175(9): Anti Commutators for Particle on a Ring

In this case use the results:

$$\left.\begin{aligned} x &= r\cos\phi, \quad y = r\sin\phi, \\ \frac{\partial\psi}{\partial x} &= \frac{\partial\psi}{\partial r}\cos\phi - \frac{\sin\phi}{r}\frac{\partial\psi}{\partial\phi} \\ \frac{\partial\psi}{\partial y} &= \frac{\partial\psi}{\partial r}\sin\phi + \frac{\cos\phi}{r}\frac{\partial\psi}{\partial\phi} \end{aligned}\right\} \quad -(1)$$

When r is a constant:

$$\frac{\partial\psi}{\partial x} = -\frac{\sin\phi}{r}\frac{\partial\psi}{\partial\phi}, \quad \frac{\partial\psi}{\partial y} = \frac{\cos\phi}{r}\frac{\partial\psi}{\partial\phi} \quad -(2)$$

Therefore:

$$\left\{x, \frac{d}{dy}\right\}\psi = 2x\frac{d\psi}{dy} + \left(\frac{dx}{dy}\right)\psi$$

$$= 2\cos^2\phi\frac{d\psi}{d\phi} + \left(\frac{\cos\phi}{r}\frac{d}{d\phi}(r\cos\phi)\right)\psi$$

$$= 2\cos^2\phi\frac{d\psi}{d\phi} - \sin\phi\cos\phi\,\psi \quad -(3)$$

& $$\left\{y, \frac{d}{dx}\right\}\psi = 2y\frac{d\psi}{dx} + \left(\frac{dy}{dx}\right)\psi$$

$$= -2\sin^2\phi\frac{d\psi}{d\phi}\left(\frac{\sin\phi}{r}\frac{d}{d\phi}(r\sin\phi)\right)\psi$$

$$= -2\sin^2\phi\frac{d\psi}{d\phi} - \sin\phi\cos\phi\,\psi \quad -(4)$$

So: $$\boxed{\left(\left\{x, \frac{d}{dy}\right\} - \left\{y, \frac{d}{dx}\right\}\right)\psi = 2\frac{d\psi}{d\phi}} \quad -(5)$$

The wave function is:

$$\phi = \left(\frac{1}{2\pi}\right)^{1/2} \exp\left(i m_J \phi\right) \quad -(6)$$

So $$\frac{d\phi}{d\phi} = i m_J \left(\frac{1}{2\pi}\right)^{1/2} \exp\left(i m_J \phi\right). \quad -(7)$$

Therefore:

$$\left\langle \frac{d\phi}{d\phi} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \phi^* \frac{d\phi}{d\phi} \phi \, d\phi \quad -(8)$$

$$= \left(\frac{1}{2\pi}\right)^{3/2} i m_J \int_0^{2\pi} e^{i m_J \phi} \, d\phi$$

$$= \left(\frac{1}{2\pi}\right)^{3/2} e^{i m_J \phi} \Big|_0^{2\pi}$$

$$= 0$$

So
$$\boxed{\left\langle \left(\left[x, \hat{P}_y \right] - \left[y, \hat{P}_x \right] \right) \right\rangle = 0} \quad -(9)$$

Now consider:

$$\left[x, \frac{d}{dx} \right] \phi = 2x \frac{d\phi}{dx} + \phi$$

$$= -2 \sin\phi \cos\phi \frac{d\phi}{d\phi} + \phi$$

$$= -\sin(2\phi) \frac{d\phi}{d\phi} + \phi \quad -(10)$$

) and

$$\left[y, \frac{d}{dy} \right] \phi = 2y \frac{d\phi}{dy} + \phi$$

$$= \sin(2\phi) \frac{d\phi}{d\phi} + \phi \quad -(11)$$

then:

$$\boxed{\left(\left[y, \frac{d}{dy} \right] - \left[x, \frac{d}{dx} \right] \right) \phi = 2 \sin(2\phi) \frac{\partial \phi}{\partial \phi}} \quad -(12)$$

The expectation value is:

$$\left\langle \sin(2\phi) \frac{\partial \phi}{\partial \phi} \right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \sin(2\phi) \frac{\partial \phi}{\partial \phi} \, d\phi \quad -(13)$$

$$= \left(\frac{1}{2\pi} \right)^{3/2} i m_J \int_0^{2\pi} e^{i m_J \phi} \sin 2\phi \, d\phi$$

Use the result:

$$\int e^{bx} \sin ax \, dx = \frac{1}{a^2 + b^2} e^{bx} \left(b \sin ax - a \cos ax \right) \quad -(14)$$

to find:

$$\int_0^{2\pi} e^{i m_J \phi} \sin 2\phi \, d\phi = \frac{1}{2 - m_J^2} e^{i m_J \phi} \left(i m_J \sin 2\phi - 2 \cos 2\phi \right) \Big|_0^{2\pi} \quad -(15)$$

$$= 0$$

$$\boxed{\left\langle \left[y, p_y \right] - \left[x, p_x \right] \right\rangle = 0} \quad -(16)$$

4)

However:

$$\left(\{y, p_y\} + \{x, p_x\}\right)\phi = -2i\hbar\phi \quad -(17)$$

Now use:

$$[x^2, p^2]\phi = 2i\hbar\{x, p\}\phi \quad -(18)$$

from note 175(7). Therefore:

$$\boxed{\begin{aligned}\left([x^2, p_x^2] + [y^2, p_y^2]\right)\phi &= 4\hbar^2\phi \quad -(19)\\ \left([y^2, p_y^2] - [x^2, p_x^2]\right)\phi &= 4\hbar^2\sin(2\phi)\frac{\partial\phi}{\partial\phi} \quad -(20)\end{aligned}}$$

Therefore:

$$\boxed{\begin{aligned}\left\langle [x^2, p_x^2] + [y^2, p_y^2]\right\rangle &= 4\hbar^2 \quad -(21)\\ \left\langle [y^2, p_y^2] - [x^2, p_x^2]\right\rangle &= 0 \quad -(22)\end{aligned}}$$

with $[x, p_x]\phi = [y, p_y]\phi = i\hbar\phi$, $-(23)$

According to Copenhagen, if x^2 and y^2 are specified precisely, p_x^2 and p_y^2 are unknowable from Eq. (21), but at the same time specified precisely from Eq. (22). Copenhagen becomes an absurd philosophy

): Results to Date

1) $\langle \{ x, p_x \} \rangle = 0.$

for the particle in a box and harmonic oscillator for all wave functions.

2) For the particle on a ring, and for planar rotation:

$$\left\langle \{ x, p_y \} - \{ y, p_x \} \right\rangle = 0$$

$$\left\langle \{ y, p_y \} - \{ x, p_x \} \right\rangle = 0$$

and these combination of anti-commutators are zero.

Eq. (5) is an equation of motion:

$$\left(\{ x, p_y \} - \{ y, p_x \} \right) \psi = -2i\hbar \frac{d\psi}{d\phi}$$

i.e. $$\boxed{\frac{d\psi}{d\phi} = \frac{i}{2\hbar} \left(\{ x, p_y \} - \{ y, p_x \} \right) \psi}$$

which is true for all rotational motion in a plane.