) 173(1): Development of the Spin-Orbit Term of the Fermion Equation.

Background Reading

1) E. Herzbacher, "Quantum Mechanics" (Wiley, 3rd. ed., 1970 and subsequent).

2) L. H. Ryder, "Quantum Field Theory" (CUP, 2nd. ed., 1996).

The Pauli matrices used in the fermion equation are as defined in Eq. (12.48) of Herzbacher:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad -(1)$$

This is different from the definition sometimes used in some sources, where the sign of σ^2 is reversed, but the same as the definition in Ryder, ref (2). So with the standard definition (1) the fermion equation is:

$$\sigma^0 \hat{E} \phi \sigma^0 - c\sigma^3 \left(\hat{p}_X \phi \sigma^1 + \hat{p}_Y \phi \sigma^2 + \hat{p}_Z \phi \sigma^3 \right) = mc^2 \sigma^1 \phi \quad -(2)$$

i.e.

$$\sigma^0 \hat{E} \phi \sigma^0 - c\sigma^3 \phi \, \underline{\sigma} \cdot \underline{\hat{p}} = mc^2 \sigma^1 \phi \quad -(3)$$

where

$$\underline{\sigma} \cdot \underline{p} = \begin{bmatrix} p_Z & p_X - ip_Y \\ p_X + ip_Y & -p_Z \end{bmatrix}, \quad -(4)$$

$$\phi = \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} \quad -(5)$$

2) The commutation relation of the Pauli matrices are exemplified as follows:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \quad -(6)$$

i.e. $$\sigma^1\sigma^2 - \sigma^2\sigma^1 = 2i\sigma^3 \quad -(7)$$

or $$\left[\frac{\sigma^1}{2}, \frac{\sigma^2}{2}\right] = i\frac{\sigma^3}{2} \quad -(8)$$

so the spin or intrinsic angular momentum is:

$$\underline{S} = \frac{1}{2}\hbar\,\underline{\sigma}. \quad -(9)$$

It is advantageous to consider the $\hat{p}_Z$ component of eq. (2), which simplifies the equation without loss of generality to:

$$\sigma^0\hat{E}\phi\sigma^0 - c\sigma^3\hat{p}_Z\phi\sigma^3 = mc^2\sigma^1\phi \quad -(10)$$

It is seen that:

$$(\sigma^3\hat{p}_3\phi)\sigma^3 = \sigma^3(\hat{p}_3\phi\sigma^3) \quad -(11)$$

where $$\hat{p}_3 = -\hat{p}_Z \quad -(12)$$

so $$(\sigma^0\hat{E}\phi)\sigma^0 + c(\sigma^3\hat{p}_3\phi)\sigma^3 = mc^2\sigma^1\phi. \quad -(13)$$

Defining $$\hat{p}_0 = \hat{E}_0 / c \quad -(14)$$

gives $$(\sigma^0\hat{p}_0\phi)\sigma^0 + (\sigma^3\hat{p}_3\phi)\sigma^3 = mc\sigma^1\phi \quad -(15)$$

In condensed notation, eq. (15) is:

$$\boxed{\left(\sigma^\mu \hat{p}_\mu \phi\right)\sigma^\mu = mc\sigma^1\phi} \quad -(16)$$

In position representation:

$$\left(\sigma^\mu \partial_\mu \phi\right)\sigma^\mu = -\frac{imc}{\hbar}\sigma^1\phi \quad -(17)$$

In eqs. (16) and (17), summation over the repeated indices is implied inside the brackets, with:

$$\boxed{\mu = 0, 3.} \quad -(18)$$

Eq. (17) is similar to the first order Dirac equation:

$$\gamma^\mu \partial_\mu \phi = -\frac{imc}{\hbar}\phi \quad -(19)$$

but eq. (17) is one in 2×2 matrices only.

Proof of Eq. (11)

$$\left(\sigma^3 \hat{p}_3 \phi\right)\sigma^3 = \left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\hat{p}_3\begin{bmatrix}\phi_1^R & \phi_2^R\\ \phi_1^L & \phi_2^L\end{bmatrix}\right)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$= \hat{p}^3\begin{bmatrix}\phi_1^R & \phi_2^R\\ -\phi_1^L & -\phi_2^L\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \hat{p}^3\begin{bmatrix}\phi_1^R & -\phi_2^R\\ -\phi_1^L & \phi_2^L\end{bmatrix} \quad -(20)$$

$$\sigma^3\left(\hat{p}_3\phi\sigma^3\right) = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\left(\hat{p}_3\begin{bmatrix}\phi_1^R & \phi_2^R\\ \phi_1^L & \phi_2^L\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)$$

$$= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\hat{p}_3\begin{bmatrix}\phi_1^R & -\phi_2^R\\ \phi_1^L & -\phi_2^L\end{bmatrix} = \hat{p}^3\begin{bmatrix}\phi_1^R & -\phi_2^R\\ -\phi_1^L & \phi_2^L\end{bmatrix} \quad -(21)$$

Q.E.D.

