172(7): The $g = 2$ Factor of the Electron from the Pauli Schroedinger Equation.

It is little known that the g factor of the electron is obtainable from the Schroedinger equation using the Pauli matrices as a basis set. The $g = 2$ factor is in fact an approximation of the Dirac equation in the weak field, non-relativistic limit, as in note 172(6). Consider the hamiltonian operator:

$$\hat{H} = \frac{1}{2m} (\underline{\sigma} \cdot \underline{p}) \cdot (\underline{\sigma} \cdot \underline{p}) + \sigma^0 V \quad -(1)$$

where V is the potential energy. Then the Schroedinger Pauli equation is:

$$\hat{H} \psi = E \psi \quad -(2)$$

In the presence of a static magnetic field:

$$\hat{H} = \frac{1}{2m} \underline{\sigma} \cdot (\underline{p} - e\underline{A}) \underline{\sigma} \cdot (\underline{p} - e\underline{A}) + V\sigma^0$$

$$= \frac{1}{2m} \underline{\sigma} \cdot \underline{\pi} \, \underline{\sigma} \cdot \underline{\pi} + \sigma^0 V \quad -(3)$$

$$= \frac{1}{2m} \left((\underline{p} - e\underline{A})^2 + i\underline{\sigma} \cdot (\underline{p} - e\underline{A}) \times (\underline{p} - e\underline{A}) \right)$$

$$= \frac{1}{2m} \left((\underline{p} - e\underline{A})^2 - ei (\underline{p} \times \underline{A} + \underline{A} \times \underline{p}) \cdot \underline{\sigma} \right) \quad -(3)$$

Now use

$$\underline{p} = -i\hbar \underline{\nabla}. \quad -(4)$$

so

$$(\underline{p} \times \underline{A} + \underline{A} \times \underline{p}) \psi$$

$$= -i\hbar \left(\underline{\nabla} \times (\underline{A}\phi) + \underline{A} \times \underline{\nabla}\phi \right)$$

$$= -i\hbar \left((\underline{\nabla} \times \underline{A})\phi + \underline{\nabla}\phi \times \underline{A} + \underline{A} \times \underline{\nabla}\phi \right)$$

$$= -i\hbar (\underline{\nabla} \times \underline{A})\phi = -i\hbar \underline{B}\phi \quad -(5)$$

So:

$$\hat{H} = \frac{1}{2m}(\underline{p} - e\underline{A})^2 - \left(\frac{e\hbar}{2m}\right)\underline{\sigma}\cdot\underline{B} \quad -(6)$$

and the interaction of the electron with the magnetic field is

$$\boxed{\hat{H}_{int} = -\frac{e\hbar}{2m}\underline{\sigma}\cdot\underline{B}} \quad -(7).$$

which is the same result as obtained from the Dirac eq. in note 172(6).

A charge e circulating in a closed orbit of angular momentum $\underline{l}$ has a magnetic dipole moment:

$$\underline{m} = \frac{e}{2m}\underline{l} \quad -(8)$$

The anomalous Zeeman effect shows that

$$\underline{m} = \frac{e\hbar}{2m}\underline{\sigma} = \frac{e}{m}\underline{S} \quad -(9)$$

where the spin angular momentum is:

$$\underline{S} = \frac{1}{2}\hbar\underline{\sigma} \quad -(10)$$

If the angular momentum $\underline{S}$ behaved in the same way

3) is the angular momentum $\underline{S}$ then:

$$\underline{m} = ? \frac{e}{2m} \underline{S} \quad -(11)$$

but the experimental result is:

$$\underline{m} = 2\left(\frac{e}{2m}\right)\underline{S} = \frac{e\hbar}{2m}\underline{\sigma} \quad -(12)$$

It is often claimed that only the Dirac equation produces this result, but as shown in eq. (7), it is also obtained from the Schroedinger Pauli equation, and from the ECE fermion equation. It is also claimed that only the standard representation of the Dirac equation produces eq. (12), but as shown in note 172(6), so does the chiral representation.

Conclusions

1) The factor 2 in the numerator of eq. (12) is known as the g factor of the electron or Landé factor. It is exactly 2 only in the non-relativistic limit. In the relativistic quantum mechanics it is no longer exactly 2. As in this note it can be derived from the Schroedinger equation with Pauli matrices. As in note 172(6) it can also be derived from the ECE fermion equation, and chiral rep. of the Dirac equation.

2) Negative energy states are eliminated by using the ECE fermion equation.