

B(5): Covariant Form of Dirac Equation:

This is

$$\boxed{\pi_\mu \not{\psi} \sigma^\mu - mc \sigma^0 \psi = 0} \quad - (1)$$

where

$$\pi_\mu = (\pi_0, \pi_1, \pi_2, \pi_3) \quad - (2)$$

$$\pi_0 = \sigma^0 p_0, \quad \pi_i = \sigma^i p_i \quad - (3)$$

Here

$$p_\mu = (p_0, p_1, p_2, p_3) = \left(\frac{E}{c}, -\underline{p} \right) \quad - (4)$$

The wave function is the bispinor:

$$\psi = \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} \quad - (5)$$

The Pauli spinors are:

$$\psi^R = \begin{bmatrix} \psi_1^R \\ \psi_2^R \end{bmatrix}, \quad \psi^L = \begin{bmatrix} \psi_1^L \\ \psi_2^L \end{bmatrix} \quad - (6)$$

and the Pauli matrices are:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad - (7)$$

It is proposed to name π_μ "the fermion operator".
The fermion operator's position representation is
defined by:

$$\not{x}_\mu = -\frac{i}{\hbar} \pi_\mu \quad - (8)$$

so:

$$\boxed{i\hbar \not{x}_\mu \not{\psi} \sigma^\mu - mc \sigma^0 \psi = 0} \quad - (9)$$

2) It is proposed to name of a "fermion spinor".
 In comparison to covariant format of Dirac equation is:

$$(\gamma^\mu p_\mu - mc) \psi_D = 0 \quad - (10)$$

where

$$\psi_D = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} \quad - (11)$$

is Dirac spinor, and γ^μ are Dirac matrices.

The complete detail of development of eq. (1) is given in note 172(8). It has been shown in notes 173 that eq. (1) produces the same formalism as eq. (10) but without "negative energy". The latter does not exist in nature, and is a purely mathematical artifice.

The adjoint fermion spinor is

$$\psi^+ = \begin{bmatrix} \phi_1^R* & \phi_1^L* \\ \phi_2^R* & \phi_2^L* \end{bmatrix} \quad - (12)$$

where * denotes "complex conjugate". The adjoint fermion equation is:

$$\boxed{-i\hbar \gamma_\mu \psi^+ \sigma^\mu - mc \sigma^1 \psi^+ = 0} \quad - (13)$$

The probability four-current of the fermion equation is defined as:

$$3) \quad j^\mu = \text{Trace} (\psi \sigma^\mu \psi^\dagger + \psi^\dagger \sigma^\mu \psi) \quad - (14)$$

This current is conserved and rigorously non-negative:

$$\partial_\mu j^\mu = 0, \quad j^\mu > 0 \quad - (15)$$

Proof

Multiply eq. (1) from the right with ψ^\dagger :

$$i \partial_\mu \psi \sigma^\mu \psi^\dagger = m c \sigma^0 \psi \psi^\dagger \quad - (16)$$

Multiply eq. (3) from the right with ψ :

$$- i \partial_\mu \psi^\dagger \sigma^\mu \psi = m c \sigma^0 \psi^\dagger \psi \quad - (17)$$

Subtract eq. (17) from eq. (16):

$$i \partial_\mu (\psi \sigma^\mu \psi^\dagger + \psi^\dagger \sigma^\mu \psi) = m c \sigma^0 (\psi \psi^\dagger - \psi^\dagger \psi) \quad - (18)$$

By definition:

$$\psi \psi^\dagger - \psi^\dagger \psi = \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} \begin{bmatrix} \psi_1^{R*} & \psi_1^{L*} \\ \psi_2^{R*} & \psi_2^{L*} \end{bmatrix} - \begin{bmatrix} \psi_1^{R*} & \psi_1^{L*} \\ \psi_2^{R*} & \psi_2^{L*} \end{bmatrix} \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} \quad - (19)$$

$$\text{so } \text{Trace } \psi \psi^\dagger = \text{Trace } \psi^\dagger \psi = \psi_1^R \psi_1^{R*} + \psi_2^R \psi_2^{R*} + \psi_1^L \psi_1^{L*} + \psi_2^L \psi_2^{L*} \quad - (20)$$

1. The result (20) is obtained from the Dirac equation
 by:

$$j^\mu = \bar{\psi}_D \gamma^\mu \psi_D \quad - (21)$$

where the Dirac adjoint spinor is:

$$\bar{\psi}_D = \psi^\dagger \gamma^0 \quad - (22)$$

So:

$$\text{Trace } \psi \psi^\dagger = \text{Trace } \psi^\dagger \psi = \bar{\psi}_D \gamma^0 \psi_D \quad - (23)$$

From eq. (20):

$$\text{Trace } (\psi \psi^\dagger - \psi^\dagger \psi) = 0 \quad - (24)$$

so from eq. (18):

$$i \oint_\mu \text{Trace } (\psi \sigma^\mu \psi^\dagger + \psi^\dagger \sigma^\mu \psi) = 0 \quad - (25)$$

The momentum representation of eq. (25) is:

$$\pi_\mu \text{Trace } (\psi \sigma^\mu \psi^\dagger + \psi^\dagger \sigma^\mu \psi) = 0 \quad - (26)$$

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$$\pi_\mu \neq 0 \quad - (27)$$

follows that

$$\oint_\mu (\psi \sigma^\mu \psi^\dagger + \psi^\dagger \sigma^\mu \psi) = 0 \quad - (28)$$

E.D.