112(1): Chiral and Standard Representations of the Dirac Equation

The chiral rep of the Dirac equation is described by: L. H. Ryder, "Quantum Field Theory" (CUP, 1996, 2nd. ed.), p. 41 ff. It is obtained from group theory. I extended it in many papers and monographs. Consider the two spinors:

$$\phi^R = \begin{bmatrix} \phi_1^R \\ \phi_2^R \end{bmatrix}, \quad \phi^L = \begin{bmatrix} \phi_1^L \\ \phi_2^L \end{bmatrix} \quad -(1)$$

Under the Lorentz transform:

$$\phi^R \rightarrow \exp\left(\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right)\phi^R \quad -(2)$$

$$\phi^L \rightarrow \exp\left(-\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right)\phi^L \quad -(3)$$

Let the original spinors refer to a fermion at rest, then:

$$\phi^R(0) = \phi^L(0). \quad -(4)$$

Eqs. (2) and (3) become:

$$\phi^R(\underline{p}) = \exp\left(\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right)\phi^R(0) \quad -(4)$$

$$\phi^L(\underline{p}) = \exp\left(-\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right)\phi^L(0) \quad -(5)$$

Here:

$$\exp\left(\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right) = \cosh\frac{\phi}{2} + \underline{\sigma}\cdot\underline{n}\,\sinh\frac{\phi}{2} \quad -(6)$$

$$\exp\left(-\frac{1}{2}\underline{\sigma}\cdot\underline{\phi}\right) = \cosh\frac{\phi}{2} - \underline{\sigma}\cdot\underline{n}\,\sinh\frac{\phi}{2} \quad -(7)$$

where $\underline{n}$ is a unit vector in the direction of the Lorentz boost.

By definition:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad \beta = \frac{v}{c} \quad -(8)$$

$$\gamma = \cosh\phi, \quad \gamma\beta = \sinh\phi \quad -(9)$$

) and
$$\cosh^2\frac{\phi}{2} = \frac{1}{2}\left(1+\cosh\phi\right) \quad -(10)$$
$$\sinh^2\frac{\phi}{2} = \frac{1}{2}\left(\cosh\phi - 1\right) \quad -(11)$$

So
$$\cosh^2\frac{\phi}{2} = \frac{1}{2}\left(1+\gamma\right), \quad \sinh^2\frac{\phi}{2} = \frac{1}{2}\left(\gamma-1\right) \quad -(12)$$

Therefore:
$$\phi^R(\underline{p}) = \left[\left(\frac{\gamma+1}{2}\right)^{1/2} + \frac{\underline{\sigma}\cdot\underline{p}}{|\underline{p}|}\left(\frac{\gamma-1}{2}\right)^{1/2}\right]\phi^R(0) \quad -(13)$$

where
$$\gamma = \frac{E}{mc^2} \quad -(14)$$

where E is the total energy. So:
$$\phi^R(\underline{p}) = \left[\left(\frac{1}{2}\left(\frac{E}{mc^2}+1\right)\right)^{1/2} + \frac{\underline{\sigma}\cdot\underline{p}}{|\underline{p}|}\left(\frac{1}{2}\left(\frac{E}{mc^2}-1\right)\right)^{1/2}\right]\phi^R(0)$$
$$= \left[\frac{E + mc^2 + \frac{\underline{\sigma}\cdot\underline{p}}{|\underline{p}|}\left(E^2 - m^2c^4\right)^{1/2}}{\left(2mc^2\left(E+mc^2\right)\right)^{1/2}}\right]\phi^R(0) \quad -(15)$$

Now use:
$$\left(p^2\right)^{1/2} = |\underline{p}| = c\left(E^2 - m^2c^4\right)^{1/2} \quad -(16)$$

so:
$$\phi^R(\underline{p}) = \left[\frac{E + mc^2 + c\,\underline{\sigma}\cdot\underline{p}}{\left(2mc^2\left(E+mc^2\right)\right)^{1/2}}\right]\phi^R(0) \quad -(17)$$

Similarly:
$$\phi^L(\underline{p}) = \left[\frac{E + mc^2 - c\,\underline{\sigma}\cdot\underline{p}}{\left(2mc^2\left(E+mc^2\right)\right)^{1/2}}\right]\phi^L(0) \quad -(18)$$

3) From eqs. (4), (17) and (18):

$$(E + mc^2 - c\underline{\sigma}\cdot\underline{p})\,\phi^R(\underline{p}) = (E + mc^2 + c\underline{\sigma}\cdot\underline{p})\,\phi^L(\underline{p}) \quad -(19)$$

It is seen that:

$$\boxed{\phi^R(\underline{p}) = \left(\frac{E + c\underline{\sigma}\cdot\underline{p}}{mc^2}\right)\phi^L(\underline{p})} \quad -(20)$$

Check

From eq. (20) in eq. (19)

$$E + mc^2 + c\underline{\sigma}\cdot\underline{p} = \frac{(E + mc^2 - c\underline{\sigma}\cdot\underline{p})(E + c\underline{\sigma}\cdot\underline{p})}{mc^2}$$

$$= \frac{(E - c\underline{\sigma}\cdot\underline{p})(E + c\underline{\sigma}\cdot\underline{p})}{mc^2} + \frac{mc^2}{mc^2}(E + c\underline{\sigma}\cdot\underline{p})$$

$$= E^2 - c^2p^2 + E + c\underline{\sigma}\cdot\underline{p}$$

$$= mc^2 + E + c\underline{\sigma}\cdot\underline{p}$$ ✓ Q.E.D.

