170(4): RMS Poynting Theorem

The vacuum electric field is defined by:

$$\langle \underline{E}_{vac} \rangle = 0, \quad \langle E_{vac}^2 \rangle^{1/2} \neq 0, \quad -(1)$$

i.e only its root mean square is non-zero. The Poynting theorem:

$$\underline{E}_{vac} \cdot \left(\underline{\nabla} \times \underline{H} - \frac{\partial \underline{D}}{\partial t} \right) = \underline{J} \cdot \underline{E}_{vac} \quad -(2)$$

refers to the instantaneous value of $\underline{E}_{vac}$, i.e. its unaveraged value. Square both side of eq. (2) to obtain:

$$(\underline{J} \cdot \underline{E}_{vac})^2 = \left(\underline{E}_{vac} \cdot \left(\underline{\nabla} \times \underline{H} - \frac{\partial \underline{D}}{\partial t} \right) \right)^2$$

$$= \left(\frac{\partial u}{\partial t} + \underline{\nabla} \cdot \underline{S} \right)^2 \quad -(3)$$

and average over both sides:

$$\boxed{\left\langle \left(\underline{J} \cdot \underline{E}_{vac} \right)^2 \right\rangle^{1/2} = \left\langle \left(\frac{\partial u}{\partial t} + \underline{\nabla} \cdot \underline{S} \right)^2 \right\rangle^{1/2} \quad -(4)}$$

The Stokes theorem for the Poynting vector is developed similarly:

$$\int \underline{\nabla} \cdot \underline{S} \, d^3x = \oint \underline{S} \cdot d\underline{A} \quad -(5)$$

$$\left\langle \left(\int \underline{\nabla} \cdot \underline{S} \, d^3x \right)^2 \right\rangle^{1/2} = \left\langle \left(\oint \underline{S} \cdot d\underline{A} \right)^2 \right\rangle^{1/2} \quad -(6)$$