172(4): Eigenvalues of the ECE Fermion Equation for the Static Fermion.

In this case the ECE fermion equation is:

$$\sigma^0 \hat{E} \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} = \sigma^1 mc^2 \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} \quad -(1)$$

i.e.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \hat{E} \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} mc^2 \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_2^L & \phi_2^L \end{bmatrix} \quad -(2)$$

which is:

$$\left.\begin{aligned} \hat{E}\phi_1^R &= mc^2 \phi_1^L \\ \hat{E}\phi_2^R &= mc^2 \phi_2^L \\ \hat{E}\phi_1^L &= mc^2 \phi_1^R \\ \hat{E}\phi_2^L &= mc^2 \phi_2^R \end{aligned}\right\} \quad -(3)$$

with

$$\phi^R(0) = \phi^L(0) \quad -(4)$$

so

$$\left.\begin{aligned} \hat{E}\phi_1^R &= mc^2 \phi_1^R \\ \hat{E}\phi_1^L &= mc^2 \phi_1^L \\ \hat{E}\phi_2^R &= mc^2 \phi_2^R \\ \hat{E}\phi_2^L &= mc^2 \phi_2^L \end{aligned}\right\} \quad -(5)$$

with

$$\hat{E} = i\hbar \frac{\partial}{\partial t} \quad -(6)$$

It is seen that there are no negative energy states, and

$$\phi_1^R = \phi_2^R = \phi_1^L = \phi_2^L = e^{-i\omega_0 t} \quad -(7)$$

where

$$\omega_0 = \frac{mc^2}{\hbar} \quad -(8)$$

2) is the rest angular frequency of the fermion.

The Dirac equation of the rest fermion in chiral representation is:

$$\gamma^0 \hat{E} \psi = mc^2 \psi \quad -(9)$$

i.e.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \hat{E} \begin{bmatrix} \psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L \end{bmatrix} = mc^2 \begin{bmatrix} \psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L \end{bmatrix} \quad -(10)$$

which gives eqs. (3) again, but in a more complicated way than the ECE fermion equation. The latter is preferred by Ockham's Razor.

The eigenfunction in eq. (1) is the tetrad:

$$\psi^a_\mu = \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} \quad -(11)$$

defined by

$$\psi^a = \psi^a_\mu \psi^\mu \quad -(12)$$

where:

$$\psi^a = \begin{bmatrix} \psi^R \\ \psi^L \end{bmatrix}, \quad \psi^\mu = \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix}. \quad -(13)$$

Therefore:

$$\begin{bmatrix} \psi^R \\ \psi^L \end{bmatrix} = e^{-i\omega_0 t} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} \quad -(14)$$

$$= e^{-i\omega_0 t} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix}$$

3) i.e.

$$\begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix} = e^{-i\omega_0 t} \left(\sigma^0 + \sigma^1 \right) \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \quad -(15)$$

where the σ^0 and σ^1 Pauli matrices are:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad -(16)$$

Note that these are the same Pauli matrices as those in eq. (1). Therefore:

$$\boxed{q_\mu^a = e^{-i\omega_0 t} \left(\sigma^0 + \sigma^1 \right)} \quad -(17)$$

In order to find an expression for the basic spinor

$$\phi = \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \quad -(18)$$

the SU(2) representation space is used. This rep space is defined by:

$$u^+ = u^{-1}, \quad \det u = 1. \quad -(19)$$

i.e.

$$u = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}, \quad |a|^2 + |b^2| = 1. \quad -(20)$$

If

$$H = \begin{bmatrix} \phi^1\phi^2 & -\phi^1\phi^1 \\ \phi^2\phi^2 & -\phi^1\phi^2 \end{bmatrix} \quad -(21)$$

then:

$$H \rightarrow u H u^+ \quad -(22)$$

under SU(2) transformation.

4) The Pauli matrices may be used to construct a traceless 2×2 matrix transforming under SU(2) like H :

$$h = \underline{\sigma} \cdot \underline{r} = \begin{bmatrix} Z & X - iY \\ X + iY & -Z \end{bmatrix} \quad - (23)$$

and this is the helicity of the fermion. It is opposite in sign for the two fermion spins, right handed and left handed. So helicity is a concept like chirality or mirror image or handedness. It is seen that

$$\det h = \det h' = X^2 + Y^2 + Z^2 = X'^2 + Y'^2 + Z'^2 \quad - (24)$$

when:

$$h \rightarrow U h U^+ = h'$$

An SU(2) transformation on $\begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix}$ is equivalent to an O(3) transformation on $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$.

A possible choice of H is :

$$H = \begin{bmatrix} \phi^1 \phi^2 & -\phi^1 \phi^1 \\ \phi^2 \phi^2 & -\phi^1 \phi^2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & -(1-i) \\ 1+i & -\sqrt{2} \end{bmatrix} \quad - (25)$$

i.e.

$$\boxed{\begin{aligned} \phi^1 &= (1-i)^{1/2} \\ \phi^2 &= (1+i)^{1/2} \end{aligned}} \quad - (26)$$

5) Using de Moivre's Theorem:

$$z^n = \cos n\theta + i \sin n\theta \quad -(27)$$

and

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right) -(28)$$

then

$$\psi^2 = (1+i)^{1/2} = 2^{1/4}\left(\cos\frac{\pi}{8} + i \sin\frac{\pi}{8}\right). -(29)$$

Using

$$\psi^1 \psi^2 = \sqrt{2} \quad -(30)$$

then

$$\psi^1 = (1-i)^{1/2} = 2^{1/4}\left(\cos\frac{\pi}{8} - i \sin\frac{\pi}{8}\right). -(31)$$

By definition:

$$\begin{bmatrix} \psi^R \\ \psi^L \end{bmatrix} = \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} \quad -(32)$$

so

$$\begin{bmatrix} \psi^R \\ \psi^L \end{bmatrix} = e^{-i\omega t} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (1-i)^{1/2} \\ (1+i)^{1/2} \end{bmatrix} -(33)$$

so

$$\boxed{\begin{aligned} \psi^R = \psi^L &= e^{-i\omega t}\left((1-i)^{1/2} + (1+i)^{1/2}\right) \\ &= 2^{5/4} \cos\frac{\pi}{8}\, e^{-i\omega t} \end{aligned}}$$

When the fermion starts to move in Z, its antifermion becomes distinguishable. This will be the subject of the next note.