) Spin - orbit interaction is intrinsic to the fermion
equation as can be seen from the identity:

$$\underline{\sigma} \cdot \underline{\hat{p}} = \left(\underline{\sigma} \cdot \frac{\underline{r}}{r} \right) \left(\frac{\underline{r}}{r} \cdot \underline{\hat{p}} + i \frac{\underline{\sigma} \cdot \underline{L}}{r} \right) \quad -(22)$$

where

$$\underline{\hat{p}} = -i\hbar \underline{\nabla} \quad -(23)$$

Proof of Eq. (22)

use:

$$(\underline{\sigma} \cdot \underline{r})(\underline{\sigma} \cdot \underline{p}) = \underline{r} \cdot \underline{p} + i \underline{\sigma} \cdot \underline{r} \times \underline{p}$$
$$= \underline{r} \cdot \underline{p} + i \underline{\sigma} \cdot \underline{L} \quad -(24)$$

and

$$(\underline{\sigma} \cdot \underline{r})(\underline{\sigma} \cdot \underline{r}) = r^2 \quad -(25)$$

So:

$$(\underline{\sigma} \cdot \underline{r})(\underline{\sigma} \cdot \underline{r})(\underline{\sigma} \cdot \underline{p}) = r^2 \underline{\sigma} \cdot \underline{p} \quad -(26)$$
$$= \underline{\sigma} \cdot \underline{r} \left(\underline{r} \cdot \underline{p} + i \underline{\sigma} \cdot \underline{L} \right)$$

and eq. (22) follows, Q.E.D.

Therefore:

$$\sigma^3 \hat{p}_Z = \sigma^3 \frac{r_Z}{r} \left(\frac{r_Z}{r} \hat{p}_Z + i \sigma^3 \frac{\hat{L}_Z}{r_Z} \right)$$
$$= \sigma^3 \left(\hat{p}_Z + i \frac{\hat{L}_Z}{r_Z} \right) \quad -(27)$$

i.e.

$$\sigma^3 \hat{p}_Z = \sigma^3 \left(\hat{p}_Z + i \frac{\hat{L}_Z}{Z} \right) \quad -(28)$$

Eq. (15) is therefore:

$$\left(\sigma^0 \hat{p}_0 \phi\right)\sigma^0 - \left(\sigma^3 \hat{p}_Z \phi\right)\sigma^3 = mc\sigma^1\phi \quad -(29)$$

or
$$\hat{p}_0 \phi = \left(\sigma^3 \hat{p}_Z \phi\right)\sigma^3 + mc\sigma^1\phi$$

$$\boxed{\hat{p}_0 \phi = \left(\sigma^3\left(\hat{p}_Z + i\frac{\hat{L}_Z}{Z}\right)\phi\right)\sigma^3 + mc\sigma^1\phi}$$

$-(30)$

In position representation:

$$i\hbar\frac{\partial\phi}{\partial t} = c\left(\sigma^3\left(\frac{\hbar}{i}\frac{\partial}{\partial Z} + i\frac{\hat{L}_Z}{Z}\right)\phi\right)\sigma^3 + mc^2\sigma^1\phi$$

$-(31)$

Written out in full, eq. (31) is:

$$i\hbar\frac{\partial\phi}{\partial t}\begin{bmatrix}1&0\\0&1\end{bmatrix} = c\left(\begin{bmatrix}1&0\\0&-1\end{bmatrix}\left(\frac{\hbar}{i}\frac{\partial}{\partial Z} + i\frac{\hat{L}_Z}{Z}\right)\begin{bmatrix}\phi_1^R&\phi_2^R\\\phi_1^L&\phi_2^L\end{bmatrix}\right)\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$
$$+ mc^2\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}\phi_1^R&\phi_2^R\\\phi_1^L&\phi_2^L\end{bmatrix} \quad -(32)$$

i.e.
$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix}\phi_1^R&\phi_2^R\\\phi_1^L&\phi_2^L\end{bmatrix} = c\left(\frac{\hbar}{i}\frac{\partial}{\partial Z} + i\frac{\hat{L}_Z}{Z}\right)\begin{bmatrix}\phi_1^R&-\phi_2^R\\-\phi_1^L&\phi_2^L\end{bmatrix}$$
$$+ mc^2\begin{bmatrix}\phi_1^L&\phi_2^L\\\phi_1^R&\phi_2^R\end{bmatrix} \quad -(33)$$

This gives four equations:

$$i\hbar \frac{\partial \phi_1^R}{\partial t} - \left(\frac{c\hbar}{i}\frac{\partial}{\partial Z} + i\frac{\hat{L}_Z}{Z}\right)\phi_1^R = mc^2 \phi_1^L \quad -(34)$$

$$i\hbar \frac{\partial \phi_2^R}{\partial t} + \left(\frac{c\hbar}{i}\frac{\partial}{\partial Z} + i\frac{\hat{L}_Z}{Z}\right)\phi_2^R = mc^2 \phi_2^L \quad -(35)$$

$$i\hbar \frac{\partial \phi_1^L}{\partial t} + \left(\frac{c\hbar}{i}\frac{\partial}{\partial Z} + i\frac{\hat{L}_Z}{Z}\right)\phi_1^L = mc^2 \phi_1^R \quad -(36)$$

$$i\hbar \frac{\partial \phi_2^L}{\partial t} - \left(\frac{c\hbar}{i}\frac{\partial}{\partial Z} + i\frac{\hat{L}_Z}{Z}\right)\phi_2^L = mc^2 \phi_2^R \quad -(37)$$

These may be used to describe the free fermion and its interaction with other fields.

Note carefully that eqns. (34) to (37) are valid in the general spacetime and are generally covariant.
