1/ 172(9) : Probability Current of the Fermion Equation

This is defined as:

$$\boxed{j^\mu = \text{Trace}\,(\phi\,\sigma^\mu\,\phi^+)} \quad -(1)$$

Note carefully that the Pauli matrices used in the fermion equation are:

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad -(2)$$

with:

$$\left[\frac{\sigma^1}{2}, \frac{\sigma^2}{2}\right] = i\,\frac{\sigma^3}{2} \quad -(3)$$

$$\left[\frac{\sigma^3}{2}, \frac{\sigma^1}{2}\right] = -i\,\frac{\sigma^2}{2} \quad -(4)$$

$$\left[\frac{\sigma^2}{2}, \frac{\sigma^3}{2}\right] = i\,\frac{\sigma^1}{2} \quad -(5)$$

so the ECE Fermion equation is:

$$\sigma^0 \hat{E}\,\phi\,\sigma^0 + c\,\sigma^3\left(\hat{p}_i\,\phi\,\sigma^i\right) = mc^2\,\sigma^1\phi \quad -(6)$$

where

$$\hat{p}_i\,\phi\,\sigma^i = \hat{p}_1\,\phi\,\sigma^1 + \hat{p}_2\,\phi\,\sigma^2 + \hat{p}_3\,\phi\,\sigma^3 \quad -(7)$$

The σ^2 Pauli matrix is the opposite sign of the usual Pauli matrix. This means that the Pauli matrix basis set is the mirror image of the usual one.

In eq. (1):

$$\phi = \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix}, \quad \phi^+ = \begin{bmatrix} \phi_1^{R*} & \phi_1^{L*} \\ \phi_2^{R*} & \phi_2^{L*} \end{bmatrix} \quad -(8)$$

where $*$ denotes complex conjugate. Therefore:

$$j^0 = \text{Trace}\,(\phi\,\sigma^0\,\phi^+)$$

$$= \phi_1^R \phi_1^{R*} + \phi_2^R \phi_2^{R*} \quad -(9)$$

$$+ \phi_1^L \phi_1^{L*} + \phi_2^L \phi_2^{L*}$$

which is always positive as required of a probability in the Born interpretation. It is seen that j^0 is conserved:

$$\partial_0 j^0 = 0 \quad -(10)$$

and this is true in a general spacetime.

If we investigate:

$$\partial_\mu j^\mu = \text{Trace}\,\partial_\mu(\phi\,\sigma^\mu\,\phi^+) \quad -(11)$$

it is seen that:

$$\partial_\mu j^\mu = \text{Trace}\left((\partial_\mu\phi)(\sigma^\mu\phi^+) + \phi\,\partial_\mu(\sigma^\mu\phi^+)\right) \quad -(12)$$

$$= \text{Trace}\left(\sigma^\mu\left((\partial_\mu\phi)\phi^+ + \phi(\partial_\mu\phi^+)\right)\right)$$

If $\phi = \phi(0)\,e^{-i\phi}, \quad \phi^+ = \phi^+(0)\,e^{i\phi} \quad -(13)$

then $\boxed{\partial_\mu j^\mu = 0} \quad -(14)$

3) and the probability current is conserved.

The results (9) and (14) are the same as that for the Dirac equation in chiral representation, but are obtained using 2×2 matrices. This makes the Dirac γ matrices obsolete. To obtain eq. (9) and (14) from the Dirac equation the procedure is to use:

$$\left(i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right)\psi = 0 \quad -(15)$$

$$\bar{\psi}\left(i\gamma^\mu \overleftarrow{\partial}_\mu + \frac{mc}{\hbar}\right) = 0 \quad -(16)$$

$$j^\mu = \bar{\psi}\gamma^\mu\psi \quad -(17)$$

where

$$\bar{\psi} = \psi^+ \gamma^0 \quad -(18)$$

so

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^+\psi \quad -(19)$$

$$= \begin{bmatrix}\psi_1^{R*} & \psi_2^{R*} & \psi_1^{L*} & \psi_2^{L*}\end{bmatrix}\begin{bmatrix}\psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L\end{bmatrix}$$

$$= \psi_1^R\psi_1^{R*} + \psi_2^R\psi_2^{R*} + \psi_1^L\psi_1^{L*} + \psi_2^L\psi_2^{L*} \quad -(20)$$

The definition (1) has the additional advantage that the trace is of special significance in group theory.