1) 172(5): Simple Solution of the ECE Fermion Equation

Consider for simplicity motion in the Z axis, then the ECE fermion equation is:

$$\sigma^0 \hat{E}\phi - c\hat{p}_Z \sigma^3 \phi \sigma^3 = \sigma^1 mc^2 \phi \quad -(1)$$

in which:

$$\hat{E} = i\hbar \frac{\partial}{\partial t}, \quad \hat{p}_Z = -i\hbar \frac{\partial}{\partial Z} \quad -(2)$$

and

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad -(3)$$

The eigenfunction ϕ is a tetrad:

$$\phi = \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix}. \quad -(4)$$

Eq. (1) may be solved analytically as follows. First note that:

$$\sigma^1 \phi = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} = \begin{bmatrix} \phi_1^L & \phi_2^L \\ \phi_1^R & \phi_2^R \end{bmatrix}, \quad -(5)$$

and

$$\sigma^3 \phi \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad -(6)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_1^R & -\phi_2^R \\ \phi_1^L & -\phi_2^L \end{bmatrix} = \begin{bmatrix} \phi_1^R & -\phi_2^R \\ -\phi_1^L & \phi_2^L \end{bmatrix} \quad -(7)$$

so eq. (1) is:

$$\hat{E} \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} - c\hat{p}_Z \begin{bmatrix} \phi_1^R & -\phi_2^R \\ -\phi_1^L & \phi_2^L \end{bmatrix} = mc^2 \begin{bmatrix} \phi_1^L & \phi_2^L \\ \phi_1^R & \phi_2^R \end{bmatrix} \quad -(8)$$

i.e.

$$\hat{E}\phi_1^R - c\hat{p}_Z \phi_1^R = mc^2 \phi_1^L \quad -(9)$$

$$\hat{E}\phi_2^R + c\hat{p}_Z \phi_2^R = mc^2 \phi_2^L \quad -(10)$$

$$\hat{E}\phi_1^L + c\hat{p}_Z \phi_1^L = mc^2 \phi_1^R \quad -(11)$$

$$\hat{E}\phi_2^L - c\hat{p}_Z \phi_2^L = mc^2 \phi_2^R \quad -(12)$$

) The Dirac equation corresponding to eq. (1) is:

$$\left(\gamma^0 \hat{E} + c\gamma^3 \hat{p}_3\right)\psi_D = mc^2 \psi_D \quad -(13)$$

where:

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^3 = \begin{bmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{bmatrix} \quad -(14)$$

and

$$p_3 = -p_Z. \quad -(15)$$

Here ψ_D is the Dirac spinor. Written out in full, eq. (13) is:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \hat{E} \begin{bmatrix} \psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L \end{bmatrix} - c \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \hat{p}_Z \begin{bmatrix} \psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L \end{bmatrix} = mc^2 \begin{bmatrix} \psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L \end{bmatrix} \quad -(16)$$

which gives eqns. (9) to (12) again.

The advantage of eq. (1) is that it can be written in the same overall structure for the general spacetime, and uses 2 × 2 matrices only. On the other hand the Dirac eq. (13) uses 4 × 4 matrices which are not entirely derivable. Eq. (13) is the chiral representation - there is also a standard representation.

The structure of eq. (1) is:

$$\boxed{\sigma^0 \hat{p}_0 \psi \sigma^0 + \sigma^3 \hat{p}_3 \psi \sigma^3 = \sigma^1 mc\psi} \quad -(17)$$

where

$$\hat{p}_0 = \hat{E}/c \quad -(18)$$

and is a type of tensor product on the LHS of Pauli matrices, which are basis elements. Eq. (17)

Therefore uses only the well defined Pauli matrices:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad -(19)$$

and in eq. (17) the eigenvalue mc^2 is always positive.

Eq. (17) can be developed greatly for quantum field theory, particle theory and quantum electrodynamics.

