174(2): Details of the Solution of the Fermion Equation for the Hydrogen Atom.

The fermion equation produces the correct Lorentz transform of the right and left Pauli spinors, whereas the standard Dirac equation does not. The fermion equation is therefore the correct equation in physics. The fermion eqn. may be written as:

$$(E - e\phi + c\underline{\sigma}\cdot\hat{\underline{p}})\,\phi^L = mc^2\phi^R \quad -(1)$$

$$(E - e\phi - c\underline{\sigma}\cdot\hat{\underline{p}})\,\phi^R = mc^2\phi^L \quad -(2)$$

where ϕ^R and ϕ^L are the right and left Pauli spinors:

$$\phi^R = \begin{bmatrix} \phi_1^R \\ \phi_2^R \end{bmatrix}, \quad \phi^L = \begin{bmatrix} \phi_1^L \\ \phi_2^L \end{bmatrix} \quad -(3)$$

Eqs (1) and (2) imply that under the Lorentz transform:

$$\phi^R \rightarrow \exp\left(\frac{1}{2}\,\underline{\sigma}\cdot\underline{\phi}\right)\phi^R \quad -(4)$$

$$\phi^L \rightarrow \exp\left(-\frac{1}{2}\,\underline{\sigma}\cdot\underline{\phi}\right)\phi^L \quad -(5)$$

(L. H. Ryder, "Quantum Field Theory" (C.U.P., 1996, 2nd. ed.) pp. 41 ff.)

It is clear that the eigenvalues mc^2 are always positive.

In the hydrogen atom, ϕ is the Coulomb potential between electron and proton. The energy levels E of the H atom are found by solving eqs. (1) and (2) simultaneously. It is instructive to give

2) The details of this calculation is follows. The basics are found in:

E. Merzbacher, "Quantum Mechanics" (Wiley, 2rd. ed., 1970), pp. 603 ff.

and are collected here for ease of reference.

First note that both $\underline{\sigma}$ and $\underline{\hat{p}}$ are operators.

First use:

$$\underline{\sigma} \cdot \underline{\hat{p}} = \underline{\sigma} \cdot \frac{\underline{r}}{r} \left(\frac{\underline{r}}{r} \cdot \underline{\hat{p}} + i \frac{\underline{\sigma} \cdot \underline{\hat{L}}}{r} \right) \quad -(6)$$

where

$$\frac{\underline{r}}{r} \cdot \underline{\hat{p}} = \frac{\hbar}{i} \frac{\partial}{\partial r}. \quad -(7)$$

Note that:

$$\underline{\sigma} \cdot \frac{\underline{r}}{r} \, Y^{jm}_{j \mp \frac{1}{2}} = - Y^{jm}_{j \pm \frac{1}{2}} \quad -(8)$$

because $\underline{\sigma} \cdot \underline{r}$ is a pseudo-scalar. In the eigen-equation (8) the eigenfunctions are defined by Clebsch-Gordan coefficients. Next note that:

$$\underline{\sigma} \cdot \underline{\hat{L}} \, Y^{jm}_{j - \frac{1}{2}} = \left(j - \frac{1}{2} \right) Y^{jm}_{j - \frac{1}{2}} \quad -(9)$$

$$\underline{\sigma} \cdot \underline{\hat{L}} \, Y^{jm}_{j + \frac{1}{2}} = - \left(j + \frac{3}{2} \right) Y^{jm}_{j + \frac{1}{2}} \quad -(10)$$

and that:

$$\left(L^2 + \hbar \underline{L} \cdot \underline{\sigma} + \frac{3}{4} \hbar^2 \right) \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} = j(j+1) \hbar^2 \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} \quad -(11)$$

$$\left(L_z + \frac{1}{2} \hbar \sigma_z \right) \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} = m \hbar \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} \quad -(12)$$

Eqs. (1) and (2) may be solved by noting that

3)

$$\phi^L = \phi_s^R + \phi_s^L \quad -(13)$$

$$\phi^R = \phi_s^R - \phi_s^L. \quad -(14)$$

where:

$$\phi_s^R = F(r)\, Y_{j-1/2}^{jm} \quad -(15)$$

$$\phi_s^L = -i f(r)\, Y_{j+1/2}^{jm} \quad -(16)$$

The physically meaningful wave functions are:

$$\phi^L = F(r)\, Y_{j-\frac{1}{2}}^{jm} - i f(r)\, Y_{j+\frac{1}{2}}^{jm} \quad -(17)$$

and

$$\phi^R = F(r)\, Y_{j-\frac{1}{2}}^{jm} + i f(r)\, Y_{j+\frac{1}{2}}^{jm} \quad -(18)$$

and are complex conjugates.

If ~~the~~ correct wave functions (17) and (18) are used, the energy is always positive.

The operators are worked out as follows:

$$c\,\underline{\sigma}\cdot\underline{\hat{p}}\,\phi_s^L = c\,\underline{\sigma}\cdot\frac{\underline{r}}{r}\left(\frac{\underline{r}}{r}\cdot\underline{\hat{p}} + i\,\frac{\underline{\sigma}\cdot\underline{\hat{L}}}{r}\right)\phi_s^L \quad -(19)$$

and so on. In eq. (19):

$$\underline{\sigma}\cdot\frac{\underline{r}}{r}\,\frac{\underline{r}}{r}\cdot\underline{\hat{p}}\,\phi_s^L = -i\hbar\frac{\partial}{\partial r}\left(\underline{\sigma}\cdot\frac{\underline{r}}{r}\,\phi_s^L\right) \quad -(20)$$

where

$$\underline{\sigma}\cdot\frac{\underline{r}}{r}\,\phi_s^L = -i f(r)\,\underline{\sigma}\cdot\frac{\underline{r}}{r}\,Y_{j+1/2}^{jm} \quad -(21)$$

