1) 175 (10): Schroedinger Equation of the Particle on a Sphere.

If the sphere has constant radius this is:

$$\Lambda^2 \phi = \left(\frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \phi}{\partial \phi^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta} \right) \right) \phi$$

$$= -\left(\frac{2IE}{\hbar^2} \right) \phi \qquad -(1)$$

in standard notation of any textbook. The wavefunctions are the spherical harmonics:

$$Y(\theta, \phi) = A(\theta) B(\phi) \qquad -(2)$$

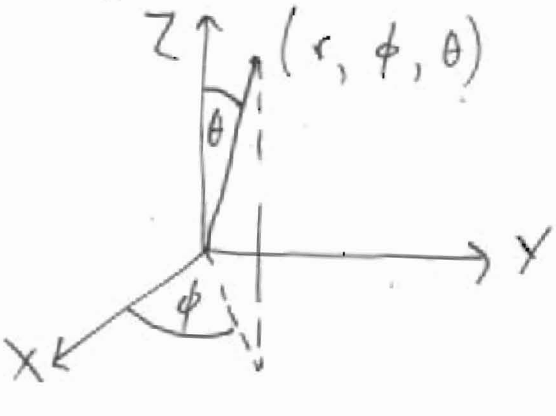
where

$$A(\theta) = \left[\frac{(2l+1)(l-|m_l|)!}{2(l+|m_l|)!} \right]^{1/2} P_l^{|m_l|}(\cos\theta)$$

$$B(\phi) = \left(\frac{1}{2\pi} \right)^{1/2} \exp(i m_l \phi)$$

l	m_l	$Y(\theta, \phi)$
0	0	$\frac{1}{(2\pi)^{1/2}}$
1	0	$\frac{1}{2}\left(\frac{3}{\pi}\right)^{1/2} \cos\theta$
1	± 1	$\mp \frac{1}{2}\left(\frac{3}{(2\pi)}\right)^{1/2} \sin\theta \exp(\pm i\phi)$

The coordinate system is defined as in the figure



2) If r is constant the gradient of ϕ is:

$$\underline{\nabla}\phi = \frac{1}{r}\frac{\partial\phi}{\partial\theta}\,\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\phi}\,\underline{e}_\phi \quad -(3)$$

where:

$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k}$$

$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j} \quad -(4)$$

so:

$$\begin{aligned}\underline{\nabla}\phi = &\frac{1}{r}\left(\cos\theta\cos\phi\frac{\partial\phi}{\partial\theta} - \frac{\sin\phi}{\sin\theta}\frac{\partial\phi}{\partial\phi}\right)\underline{i} \\ &+ \frac{1}{r}\left(\cos\theta\sin\phi\frac{\partial\phi}{\partial\theta} + \frac{\cos\phi}{\sin\theta}\frac{\partial\phi}{\partial\phi}\right)\underline{j} \quad -(5) \\ &- \frac{\sin\theta}{r}\frac{\partial\phi}{\partial\theta}\,\underline{k}\end{aligned}$$

By Schroedinger's axiom:

$$\hat{p} = -i\hbar\,\underline{\nabla} \quad -(6)$$

so

$$\hat{p}\phi = -i\hbar\,\underline{\nabla}\phi \quad -(7)$$

In three dimensions, the commutator $[x, \hat{p}_x]$ becomes:

$$[\underline{r}, \underline{\hat{p}}]\phi = \underline{r}\cdot\underline{\hat{p}}\phi - \underline{\hat{p}}\cdot(\underline{r}\phi) \quad -(8)$$

where:

$$\underline{r} = X\,\underline{i} + Y\,\underline{j} + Z\,\underline{k} \quad -(9)$$

$$\underline{\hat{p}} = \hat{p}_x\,\underline{i} + \hat{p}_y\,\underline{j} + \hat{p}_z\,\underline{k} \quad -(10)$$

3) Therefore:

$$[\underline{r}, \underline{\hat{p}}]\phi = -i\hbar\left(\underline{r}\cdot\underline{\nabla}\phi - \underline{\nabla}\cdot(\underline{r}\phi)\right) \quad -(11)$$

where:

$$\underline{\nabla}\cdot(\underline{r}\phi) = \phi\,\underline{\nabla}\cdot\underline{r} + \underline{r}\cdot\underline{\nabla}\phi \quad -(12)$$

in which:

$$\underline{\nabla}\cdot\underline{r} = 3 \quad -(13)$$

So:

$$[\underline{r}, \underline{\hat{p}}]\phi = -i\hbar\left(\underline{r}\cdot\underline{\nabla}\phi - \phi\,\underline{\nabla}\cdot\underline{r} - \underline{r}\cdot\underline{\nabla}\phi\right)$$

$$= 3i\hbar\phi \quad -(14)$$

i.e. $[X, \hat{p}_X]\phi = [Y, \hat{p}_Y]\phi = [Z, \hat{p}_Z]\phi$

$$= i\hbar\phi \quad -(15)$$

To simplify the calculation consider:

$$[Z, \hat{p}_Z]\phi = i\hbar\phi \quad -(16)$$

and work out the anticommutator:

$$\{Z, \hat{p}_Z\}\phi = -i\hbar\left\{Z, \frac{\partial}{\partial Z}\right\}\phi$$

$$= -i\hbar\left(Z\frac{\partial\phi}{\partial Z} + \frac{\partial}{\partial Z}(Z\phi)\right)$$

$$= -i\hbar\left(2Z\frac{\partial\phi}{\partial Z} + \phi\right) \quad -(17)$$

4) In the Copenhagen claim, if Z is known precisely in eq. (16) then $\hat{p}_Z$ is "unknowable". In Baconian philosophy, eq. (16) is the simple result of eq. (6). As in previous notes the anti-commutator is used to test the Copenhagen claim. As in note 175(7), eq. (8), and in one dimension:

$$[x^2, \hat{p}^2] \phi = 2i\hbar \{x, \hat{p}\} \phi \quad -(18)$$

In previous notes for UFT 175 it was found that $\langle [x^2, \hat{p}^2] \rangle$ is zero for all the wavefunctions of the particle in a box and harmonic oscillator, and may be zero or non-zero for rotational motion in a plane (particle on a ring). According to the Copenhagen school, if:

$$[\hat{A}, \hat{B}] = i\hat{C} \quad -(19)$$

then:

$$\delta\hat{A}\,\delta\hat{B} \geq \frac{1}{2}\langle \hat{C} \rangle \quad -(20)$$

$$= -i \langle [\hat{A}, \hat{B}] \rangle \quad -(21)$$

This is claimed to be an "uncertainty principle". If

$$\langle [\hat{A}, \hat{B}] \rangle = 0 \quad -(22)$$

then it is claimed by Copenhagen that $\hat{A}$ and $\hat{B}$ are "simultaneously knowable". Since $\langle [x^2, \hat{p}^2] \rangle$ is sometimes zero and sometimes non-zero, as in previous notes, Copenhagen is <u>clearly refuted.</u> We check this next for the particle on a sphere.