1)

175(2) : Fourier Transform of a Gaussian

If for example we consider a Gaussian function:

$$A(k) = \left(\frac{2\alpha}{\pi}\right)^{1/4} \exp\left(-\alpha(k-k_0)^2\right) \quad -(1)$$

its Fourier transform is:

$$f(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{2\alpha}{\pi}\right)^{1/4} \int e^{-\alpha(k-k_0)^2} e^{ikx}\, dk \quad -(2)$$

Define:

$$k'' = k - k_0 - \frac{ix}{2\alpha} \quad -(3)$$

then:

$$f(x) = \left(\frac{\alpha}{2\pi^3}\right)^{1/4} e^{ik_0x} \exp\left(-\frac{x^2}{4\alpha}\right) \int_{-\infty}^{\infty} e^{-\alpha k''^2}\, dk'' \quad -(4)$$

where

$$\int_{-\infty}^{\infty} e^{-\alpha k''^2}\, dk'' = \left(\frac{\pi}{\alpha}\right)^{1/2}. \quad -(5)$$

So

$$f(x) = \left(\frac{1}{2\pi\alpha}\right)^{1/4} e^{ik_0x} \exp\left(-\frac{x^2}{4\alpha}\right) \quad -(6)$$

$$= e^{ik_0x} f_1(x) \quad -(7)$$

Note that:

$$\int_{-\infty}^{\infty} |f_1(x)|^2\, dx = \left(\frac{1}{2\pi\alpha}\right)^{1/2} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\alpha}\right) dx = 1$$

The standard deviation of a Gaussian is

2)
$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x-X)^2}{2\sigma^2}\right) \quad -(8)$$

so if
$$X = 0 \quad \to (9)$$
$$p(x) = f^2(x) = \left(\frac{1}{2\pi\alpha}\right)^{1/2} \exp\left(-\frac{x^2}{2\alpha}\right) \quad -(9)$$

if
$$\sigma_x = \alpha^{1/2}. \quad -(10)$$

Also:
$$p(k) = \left(\frac{2\alpha}{\pi}\right)^{1/2} \exp\left(-2\alpha(k-k_0)^2\right) \quad -(11)$$
$$= A^2(k) \quad -(12)$$

So
$$\sigma_k = \frac{1}{(4\alpha)^{1/2}} \quad -(13)$$

Therefore
$$\sigma_x \sigma_k = \frac{1}{2} \quad -(14)$$

i.e.
$$\Delta x \, \Delta k = \frac{1}{2} \quad -(15)$$

or
$$\boxed{\Delta x \, \Delta p = \frac{\hbar}{2}} \quad -(16)$$

This is the original Bohr derivation of eq. (16), which is merely a mathematical consequence of the Fourier transform of a Gaussian.