172(10): Derivation of the Thomas Factor and Spin Orbit Coupling from the Fermion Equation.

Start with the fermion equation in the format of eq. (10) of note 172(6):

$$((E - e\phi)^2 - m^2c^4)\phi^R = c^2(\underline{\sigma}\cdot\underline{\Pi})(\underline{\sigma}\cdot\underline{\Pi})\phi^R \quad -(1)$$

where

$$\underline{\Pi} = \underline{p} - e\underline{A} \quad -(2)$$

Consider the case:

$$\underline{A} = \underline{0}, \quad \phi \neq 0 \quad -(3)$$

then:

$$\frac{1}{2m}(\underline{\sigma}\cdot\underline{p}\,\underline{\sigma}\cdot\underline{p})\phi^R = \frac{1}{2mc^2}((E - e\phi)^2 - m^2c^4)\phi^R$$
$$= \frac{1}{2mc^2}(c^2p^2 - 2e\phi E + e^2\phi^2)\phi^R \quad -(4)$$

using

$$E^2 - m^2c^4 = p^2c^2 \quad -(5)$$

So:

$$\frac{p^2}{2m}\phi^R = \left(\frac{1}{2m}(\underline{\sigma}\cdot\underline{p})(\underline{\sigma}\cdot\underline{p}) + \frac{e\phi E}{mc^2} - \frac{e^2\phi^2}{2mc^2}\right)\phi^R \quad -(6)$$

Now use

$$E = \gamma mc^2 \quad -(7)$$

and consider the non-relativistic limit:

$$v \ll c, \quad -(8)$$

in which

$$\gamma \to 1. \quad -(9)$$

2) Then:

$$\frac{p^2}{2m}\phi^R = \left(\frac{1}{2m}(\underline{\sigma}\cdot\underline{p})(\underline{\sigma}\cdot\underline{p}) + e\phi - \frac{e^2\phi^2}{2mc^2}\right) \quad -(10)$$

However: $$\frac{p^2}{2m}\phi^R = \frac{1}{2m}(\underline{\sigma}\cdot\underline{p})(\underline{\sigma}\cdot\underline{p})\phi^R \quad -(11)$$

so $$e\phi = 2mc^2 \quad -(12)$$

or $$\frac{e\phi}{2mc^2} = 1. \quad -(13)$$

Therefore:

$$\frac{p^2}{2m}\phi^R = \left(\frac{1}{2m}(\underline{\sigma}\cdot\underline{p})\frac{e\phi}{2mc^2}(\underline{\sigma}\cdot p)\right)\phi^R \quad -(14)$$

$$= \frac{e}{4m^2c^2}\left((\underline{\sigma}\cdot\underline{p})\phi(\underline{\sigma}\cdot p)\right)\phi^R$$

Referring to notes 146(3), page 7, we have:

$$\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p} = \frac{\hbar}{i}\underline{\sigma}\cdot\underline{\nabla}(\phi\,\underline{\sigma}\cdot\underline{p})$$

$$= \frac{\hbar}{i}\underline{\sigma}\cdot\left((\underline{\nabla}\phi)\underline{\sigma}\cdot\underline{p} + \phi(\underline{\sigma}\cdot\underline{p})(\underline{\sigma}\cdot\underline{p}\right)$$

$$= -\frac{\hbar}{i}(\underline{\sigma}\cdot\underline{E})(\underline{\sigma}\cdot\underline{p}) + \phi p^2 \quad -(15)$$

3) where

$$\underline{E} = -\underline{\nabla}\phi \quad -(16)$$

is the electric field strength in volts m^{-1}.

Using the algebra of Pauli matrices:

$$(\underline{\sigma}\cdot\underline{E})(\underline{\sigma}\cdot\underline{p}) = \underline{E}\cdot\underline{p} + i\underline{\sigma}\cdot\underline{E}\times\underline{p} \quad -(17)$$

So:

$$\underline{\sigma}\cdot\underline{p}\,\phi\,\underline{\sigma}\cdot\underline{p} = i\hbar\,\underline{E}\cdot\underline{p} - \hbar\,\underline{\sigma}\cdot\underline{E}\times\underline{p} + \phi p^2 \quad -(18)$$

The spin-orbit term is:

$$\boxed{\hat{H}_{so} = -\frac{e\hbar}{4m^2c^2}\,\underline{\sigma}\cdot\underline{E}\times\underline{p}} \quad -(19)$$

Eq. (19) is usually written as:

$$\hat{H}_{so} = -\frac{e\,\underline{S}\cdot\underline{E}\times\underline{p}}{2m^2c^2} \quad -(20)$$

where

$$\underline{S} = \frac{1}{2}\hbar\,\underline{\sigma} \quad -(21)$$

The Thomas factor is the extra factor 2 in denominator of eq. (20).

4) If we consider a electric field of the type:

$$\underline{E} = -\underline{\nabla}\phi = -\frac{e}{4\pi\epsilon_0}\frac{\underline{r}}{r^3}, \quad -(22)$$

$$\phi = -\frac{e}{4\pi\epsilon_0 r}, \quad -(23)$$

then:

$$\hat{H}_{SO} = -\left(\frac{e^2}{4\pi\epsilon_0 r^3}\right)\frac{\underline{S}\cdot\underline{L}}{2m^2c^2} \quad -(24)$$

where

$$\underline{L} = \underline{r}\times\underline{p} \quad -(25)$$