By inspection, a simple solution of the set of equations (9) to (12) can be found. First consider equations (9) and (11) in position representation:

$$\left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial Z} \right) \phi_1^R = -i\omega_0 \phi_1^L \quad -(19)$$

$$\left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial Z} \right) \phi_1^L = -i\omega_0 \phi_1^R \quad -(20)$$

$$\omega_0 = mc^2 / \hbar \quad -(21)$$

where

is the rest frequency of the fermion.

If:

$$\phi_1^R = \exp\left(-i\left(\omega_0 t + \kappa Z\right)\right) \quad -(22)$$

then:

$$\phi_1^R = \frac{\omega_0}{\omega_0 + c\kappa} \phi_1^L \quad -(23)$$

$$\phi_1^L = \frac{\omega_0 + c\kappa}{\omega_0} \phi_1^R \quad -(24)$$

from eq. (19).

If

$$\phi_1^L = \exp\left(i\left(\omega_0 t + \kappa Z\right)\right) \quad -(25)$$

i) then

$$\phi_1^L = \frac{\omega_0}{ck - \omega_0}\phi_1^R \quad -(26)$$

from eq. (20). From eqs. (24) and (26):

$$\frac{\omega_0 + ck}{\omega_0} = \frac{\omega_0}{ck - \omega_0} \quad -(27)$$

so

$$\omega_0^2 = (ck + \omega_0)(ck - \omega_0) \quad -(28)$$

i.e.

$$2\omega_0^2 = c^2 k^2 \quad -(29)$$

$$\boxed{\omega_0 = \left(\frac{c}{\sqrt{2}}\right) k} \quad -(30)$$

This is a possible solution with:

$$\boxed{\begin{aligned} \phi_1^R &= e^{-i\phi} \\ \phi_1^L &= e^{i\phi} \\ \phi &= \omega_0 t + kZ \end{aligned}} \quad -(31)$$

Another solution is:

$$\left.\begin{aligned} \phi_1^R &= \exp(-i(\omega_0 t - kZ)) \\ \phi_1^L &= \exp(i(\omega_0 t - kZ)) \end{aligned}\right] -(32)$$

which again gives eq. (30).

For these solutions the real parts of ϕ_1^R and ϕ_1^L are the same, and from eqs. (9) to (12) it is seen that the energy eigenvalue mc^2 is always positive

Anti-Fermion Equations

This is found by the parity operations:

$\hat{P}(p_z) = -p_z$; $\hat{P}(\psi_1^R) = \psi_1^L$; $\hat{P}(\psi_2^R) = \psi_2^L$;
$\hat{P}(\psi_1^L) = \psi_1^R$; $\hat{P}(\psi_2^L) = \psi_2^R$. So it is found that:

$$\hat{P}\begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} = \begin{bmatrix} \psi_1^L & \psi_2^L \\ \psi_1^R & \psi_2^R \end{bmatrix} \quad -(33)$$

i.e.

$$\boxed{\hat{P}\psi = \sigma^1\psi} \quad -(34)$$

so the ECE fermion equation (17) is:

$$\boxed{\sigma^0 \hat{p}_0 \psi \sigma^0 + \sigma^3 \hat{p}_3 \psi \sigma^3 = mc \underline{\hat{P}} \psi} \quad -(35)$$

The ECE anti-fermion equation is: $\quad -(36)$

$$\hat{E}\begin{bmatrix} \psi_1^L & \psi_2^L \\ \psi_1^R & \psi_2^R \end{bmatrix} + c\hat{p}_z \begin{bmatrix} \psi_1^L & -\psi_2^L \\ -\psi_1^R & \psi_2^R \end{bmatrix} = mc^2 \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix}$$

i.e.

$$\boxed{\sigma^0 \hat{p}_0 (\underline{\hat{P}}\psi)\sigma^0 - \sigma^3 \hat{p}_3 (\underline{\hat{P}}\psi)\sigma^3 = mc\psi} \quad -(37)$$

The energy eigenvalue mc^2 is again positive.
