -(32)$$

$$= \frac{2}{L}\left(\frac{x^2}{4} - \frac{x}{4a}\sin 2ax - \frac{1}{8a^2}\cos 2ax\right|_0^L$$

where:

$$a = \frac{n\pi}{L} \quad -(33)$$

So:

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$$\langle \hat{x} \rangle = \frac{2}{L}\left(\frac{L^2}{4} - \frac{L}{4a}\sin(2aL) - \frac{1}{8a^2}\cos(2aL) + \frac{1}{8a^2}\right)$$

$$= \frac{L}{2} - \frac{L}{n\pi}\sin(2n\pi) + \frac{1}{4a^2L}\left(1-\cos(2n\pi)\right)$$

$$= \frac{L}{2} \quad -(34)$$

So:

$$\langle [\hat{p}, \hat{x}^2] \rangle = -i\hbar L \quad -(35)$$

Comparing eqs. (30) and (35) it is seen that different exact solutions of the Schroedinger equation give different results. For the zero point harmonic oscillator $\hat{p}$ and $\hat{x}^2$ are simultaneously measurable but for the particle in a box they are not. This makes no sense in the Copenhagen interpretation because the same operators $\hat{p}$ and $\hat{x}^2$ are being used.

<u>Conclusion</u>

The so called Heisenberg uncertainty principle is merely a result of $\hat{p} = -i\hbar\,\partial/\partial x$, and has no further significance.

If we use the relation:

7) $$\hat{H} = \hat{p}^2 / 2m \quad -(36)$$

it is seen that eq. (9) means:

$$\langle [\hat{x}^2, \hat{H}] \rangle = 2m \left(2\hbar^2 + 4i\hbar \langle \hat{x}\hat{p} \rangle \right) \quad -(37)$$

and eq. (25) means:

$$\langle [\hat{H}, \hat{x}] \rangle = -4m\hbar i \langle \hat{p} \rangle. \quad -(38)$$

It is known that:

$$\frac{d}{dt} \langle \hat{\Omega} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{\Omega}] \rangle \quad -(39)$$

Therefore for the harmonic oscillator:

$$\langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle = \frac{\hbar^2}{4} \quad -(40)$$

is simply a relation between constants of motion:

$$\frac{d}{dt} \langle \hat{x}^2 \rangle = 0, \quad \frac{d}{dt} \langle \hat{p}^2 \rangle = 0, \quad -(41)$$

$$\langle [\hat{x}^2, \hat{H}] \rangle = 0, \quad \langle [\hat{p}^2, \hat{H}] \rangle = 0 \quad -(42)$$

According to Heisenberg, if $[\hat{A}, \hat{B}] = \hat{C}$ then

$$\delta\hat{A}\, \delta\hat{B} \geq \frac{1}{2} | \langle \hat{C} \rangle | \quad -(43)$$

for all operators. This is not true if one of the operators is $\hat{H}$.