1)

167(3): The Metric from Plane Waves

This note is intended as an example of eq. (30) of note 167(2):

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \omega^a_{\mu b} A^b_\nu - \omega^a_{\nu b} A^b_\mu \quad -(1)$$

and:

$$\partial_\mu F^{a\mu\nu} = \mu_0 J^{a\nu} \quad -(2)$$

The simplest case is, for each a:

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} = \begin{bmatrix} 0 & -E_X/c & -E_Y/c & -E_Z/c \\ E_X/c & 0 & -B_Z & B_Y \\ E_Y/c & B_Z & 0 & -B_X \\ E_Z/c & -B_Y & B_X & 0 \end{bmatrix} \quad -(3)$$

in which case:

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad -(4)$$

$$\underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \mu_0 \underline{J} \quad -(5)$$

Consider a metric of the type:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(6)$$

then,

$$g^{00} = 1, \; g^{11} = g^{22} = g^{33} = -1 \quad -(7)$$

all other elements zero. It follows that:

$$F^{01} = g^{00} g^{11} F_{01} = -F_{01} \quad -(8)$$

etc.

$$F^{12} = F_{12} \quad -(9)$$

etc.

2) Therefore:

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_z \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix} \quad -(10)$$

The Hodge dual is:

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \quad -(11)$$

$$= \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{bmatrix} \quad -(12)$$

and

$$\tilde{F}_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0 \end{bmatrix} \quad -(13)$$

so

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad -(14)$$

gives

$$\underline{\nabla} \cdot \underline{B} = 0 \quad -(15)$$

$$\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(16)$$

In general, the metric is defined by:

$$g_{\mu\nu} = V^a_\mu V^b_\nu \eta_{ab} \quad -(17)$$

3) If the tetrads are complex valued:

$$g_{\mu\nu} = q^a_\mu \left(q^b_\nu\right)^* \eta_{ab} \quad -(18)$$

Now seek solutions of eq. (18) such that:

$$g_{\mu\nu} = \text{diag}\,(1, -1, -1, -1) \quad -(19)$$

$$\eta_{ab} = \text{diag}\,(1, -1, -1, -1) \quad -(20)$$

A possible solution is the plane wave:

$$\underline{q}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) \exp\,(i\phi) \quad -(21)$$

$$\underline{q}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) \exp\,(-i\phi) \quad -(22)$$

in which:

$$\left.\begin{aligned} q^{(1)}_1 &= -q^{(1)}_X = -\frac{1}{\sqrt{2}} e^{i\phi} \\ q^{(1)}_2 &= -q^{(1)}_Y = \frac{i}{\sqrt{2}} e^{i\phi} \\ q^{(2)}_1 &= -q^{(2)}_X = -\frac{1}{\sqrt{2}} e^{-i\phi} \\ q^{(2)}_2 &= -q^{(2)}_Y = -\frac{i}{\sqrt{2}} e^{-i\phi} \end{aligned}\right\} \quad -(23)$$

so:

$$g_{11} = 1 = q^{(1)}_1 \left(q^{(1)}_1\right)^* + q^{(1)}_2 \left(q^{(1)}_2\right)^* \quad -(24)$$

$$g_{22} = 1 = q^{(2)}_1 \left(q^{(2)}_1\right)^* + q^{(2)}_2 \left(q^{(2)}_2\right)^* \quad -(25)$$

The plane wave solution is that of the free space equations:

4)

$$\left.\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= \underline{0} \\ \underline{\nabla} \cdot \underline{E} &= 0 \\ \underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} &= \underline{0} \end{aligned}\right\} \quad -(26)$$

i.e. $$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu F^{\mu\nu} = 0. \quad -(27)$$

The solution is:

$$\underline{E}^{(1)} = \underline{E}^{(2)*} = \frac{E^{(0)}}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) e^{i\phi} \quad -(28)$$

$$\underline{B}^{(1)} = \underline{B}^{(2)*} = \frac{B^{(0)}}{\sqrt{2}} \left(i \underline{i} - \underline{j} \right) e^{i\phi} \quad -(29)$$

In the case of eqs (4) and (5), with finite charge and current, the solution is no longer a plane wave, and is for example the solution of Lienard and Wiechert. If $\underline{P}$ and $\underline{M}$ are neglected, the general solution are, for each a:

$$\underline{D} = \epsilon_0 \left(g^{00} g^{11} E_x \underline{i} + g^{00} g^{22} E_y \underline{j} + g^{00} g^{33} E_Z \underline{k} \right) \quad -(30)$$

$$\underline{H} = \frac{1}{\mu_0} \left(g^{22} g^{33} B_x \underline{i} + g^{11} g^{33} B_y \underline{j} + g^{22} g^{11} B_Z \underline{k} \right) \quad -(31)$$

These equations come from:

$$G^{\mu\nu} = \epsilon_0 c \, g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \quad -(32)$$

5) where:

$$G^{\mu\nu} = \begin{bmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z/c & H_y/c \\ D_y & H_z/c & 0 & -H_x/c \\ D_z & -H_y/c & H_x/c & 0 \end{bmatrix} \quad -(33)$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_z \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix} \quad -(34)$$

so:

$$\left.\begin{aligned} D_x &= -\epsilon_0 E_x, \quad D_y = -\epsilon_0 E_y, \quad D_z = -\epsilon_0 E_z \\ H_x &= \frac{1}{\mu_0} B_x, \quad H_y = \frac{1}{\mu_0} B_y, \quad H_z = \frac{1}{\mu_0} B_z \end{aligned}\right] \quad -(35)$$

if

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \quad -(36)$$

otherwise:

$$\boxed{\left.\begin{aligned} D_x &= \epsilon_0 g^{00} g^{11} E_x, & H_x &= \frac{1}{\mu_0} g^{22} g^{33} B_x, \\ D_y &= \epsilon_0 g^{00} g^{22} E_y, & H_y &= \frac{1}{\mu_0} g^{11} g^{33} B_y, \\ D_z &= \epsilon_0 g^{00} g^{33} E_z, & H_z &= \frac{1}{\mu_0} g^{22} g^{11} B_z. \end{aligned}\right]} \quad -(37)$$

For example, the Helmholtz equation is given by:

$$\boxed{\begin{aligned} g^{00} g^{11} = g^{00} g^{22} = g^{00} g^{33} &= \frac{\epsilon}{\epsilon_0} \\ g^{22} g^{33} = g^{11} g^{33} = g^{22} g^{11} &= \frac{\mu}{\mu_0} \\ \underline{J} = \underline{0}, \quad \rho &= 0 \end{aligned}} \quad -(38)$$

with the metric (36).