So:

$$\boxed{\begin{aligned} mc^2\phi^R(\underline{p}) &= (E + c\underline{\sigma}\cdot\underline{p})\,\phi^L(\underline{p}) \\ mc^2\phi^L(\underline{p}) &= (E - c\underline{\sigma}\cdot\underline{p})\,\phi^R(\underline{p}) \end{aligned}} \quad -(21)$$

When the particle is at rest:

$$\left.\begin{aligned} E\phi^L(\underline{p}) &= mc^2\phi^R(\underline{p}) \\ E\phi^R(\underline{p}) &= mc^2\phi^L(\underline{p}) \end{aligned}\right\} \quad -(22)$$

In this chiral representation there is no negative energy. The eigenvalue are $\pm$ of mc^2.

In the original (standard) derivation by Dirac

4) eqns. (22) are:

$$\left.\begin{aligned} E\,\phi^R(\underline{p}) &= mc^2\,\phi^R(\underline{p}) \\ E\,\phi^L(\underline{p}) &= -mc^2\,\phi^L(\underline{p}) \end{aligned}\right\} \quad -(23)$$

and there is an eigenvalue $-mc^2$, or negative energy. This gave rise to the Dirac sea interpretation, which was quickly abandoned in the thirties.

Eqns. (21) may be written as the Dirac equation, but much more incisively as the ECE fermion equation.

1) Dirac Equation

Write eqns. (21) as:

$$\begin{bmatrix} -mc^2 & E + c\underline{\sigma}\cdot\underline{p} \\ E - c\underline{\sigma}\cdot\underline{p} & -mc^2 \end{bmatrix}\begin{bmatrix} \phi^R(\underline{p}) \\ \phi^L(\underline{p}) \end{bmatrix} = 0 \quad -(24)$$

which is

$$(\gamma^\mu p_\mu - mc)\,\phi = 0, \quad -(25)$$

the covariant form of the Dirac equation. Here:

$$\phi = \begin{bmatrix} \phi^R \\ \phi^L \end{bmatrix}, \quad p_\mu = \left(\frac{E}{c}, -\underline{p}\right) \quad -(26)$$

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} \quad -(27)$$

$$\gamma^\mu p_\mu = \gamma^0 p_0 - \gamma^i p_i . \quad -(28)$$

So the Dirac equation is the Lorentz transform of the Pauli spinor as in eqns. (2) and (3). The underlying group theory is that of the Lorentz group extended by parity. So eq. (3) is generated from eq. (2) by the parity operator.

) Note carefully that the concept of negative energy is nowhere used in the derivation of eq. (25) from eqs. (2) and (3).

Written out in full:

$$\phi = \begin{bmatrix} \phi_1^R \\ \phi_2^R \\ \phi_1^L \\ \phi_2^L \end{bmatrix}, \quad \gamma^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad -(29)$$

$$\gamma^1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \gamma^3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad -(30)$$

Note carefully that these 4×4 matrices are different from those used originally by Dirac.

2) ECE Fermion Equation

As shown in note 111(2), this is:

$$\sigma^0 \hat{E} \phi - c \hat{p}_Z \sigma^3 \phi \sigma^3 = \sigma^1 mc^2 \phi \quad -(31)$$

if attention is restricted to the Z axis for clarity and simplicity. In eq. (31):

$$\phi = \begin{bmatrix} \phi_1^R & \phi_2^R \\ \phi_1^L & \phi_2^L \end{bmatrix} := \begin{bmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{bmatrix} \quad -(32)$$

The ECE fermion equation (31) is again the result of a Lorentz transform in special relativity and its eigenvalue mc^2 is positive. In the general spacetime ϕ is a tetrad. So eq. (31) holds for all spacetimes. It is a factorization of:

$$(\square + R) \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} = 0 \quad -(33)$$

i.e. of the ECE wave equation:

$$(\square + R)\, q_\mu^a = 0 \quad -(34)$$

where

$$q_\mu^a = \begin{bmatrix} \psi_1^R & \psi_2^R \\ \psi_1^L & \psi_2^L \end{bmatrix} \quad -(35)$$

In eq. (31) the limit:

$$R = \left(\frac{mc}{\hbar}\right)^2 \quad -(36)$$

has been used.

The Dirac eq. (25) is a factorization of:

$$(\square + R) \begin{bmatrix} \psi_1^R \\ \psi_2^R \\ \psi_1^L \\ \psi_2^L \end{bmatrix} = 0 \quad -(37)$$

which is equivalent to eq. (33).