1) 174(6): Angular Part of the Wavefunction from the Fermion Equation for Atomic H.

We use the ket notation, which is related to the spherical harmonic notation as follows:

$$| j m_j \, l_+ \rangle = Y^{jm}_{j+\frac{1}{2}} \quad -(1)$$

$$| j m_j \, l_- \rangle = Y^{jm}_{j-\frac{1}{2}} \quad -(2)$$

where

$$l_\pm = j \pm \frac{1}{2} . \quad -(3)$$

The basic problem is to solve:

$$\left(\hat{L}_Z + \frac{\hbar}{2} \hat{\sigma}_Z \right) \begin{bmatrix} \phi_S^R \\ \phi_S^L \end{bmatrix} = m\hbar \begin{bmatrix} \phi_S^R \\ \phi_S^L \end{bmatrix} -(4)$$

and

$$\left(\hat{L}^2 + \hbar \underline{\hat{L}} \cdot \underline{\hat{\sigma}} + \frac{3}{4} \hbar^2 \right) \begin{bmatrix} \phi_S^R \\ \phi_S^L \end{bmatrix} = j(j+1)\hbar^2 \begin{bmatrix} \phi_S^R \\ \phi_S^L \end{bmatrix} . -(5)$$

There are two angular momenta present, the orbital and spin angular momenta of the electron. The permitted states of the total angular momentum are given by the Clebsch Gordan series:

$$j = j_1 + j_2, \, j_1 + j_2 - 1, \dots, |j_1 - j_2| -(6)$$

$$m_j = m_{j1} + m_{j2} . \quad -(7)$$

When

$$j_2 = \frac{1}{2} \quad -(8)$$

then

$$\boxed{j = l + \frac{1}{2} \text{ or } l - \frac{1}{2}} \quad -(9)$$

which is eq. (3).

2) We have:

$$\hat{\underline{\sigma}} \cdot \frac{\underline{r}}{r} \, | j m_j \, l_\pm \rangle = - \, | j m_j \, l_\mp \rangle \quad -(10)$$

so the pseudoscalar operator $\hat{\underline{\sigma}} \cdot \underline{r} / r$ changes the spin of the electron from half to minus half.

Furthermore:

$$\hat{\underline{\sigma}} \cdot \hat{\underline{L}} \, | j m_j \, l_+ \rangle = -(l_+ + 1) \, | j m_j \, l_+ \rangle \quad -(11)$$

and
$$\hat{\underline{\sigma}} \cdot \underline{L} \, | j m_j \, l_- \rangle = l_- \, | j m_j \, l_- \rangle . \quad -(12).$$

These results imply that:

$$\hat{\underline{L}} \cdot \hat{\underline{S}} \, | j m_j \, l \rangle = \hat{\underline{L}} \cdot \hat{\underline{S}} \, | n l s ; j m_j \rangle$$
$$= \frac{1}{2} \hbar^2 \left(j(j+1) - l(l+1) - s(s+1) \right) | j m_j \, l \rangle \quad -(13)$$

where:
$$\hat{\underline{S}} = \frac{1}{2} \hbar \, \hat{\underline{\sigma}} \quad -(14)$$

Proof If
$$l_- = j - \frac{1}{2} \quad -(15)$$
then
$$s = -\frac{1}{2} \quad -(16)$$

so:

$$j(j+1) - l(l+1) - s(s-1)$$
$$= j(j+1) - \left(j - \frac{1}{2}\right)\left(j + \frac{1}{2}\right) - \frac{3}{4}$$

3)
$$= j^2 + j - j^2 + \frac{1}{4} - \frac{3}{4}$$
$$= j - \frac{1}{2} = l_- \quad -(17)$$

Q.E.D. Similarly, if
$$l = j + \frac{1}{2} \quad -(18)$$
then
$$s = \frac{1}{2} \quad -(19)$$

and
$$j(j+1) - l(l+1) - s(s+1) = j(j+1) - (j+\tfrac{1}{2})(j+\tfrac{3}{2}) - \tfrac{3}{4}$$
$$= j^2 + j - j^2 - 2j - \frac{3}{4} - \frac{3}{4}$$
$$= -j - \frac{3}{2} = -\left(j + \frac{1}{2} + 1\right)$$
$$= -(l_+ + 1) \quad -(20).$$

