1) 173(4): The Semi Operator Method of Relativistic Quantum Mechanics.

Consider the fermion equation:

$$\sigma^0 \hat{E} \phi \sigma^0 - c\sigma^3 \left(\hat{p}_x \phi \sigma^1 + \hat{p}_y \phi \sigma^2 + \hat{p}_z \phi \sigma^3 \right)$$
$$= mc^2 \sigma^1 \phi \qquad -(1)$$

of relativistic quantum mechanics. Here:

$$\phi = \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} \qquad -(2)$$

Eq. (1) may be written as:

$$\left(\hat{E} + c\underline{\sigma} \cdot \hat{\underline{p}} \right) \phi^L = mc^2 \phi^R \quad -(3)$$
$$\left(\hat{E} - c\underline{\sigma} \cdot \hat{\underline{p}} \right) \phi^R = mc^2 \phi^L \quad -(4)$$

where

$$\phi^L = \begin{bmatrix} \phi_1^L \\ \phi_2^L \end{bmatrix}, \quad \phi^R = \begin{bmatrix} \phi_1^R \\ \phi_2^R \end{bmatrix} \qquad -(5)$$

are the Pauli spinors. The Pauli matrices are:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$-(6)$$

Therefore relativistic quantum mechanics may be developed without the Dirac matrices, and without negative energy, and without the Dirac spinor and Dirac equation.

Eq (1) is automatically covariant because it expresses the Lorentz transformation of the Pauli spinor (Ryder, "Quantum Field Theory").

2) Eqns. (3) and (4) may be written as:

$$(E + c\underline{\sigma}\cdot\underline{p})(E - c\underline{\sigma}\cdot\underline{p})\,\phi^R = m^2c^4\phi^R \quad -(7)$$

$$(E - c\underline{\sigma}\cdot\underline{p})(E + c\underline{\sigma}\cdot\underline{p})\,\phi^L = m^2c^4\phi^L \quad -(8)$$

i.e

$$(E^2 - c^2\underline{\sigma}\cdot\underline{p}\;\underline{\sigma}\cdot\underline{p})\,\phi^R = m^2c^4\phi^R \quad -(9)$$

$$(E^2 - c^2\underline{\sigma}\cdot\underline{p}\;\underline{\sigma}\cdot\underline{p})\,\phi^L = m^2c^4\phi^L \quad -(10)$$

In quantum mechanics:

$$\hat{E} = i\hbar\frac{\partial}{\partial t},\quad \hat{\underline{p}} = -i\hbar\underline{\nabla}. \quad -(11)$$

The <u>semi-operator method</u> of solving eqns. (9) and (10) starts from writing:

$$E^2 = E\hat{E} \quad -(12)$$

$$= i\hbar E\frac{\partial}{\partial t}$$

where

$$E = \gamma mc^2 \quad -(13)$$

so

$$\boxed{E^2 = i\hbar\gamma mc^2\frac{\partial}{\partial t} = E\hat{E}} \quad -(14)$$

Therefore eq. (9) for example becomes:

$$(\gamma mc^2\hat{E} - c^2\underline{\sigma}\cdot\underline{p}\;\sigma\cdot\underline{p})\,\phi^R = m^2c^4\phi^R \quad -(15)$$

i.e

$$\boxed{\hat{E}\phi^R = H\phi^R} \quad -(16)$$

3) where

$$H = \frac{1}{E}\left(c^2 \underline{\sigma}\cdot\underline{p}\ \underline{\sigma}\cdot\underline{p} + m^2c^4\right) \quad -(17)$$

is the Hamiltonian.

So the non-linear eq. (9) has been reduced straightforwardly to the linear eq. (16) without use of the Dirac matrices. This is a major advance in quantum mechanics.

For a static particle eq. (17) reduces to

$$H = mc^2 \quad -(18)$$

so eq. (16) is:

$$i\hbar \frac{\partial \phi^R}{\partial t} = mc^2 \phi^R. \quad -(19)$$

Similarly, for a static particle:

$$i\hbar \frac{\partial \phi^L}{\partial t} = mc^2 \phi^L. \quad -(20)$$

For a static particle:

$$\phi^R(0) = \phi^L(0) \quad -(21)$$

(Ryder, "Quantum Field Theory").

For a moving particle:

$$E = \gamma mc^2, \quad -(22)$$

$$\underline{p} = \gamma m \underline{v}, \quad -(23)$$

so:

$$\boxed{H = \frac{\gamma}{m}\underline{\sigma}\cdot\underline{v}\ \underline{\sigma}\cdot\underline{v} + \frac{mc^2}{\gamma}} \quad -(24)$$

), so the fermion equation becomes:

$$i\hbar \frac{\partial \phi^R}{\partial t} = H \phi^R \quad -(25)$$

$$i\hbar \frac{\partial \phi^L}{\partial t} = H \phi^L \quad -(26)$$

with H given by eq. (24).

