70(3): Calculation of the Root Mean Square Vacuum Electric Field

From eq. (14) of note 170(1):

$$\langle(\delta \underline{r})^2\rangle_{vac} = \frac{1}{2\epsilon_0\pi^2}\left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar}{mc}\right)^2\log_e\left(\frac{4\epsilon_0\hbar c}{e^2}\right) \quad -(1)$$

$$= \left(\frac{2\alpha}{\pi}\log_e\frac{1}{\pi\alpha}\right)\left(\frac{\hbar}{mc}\right)^2$$

$$= A\lambda_c^2 \quad -(2)$$

where:

$$A = \frac{2\alpha}{\pi}\log_e\frac{1}{\pi\alpha}, \quad \lambda_c = \frac{\hbar}{mc} \quad -(3)$$

and the fine structure constant is:

$$\alpha = \frac{e^2}{4\pi\hbar c\epsilon_0} = 0.007297351$$

$$1/\alpha = 137.0360.$$

So

$$\boxed{\langle(\delta \underline{r})^2\rangle_{vac} = A\lambda_c^2} \quad -(4)$$

We have:

$$\lambda_c = \frac{1}{2\pi}\left(\frac{h}{mc}\right)$$

$$= \frac{2.426309\times10^{-12}}{2\pi}\ m$$

~~$\lambda_c = 3.55144 \times 10^{-12}\ m$~~

$$\boxed{\lambda_c = 3.86159\times10^{-13}\ m} \quad -(5)$$

The electric field is defined by the zero point

2) energy $\epsilon_0 E_{0\kappa}^2 V = \frac{1}{2}\hbar c\kappa = \frac{1}{2}\hbar\omega$ — (6)

so:
$$E_{0\kappa} = \left(\frac{\hbar c\kappa}{2\epsilon_0 V}\right)^{1/2} \quad -(7)$$

where V is the volume of radiation. In analogy with the harmonic oscillator in quantum mechanics eq. (6) denotes the vacuum energy in joules. From eq. (7):

$$E_{0\kappa} = \left(\frac{\hbar c}{2\epsilon_0 V}\right)^{1/2} \kappa^{1/2} \quad -(8)$$

so:
$$E_{0\kappa}^2 = \left(\frac{\hbar c}{2\epsilon_0 V}\right)\kappa \quad -(9)$$

The complete electric field is described by:

$$\underline{E}_\kappa = \underline{E}_{0\kappa}\left(a_\kappa e^{-i\phi} + a_\kappa^* e^{i\phi}\right) \quad -(10)$$

with
$$\delta\underline{r} = \frac{e}{mc^2\kappa^2}\underline{E}_\kappa \quad -(11)$$

and
$$\langle \delta r^2\rangle_{vac} = \sum_\kappa \left(\frac{e}{mc^2\kappa^2}\right)^2 E_{0\kappa} \quad -(12)$$

From eq. (10):
$$\langle E_\kappa^2\rangle^{1/2} = \langle E_{0\kappa}^2\rangle^{1/2} \quad -(13)$$

From eqs. (11) and (13):
$$\langle E_\kappa^2\rangle^{1/2} = \frac{mc^2}{e}\kappa^2 \langle \delta r^2\rangle_{vac}^{1/2} \quad -(14)$$

From eqs. (4) and (14):

$$\langle E_\kappa^2 \rangle^{1/2} = \left(\frac{mc^2}{e} A^{1/2} \lambda_c \right) \kappa^2 \quad -(15)$$

From eq. (9):

$$\kappa^2 = \left(\frac{2\epsilon_0 V}{\hbar c} \right) \langle E_\kappa^2 \rangle \quad -(16)$$

so from eqs. (15) and (16):

$$\boxed{\langle E_\kappa^2 \rangle^{1/2} = B^{-1/3}} \quad -(17)$$

where

$$B = \frac{mc^2}{e} A^{1/2} \lambda_c \left(\frac{2\epsilon_0 V}{\hbar c} \right)^2, \quad -(18)$$

$$A = \frac{2\alpha}{\pi} \log_e \frac{1}{\pi\alpha}, \quad -(19)$$

$$\alpha = \frac{e^2}{4\pi\hbar c \epsilon_0} = 0.007297351 \quad -(20)$$

$$1/\alpha = 137.0360 \quad -(21)$$

Therefore

$$A = 0.01754 \quad -(22)$$

From eqs. (5) and (22):

$$\boxed{\begin{aligned} \langle \delta_r^2 \rangle_{vac}^{1/2} &= A^{1/2} \lambda_c \\ &= 5.114 \times 10^{-14} \text{ m} \end{aligned}} \quad -(23)$$

This compares with the radius of the lowest energy ..t of the H atom, the Bohr radius.

4)
$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 5.29177\times10^{-11}\,m \quad -(24)$$

Therefore:
$$B = 5.114\times10^{-14}\,\frac{mc^2}{e}\left(\frac{2\epsilon_0 V}{\hbar c}\right)^2 = 13.135\times10^{21}$$

and
$$\boxed{\langle E_{1c}^2\rangle^{1/2} = B^{-1/3} = 4.238\times10^{-8}\ \text{volts m}^{-1} \quad \text{for } V = 1.0\,m^3 \quad -(25)}$$

In the H atom the change in potential due to the vacuum electric field is:
$$\Delta V = \frac{1}{6}\langle\delta r^2\rangle_{vac}\left\langle\nabla^2\left(-\frac{e^2}{4\pi\epsilon_0 r}\right)\right\rangle_{atom}$$
$$= \left(\frac{1}{6}\langle\delta r^2\rangle_{vac}\cdot\frac{1}{2a_0}\right)\frac{e^2}{4\pi\epsilon_0 a_0^2} \quad -(26)$$

where a_0 is the Bohr radius given by eq. (24). Denote:
$$V = \frac{e^2}{4\pi\epsilon_0 a_0^2} \quad -(27)$$

then
$$\boxed{\frac{\Delta V}{V} = \frac{1}{12}\frac{\langle\delta r^2\rangle^{1/2}}{a_0} = 8.0534\times10^{-7}} \quad -(28)$$

To adapt this theory for a circuit we

$$\Delta V = \frac{1}{6} \langle \delta r^2 \rangle_{vac} \nabla^2 \left(\frac{-e^2}{4\pi\epsilon_0 r} \right) \quad -(29)$$

where

$$\langle \delta r^2 \rangle_{vac}^{1/2} = 5.114 \times 10^{-14} \text{ metres} \quad -(30)$$

and

$$\nabla^2 \frac{1}{r} = -4\pi \delta(\underline{r}) \quad -(31)$$

The Dirac delta function is defined by: —(32)

$$\int_V \delta(\underline{x} - \underline{X})\, d^3x = 1 \text{ if } \Delta V \text{ contains } \underline{x} = \underline{X}$$

$$= 0 \text{ otherwise.}$$

The delta function has the dimension of inverse volume.

So:

$$\boxed{\Delta V = \frac{1}{6} \langle \delta r^2 \rangle_{vac} \cdot \frac{e^2}{\epsilon_0} \delta(\underline{r})} \quad -(33)$$

This is the change in the Coulombic potential energy due to $\langle \delta r^2 \rangle_{vac}$.

For a given volume (Vol):

$$\Delta V \text{ (joules)} = \frac{1}{6} \langle \delta r^2 \rangle_{vac} \frac{e^2}{\epsilon_0 \, Vol} \quad -(34)$$

so

$$\boxed{\Delta V \text{(joules)} = \frac{1}{6} \langle \delta r^2 \rangle_{vac} \, e \, \underline{\nabla} \cdot \underline{E}}$$

—(35)