$$= i f(r)\, Y_{j-1/2}^{jm}$$

So:

$$\underline{\sigma}\cdot\frac{\underline{r}}{r}\,\frac{\underline{r}}{r}\cdot\underline{\hat{p}}\,\phi_s^L = \hbar\frac{df}{dr}\,Y_{j-1/2}^{jm} \quad -(22)$$

t) Similarly:

$$i\left(\underline{\sigma}\cdot\frac{\underline{r}}{r}\right)\left(\underline{\sigma}\cdot\hat{\underline{L}}\right)\left(-i f(r)\, Y^{jm}_{j+\frac{1}{2}}\right)$$

$$= -\hbar\left(\underline{\sigma}\cdot\frac{\underline{r}}{r}\right) f(r)\left(j+\frac{3}{2}\right) Y^{jm}_{j+\frac{1}{2}}$$

$$= \hbar\left(j+\frac{3}{2}\right) f(r)\, Y^{jm}_{j-\frac{1}{2}} \quad -(23)$$

Therefore:

$$c\,\underline{\sigma}\cdot\hat{\underline{p}}\,\phi^{L}_{S} = \hbar c\left(\frac{d}{dr}+\frac{j+3/2}{r}\right) f(r)\, Y^{jm}_{j-\frac{1}{2}} \quad -(24)$$

and so on.

With the transform (13) and (14), eqns (1) and (2) become:

$$(E - mc^2 - e\phi)\,\phi^{R}_{S} = c\,\underline{\sigma}\cdot\hat{\underline{p}}\,\phi^{L}_{S} \quad -(25)$$

$$(E + mc^2 - e\phi)\,\phi^{L}_{S} = c\,\underline{\sigma}\cdot\hat{\underline{p}}\,\phi^{R}_{S}, \quad -(26)$$

with the physical wave functions (17) and (18).

Note carefully that the Lorentz transforms (4) and (5) give eqns. (1) and (2) and ∴ that eqns. (25) and (26) are the result of a mathematical transform made in order to solve eqns. (1) and (2). $-(27)$

Eqns. (25) and (26) give:

$$(E + mc^2 - e\phi) f(r) + \hbar c\left(\frac{d}{dr} - \frac{j-\frac{1}{2}}{r}\right) F(r) = 0$$

$$(E - mc^2 - e\phi) F(r) - \hbar c\left(\frac{d}{dr} + \frac{j+3/2}{r}\right) f(r) = 0 \quad -(28)$$

which may be solved for E in an approximation.

5) Using computers eqs. (27) and (28) may be solved to machine precision for E.

The result of the analytical solution is:

$$E = mc^2 \left(1 + \frac{(Z\alpha)^2}{\left(\left(\left(j + \frac{1}{2}\right)^2 - (Z\alpha)^2 \right)^{1/2} + n' \right)^2} \right)^{-1/2},$$

where $\alpha = e^2 / (4\pi\epsilon_0 \hbar c)$, — (29)

where

$$n' = n - \left(j + \frac{1}{2} \right), \quad \text{— (30)}$$

and

$$j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \quad \text{— (31)}$$

$$n' = 0, 1, 2, \ldots \quad \text{— (32)}$$

Here n is the principal quantum number in the non-relativistic theory of the H atom. Here:

$$\phi = -\frac{Ze}{r}. \quad \text{— (33)}$$

— (34)

Eq. (29) is approximately:

$$E = mc^2 \left(1 - \frac{Z^2 e^4}{2n^2 \alpha^2} - \frac{(Z^2 e^4)^2}{2n^4 \alpha^4} \left(\frac{n}{j + \frac{1}{2}} - \frac{3}{4} \right) + \ldots \right)$$

The non-relativistic result is:

$$E = -\frac{Z^2 e^4 mc^2}{2n^2 \alpha^2} \quad \text{— (35)}$$

$$= -\left(\frac{m e^4}{32\pi^2 \epsilon_0^2 \hbar^2} \right) \frac{1}{n^2} \quad \text{— (36)}$$

6) The $1S_{1/2}$ state is $n=1$, $\ell=0$, $j=1/2$, and

$$E(1S_{1/2}) = mc^2\left(1-\frac{Z^2e^4}{\alpha^2}\right) \quad -(37)$$

The $2S_{1/2}$ state is $n=2$, $\ell=0$, $j=1/2$, and the $2P_{1/2}$ state is $n=2$, $\ell=1$, $j=1/2$, with the same energy:

$$E(2S_{1/2}) = E(2P_{1/2}) = \frac{mc^2}{\sqrt{2}}\left(1+\left(1-\frac{Z^2e^4}{\alpha^2}\right)^{1/2}\right)^{1/2} \quad -(38)$$

The radiative corrections cause the Lamb shift between $2S_{1/2}$ and $2P_{1/2}$.

<u>Conclusion</u> Those well known results are given by the fermion equation:

$$\hat{\pi}_\mu \psi \sigma^\mu = mc\sigma^1\psi \quad -(39)$$

by expanding it into eqs (1) and (2). The wave function is:

$$\psi = \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} \quad -(40)$$

and the Pauli spinors are:

$$\phi^L = \begin{bmatrix} \psi_1^L \\ \psi_2^L \end{bmatrix} = F(r)\,Y^{jm}_{j-\frac{1}{2}} - i g(r)\,Y^{jm}_{j+\frac{1}{2}} \quad -(41)$$

$$\phi^R = \begin{bmatrix} \psi_1^R \\ \psi_2^R \end{bmatrix} = F(r)\,Y^{jm}_{j-\frac{1}{2}} + i g(r)\,Y^{jm}_{j+1/2}. \quad -(42)$$