1)

174(7): Complete Solution of the Fermion Equation for the H Atom

Consider the fermion equation in the format:

$$(E + c\underline{\sigma}\cdot\hat{p})\phi^L = mc^2\phi^R \quad -(1)$$

$$(E - c\underline{\sigma}\cdot\hat{p})\phi^R = mc^2\phi^L \quad -(2)$$

Use the Lorentz transforms of ϕ^L and ϕ^R given by the transformations:

$$\phi^L = \phi^R_s + \phi^L_s \quad -(3)$$

$$\phi^R = \phi^R_s - \phi^L_s \quad -(4)$$

to find:

$$(E - mc^2)\phi^R_s + c\underline{\sigma}\cdot\hat{p}\,\phi^L_s = 0 \quad -(5)$$

$$(E + mc^2)\phi^L_s + c\underline{\sigma}\cdot\hat{p}\,\phi^R_s = 0. \quad -(6)$$

For the H atom the solutions are:

$$\phi^R_s = F\,|jml_-\rangle \quad -(7)$$

$$\phi^L_s = -if\,|jml_+\rangle. \quad -(8)$$

In these equations:

$$\underline{\sigma}\cdot\hat{p} = \underline{\sigma}\cdot\hat{r}\left(-i\hbar\frac{\partial}{\partial r} + i\,\underline{\sigma}\cdot\hat{\underline{L}}\right), \quad -(9)$$

with

$$\underline{\sigma}\cdot\hat{\underline{r}}\,|jml_\mp\rangle = -|jml_\pm\rangle, \quad -(10)$$

and

$$\underline{\sigma}\cdot\hat{\underline{L}}\,|jml\rangle = \hbar\left(j(j+1) - l(l+1) - s(s+1)\right)|jml\rangle \quad -(11)$$

where

$$l_\pm = j \pm \frac{1}{2} \quad -(12)$$

The structure of the H atom is in fact given completely by eq. (11), so the atomic structure is given

2) immediately by the fermion equation, fine detail included. This fact is emphasized using equation (9), which gives:

$$-(13)$$

$$\underline{\sigma}\cdot\underline{\hat{p}}\,\phi^R_s = -i\hbar c\left(\frac{\partial}{\partial r}-\frac{j-1/2}{r}\right)F\,|jml_+\rangle \quad -(14)$$

$$\underline{\sigma}\cdot\underline{\hat{p}}\,\phi^L_s = -\hbar c\left(\frac{\partial}{\partial r}+\frac{j+3/2}{r}\right)f\,|jml_-\rangle \quad -(15)$$

So eqs. (5) and (6) become:

$$(E-mc^2)F-\hbar c\left(\frac{d}{dr}+\frac{j+3/2}{r}\right)f=0 \quad -(16)$$

$$(E+mc^2)f+\hbar c\left(\frac{d}{dr}-\frac{j-1/2}{r}\right)F=0 \quad -(17)$$

which when solved simultaneously give the energy levels of H as in Merzbacher for example.

To derive eqns. (14) and (15) use:

$$\underline{\sigma}\cdot\underline{\hat{L}}\,|jml_-\rangle = \left(j-\frac{1}{2}\right)|jml_-\rangle \quad -(18)$$

$$\underline{\sigma}\cdot\underline{\hat{L}}\,|jml_+\rangle = -\left(j+\frac{3}{2}\right)|jml_+\rangle \quad -(19)$$

and note that:

$$\underline{\sigma}\cdot\underline{\hat{r}}\,\frac{\partial}{\partial r} = -\frac{\partial}{\partial r} \quad -(20)$$

$$\underline{\sigma}\cdot\underline{\hat{r}}\,\frac{1}{r} = -\frac{1}{r} \quad -(21)$$

because of parity.

Therefore:

3)

$$\underline{\sigma} \cdot \hat{\underline{p}}\, \phi_s^R = \underline{\sigma} \cdot \hat{\underline{r}} \left(-i\hbar \frac{\partial}{\partial r} + \frac{i\, \underline{\sigma} \cdot \hat{\underline{L}}}{r} \right) F \,|jm\ell_-\rangle$$

$$= -i\hbar \frac{\partial F}{\partial r} |jm\ell_+\rangle + i\hbar \frac{(j-\frac{1}{2})}{r} F\,|jm\ell_+\rangle$$

$$= -i\hbar \left(\frac{\partial}{\partial r} - \frac{(j-1/2)}{r} \right) F \,|jm\ell_+\rangle$$

$$\boxed{\underline{\sigma} \cdot \hat{\underline{p}}\, \phi_s^R = -i\hbar \left(\frac{\partial}{\partial r} - \frac{(j-1/2)}{r} \right) F \,|jm\ell_+\rangle}$$

and

$$\underline{\sigma} \cdot \hat{\underline{p}}\, \phi_s^L = -i\, \underline{\sigma} \cdot \hat{\underline{p}}\, f \,|jm\ell_+\rangle$$

$$= -i \left(\underline{\sigma} \cdot \hat{\underline{r}} \right) \left(-i\hbar \frac{\partial}{\partial r} + \frac{i\, \underline{\sigma} \cdot \hat{\underline{L}}}{r} \right) f \,|jm\ell_+\rangle$$

$$= -\hbar \frac{\partial f}{\partial r} |jm\ell_-\rangle - \hbar \frac{(j+3/2)}{r} f\,|jm\ell_-\rangle$$

$$= -\hbar \left(\frac{\partial}{\partial r} + \frac{j+3/2}{r} \right) f \,|jm\ell_-\rangle$$

$$\boxed{\underline{\sigma} \cdot \hat{\underline{p}}\, \phi_s^L = -\hbar \left(\frac{\partial}{\partial r} + \frac{j+3/2}{r} \right) f \,|jm\ell_-\rangle}$$

The complete wavefunctions ϕ^L and ϕ^R of the fermion equation are the <u>combinations</u> (3) and (4) in which ϕ^L and ϕ^R are inter-converted by parity. Therefore :

4)

$$\phi^L = \phi^R_s + \phi^L_s$$
$$= F|jml_-\rangle - if|jml_+\rangle \quad -(22)$$

$$\phi^R = \phi^R_s - \phi^L_s$$
$$= F|jml_-\rangle + if|jml_+\rangle \quad -(23)$$

and $$\hat{P}\,\phi^L = \phi^R \quad -(24)$$

Therefore the effect of parity is to produce the complex conjugate.

Using the de Moivre theorem:

$$e^{i\theta} = \cos\theta + i\sin\theta \quad -(25)$$
$$e^{-i\theta} = \cos\theta - i\sin\theta \quad -(26)$$

then: $$\boxed{\phi^L = e^{i\theta}, \quad \phi^R = e^{-i\theta}} \quad -(27)$$

where $$\cos\theta = F|jml_-\rangle \quad -(28)$$
$$\sin\theta = f|jml_+\rangle. \quad -(29)$$

So in eqns. (1) and (2):

$$(E + c\underline{\sigma}\cdot\underline{p})\,e^{i\theta} = mc^2 e^{-i\theta} \quad -(30)$$
$$(E - c\underline{\sigma}\cdot\underline{p})\,e^{-i\theta} = mc^2 e^{i\theta} \quad -(31)$$

and $$\phi^R_s = \cos\theta, \quad \phi^L_s = -i\sin\theta \quad -(32)$$