Q.E.D. The following operator identity applies:
$$\underline{\hat{\sigma}} \cdot \underline{\hat{p}} = \underline{\hat{\sigma}} \cdot \frac{\underline{r}}{r}\left(\frac{\underline{r}}{r} \cdot \underline{\hat{p}} + i\frac{\underline{\hat{\sigma}} \cdot \underline{\hat{L}}}{r}\right), \quad -(21)$$

i.e.
$$\underline{\hat{\sigma}} \cdot \underline{\hat{p}} = \underline{\hat{\sigma}} \cdot \frac{\underline{r}}{r}\left(-i\hbar\frac{\partial}{\partial r} + i\frac{\underline{\hat{\sigma}} \cdot \underline{\hat{L}}}{r}\right) \quad -(22)$$

Now consider the fermion equation in the format:
$$\phi^R = \exp\left(\frac{1}{2}\underline{\sigma} \cdot \underline{\phi}\right)\phi^R(0) \quad -(23)$$
$$\phi^L = \exp\left(-\frac{1}{2}\underline{\sigma} \cdot \underline{\phi}\right)\phi^L(0), \quad -(24)$$

which are the Lorentz transforms of the right and left Pauli spinors. Eqs. (23) and (24) imply:

i)
$$\left(E + c\,\underline{\hat{\sigma}}\cdot\underline{\hat{p}}\right)\phi^L = mc^2\phi^R \quad -(25)$$
and
$$\left(E - c\,\underline{\hat{\sigma}}\cdot\underline{\hat{p}}\right)\phi^R = mc^2\phi^L \quad -(26)$$
where:
$$\boxed{\begin{aligned}\phi^R &= \phi_S^R - \phi_S^L \\ \phi^L &= \phi_S^R + \phi_S^L\end{aligned}} \quad -(27)$$
Clearly, ϕ^R and ϕ^L are the physically meaningful wave functions because in eqs. (25) and (26) the rest energy is positive:
$$E_0 = mc^2 \quad -(28)$$
In ket notation:
$$\phi_S^R = |j m_j \ell_-\rangle \quad -(29)$$
$$\phi_S^L = |j m_j \ell_+\rangle \quad -(30)$$
so:
$$\boxed{\begin{aligned}\phi^R &= |j m_j \ell_-\rangle - |j m_j \ell_+\rangle \\ \phi^L &= |j m_j \ell_-\rangle + |j m_j \ell_-\rangle\end{aligned}} \quad -(31)$$
with
$$\int \phi_S^+ \phi_S \, d^3x = 1 \quad -(32)$$
Note that
$$\phi_S^R = \begin{bmatrix} \psi_{S1}^R \\ \psi_{S2}^R \end{bmatrix}, \quad \phi^L = \begin{bmatrix} \psi_{S1}^L \\ \psi_{S2}^L \end{bmatrix} \quad -(33)$$

5) and $\phi_s = \begin{bmatrix} \phi_s^R \\ \phi_s^L \end{bmatrix}$ – (34)

so $\phi_s^+ = \begin{bmatrix} \phi_s^{R*} & \phi_s^{L*} \end{bmatrix}$ – (35)

and $\int \left(\phi_{s1}^{R*} \phi_{s1}^{R} + \dots + \phi_{s2}^{L*} \phi_{s2}^{L} \right) d^3x = 1$ – (36)

From eqs. (31) and (33):

$$\phi_1^R = \phi_1^R(r) \left(|jm_j\, l_-\rangle - |jm_j\, l_+\rangle \right)$$
$$\phi_2^R = \phi_2^R(r) \left(|jm_j\, l_-\rangle - |jm_j\, l_+\rangle \right)$$
$$\phi_1^L = \phi_1^L(r) \left(|jm_j\, l_-\rangle + |jm_j\, l_+\rangle \right)$$
$$\phi_2^L = \phi_2^L(r) \left(|jm_j\, l_-\rangle + |jm_j\, l_+\rangle \right) \quad -(37)$$

Physical Interpretation

In order for the fundamental Lorentz transforms (23) and (24) to be obeyed correctly, the wave functions in the H atom must be the combinations (37). These are combinations of spin up and spin down states of the electron. Keeping the energy mc^2 positive implies these combinations. The eigen equation for $\underline{L} \cdot \hat{\underline{S}}$ is always eq. (13), giving the energy levels E as in note 174(4). The combinations in eq. (37) are always of different spin state of the electron.