<u>Interaction with the Electromagnetic Field</u>

Eq. (15) becomes:

$$\left((E - e\phi)(\hat{E} - e\phi) - c^2 \underline{\sigma} \cdot \underline{\Pi}\, \underline{\sigma} \cdot \underline{\Pi} \right) \phi^R = m^2 c^4 \phi^R \quad -(27)$$

so

$$\hat{E} \phi^R = H \phi^R \quad -(28)$$

where

$$H = \frac{m^2 c^4}{E - e\phi} + e\phi + c^2 \frac{\underline{\sigma} \cdot \underline{\Pi}\, \underline{\sigma} \cdot \underline{\Pi}}{E - e\phi} \quad -(29)$$

The <u>correctly relativistic</u> Landé and Thomas factors are obtained from:

$$H_3 = c^2 \underline{\sigma} \cdot \underline{\Pi} \left(E - e\phi \right)^{-1} \underline{\sigma} \cdot \underline{\Pi} \quad -(30)$$

$$= \frac{c^2}{E} \underline{\sigma} \cdot \underline{\Pi} \left(1 - \frac{e\phi}{E} \right)^{-1} \underline{\sigma} \cdot \underline{\Pi} \quad -(31)$$

$$\doteq \frac{c^2}{E} \underline{\sigma} \cdot \underline{\Pi} \left(1 + \frac{e\phi}{E} \right) \underline{\sigma} \cdot \underline{\Pi} \quad -(32)$$

$$= \frac{c^2}{E}\left(\underline{\sigma}\cdot\underline{\Pi}\right)\left(\underline{\sigma}\cdot\underline{\Pi}\right) + \frac{c^2 e}{E^2}\,\underline{\sigma}\cdot\underline{\Pi}\,\phi\,\underline{\sigma}\cdot\underline{\Pi} \quad -(33)$$

where

$$E = \gamma m c^2 \quad -(34)$$

So:

$$H_3 = \frac{1}{\gamma m}\left(\underline{\sigma}\cdot\underline{\Pi}\right)\left(\underline{\sigma}\cdot\underline{\Pi}\right) + \frac{e}{\gamma^2 m^2 c^2}\,\underline{\sigma}\cdot\underline{\Pi}\,\phi\,\underline{\sigma}\cdot\underline{\Pi} \quad -(35)$$

The first term in eq. (35) gives the relativistic Landé factor and the second term the relativistic Thomas factor.

Non-Relativistic Approximation

Write eq. (27) as:

$$\left(E - e\phi - \frac{c^2\,\underline{\sigma}\cdot\underline{\Pi}\,\underline{\sigma}\cdot\underline{\Pi}}{\hat{E} - e\phi}\right)\phi^R = \left(\frac{m^2c^4}{E - e\phi}\right)\phi^R \quad -(36)$$

and add mc^2 to each side:

$$\left(E + mc^2 - e\phi - \frac{c^2\,\underline{\sigma}\cdot\underline{\Pi}\,\underline{\sigma}\cdot\underline{\Pi}}{\hat{E} - e\phi}\right)\phi^R = \left(\frac{m^2c^4}{E - e\phi} + mc^2\right)\phi^R \quad -(37)$$

In the non-relativistic approximation:

$$E \rightarrow mc^2 \quad -(38)$$

so eq. (37) becomes:

$$\left(\left(2mc^2 - e\phi\right)\left(\hat{E} - e\phi\right) - c^2\left(\underline{\sigma}\cdot\underline{\Pi}\right)\left(\underline{\sigma}\cdot\underline{\Pi}\right)\right)\phi^R$$

$$= \left(m^2c^4 + mc^2\left(E - e\phi\right)\right)\phi^R$$

$$= \left(mc^2\left(2mc^2 - e\phi\right)\right)\phi^R \quad -(39)$$

6)

i.e
$$\hat{E}\phi^R = H\phi^R \quad -(40)$$

where
$$H = mc^2 + e\phi + c^2(\underline{\sigma}\cdot\underline{\Pi})(2mc^2 - e\phi)^{-1}(\underline{\sigma}\cdot\underline{\Pi})$$
$$\doteqdot mc^2 + e\phi + \frac{1}{2m}\underline{\sigma}\cdot\underline{\Pi}\left(1 - \frac{e\phi}{2mc^2}\right)\underline{\sigma}\cdot\underline{\Pi},$$

i.e.
$$H = mc^2 + e\phi + \frac{1}{2m}\underline{\sigma}\cdot\underline{\Pi}\,\underline{\sigma}\cdot\underline{\Pi} + \frac{1}{4m^2c^2}\underline{\sigma}\cdot\underline{\Pi}\,\phi\,\underline{\sigma}\cdot\underline{\Pi} \quad -(41)$$

The Landé ($g = 2$) factor in this limit is given by
$$H_3 = \frac{1}{2m}\underline{\sigma}\cdot\underline{\Pi}\,\underline{\sigma}\cdot\underline{\Pi} \quad -(42)$$

and the Thomas factor, spin-orbit interaction, and Darwin term from:
$$H_4 = \frac{1}{4m^2c^2}\underline{\sigma}\cdot\underline{\Pi}\,\phi\,\underline{\sigma}\cdot\underline{\Pi} \quad -(43)$$

However, the relativistic result is given by eq.(35)

These results are all obtained without Dirac matrices, and without "negative energy"