167(2): Basic Geometry

Consider the covariant derivative:

$$D_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\lambda} V^\lambda \quad -(1)$$

then its commutator is:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\kappa\mu\nu} V^\kappa - T^\kappa{}_{\mu\nu} D_\kappa V^\rho \quad -(2)$$

where $R^\rho{}_{\kappa\mu\nu}$ is the curvature and $T^\kappa{}_{\mu\nu}$ is the torsion.

By definition:

$$T^\kappa{}_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu}. \quad -(3)$$

Therefore

$$\boxed{\Gamma^\kappa_{\mu\nu} = -\Gamma^\kappa_{\nu\mu}}. \quad -(4)$$

If

$$\mu = \nu \quad -(5)$$

then

$$[D_\mu, D_\nu] = R^\rho{}_{\kappa\mu\nu} = T^\kappa{}_{\mu\nu} = \Gamma^\kappa_{\mu\nu} = 0 \quad -(6)$$

the Hodge dual of $\Gamma^\kappa_{\mu\nu}$ can be defined if $\kappa$ is held constant:

$$\Delta^\kappa_{\mu\nu} = \tilde{\Gamma}^\kappa_{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\mu\nu}{}^{\alpha\beta} \Gamma^\kappa_{\alpha\beta}. \quad -(7)$$

therefore

$$\tilde{T}^\kappa{}_{\mu\nu} = \Delta^\kappa_{\mu\nu} - \Delta^\kappa_{\nu\mu}. \quad -(8)$$

This result is given by:

$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho{}_{\kappa\mu\nu} V^\kappa - \tilde{T}^\kappa{}_{\mu\nu} D_\kappa V^\rho \quad -(9)$$

in which:

$$D_\mu V^\rho = \partial_\mu V^\rho + \Delta^\rho_{\mu\lambda} V^\kappa \quad -(10)$$

Now define the covariant derivative in terms of

2) The spin connection:

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b \quad -(11)$$

The complete vector field is constant so:

$$DV = D_\mu V^\nu dx^\mu \partial_\nu = D_\mu V^a dx^\mu e_a \quad -(12)$$

Using eq. (1) and (11) in (12):

$$\partial_\mu q^a_\nu = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \quad -(13)$$

Using eq. (10) and (11)

$$\partial_\mu q^a_\nu = \tilde{\Gamma}^a_{\mu\nu} - \Omega^a_{\mu\nu} \quad -(14)$$

where $$D_\mu V^a = \partial_\mu V^a + \Omega^a_{\mu b} V^b \quad -(15)$$

Therefore: $$\tilde{\Gamma}^a_{\mu\nu} - \Omega^a_{\mu\nu} = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \quad -(16)$$

So $$\Omega^a_{\mu\nu} = \tilde{\omega}^a_{\mu\nu}. \quad -(17)$$

Here: $$V^a = q^a_\nu V^\nu \quad -(18)$$

and:

$$\Gamma^a_{\mu\nu} = \Gamma^\kappa_{\mu\nu} q^a_\kappa \quad -(19)$$

$$\omega^a_{\mu\nu} = \omega^a_{\mu b} q^b_\nu \quad -(20)$$

$$\tilde{\Gamma}^a_{\mu\nu} = \tilde{\Gamma}^\kappa_{\mu\nu} q^a_\kappa \quad -(21)$$

$$\tilde{\omega}^a_{\mu\nu} = \tilde{\omega}^a_{\mu b} q^b_\nu \quad -(22)$$

Therefore if

3)

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \quad -(23)$$

and $$T^{\lambda}_{\mu\nu} = q^{\lambda}_{a} T^{a}_{\mu\nu} \quad -(24)$$

then $$T^{a}_{\mu\nu} = \partial_\mu q^{a}_{\nu} - \partial_\nu q^{a}_{\mu} + \omega^{a}_{\mu b} q^{b}_{\nu} - \omega^{a}_{\nu b} q^{b}_{\mu} \quad -(25)$$

$$= \left(\partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu b} q^{b}_{\nu}\right) - \left(\partial_\nu q^{a}_{\mu} + \omega^{a}_{\nu b} q^{b}_{\mu}\right)$$

$$= \Gamma^{\lambda}_{\mu\nu} q^{a}_{\lambda} - \Gamma^{\lambda}_{\nu\mu} q^{a}_{\lambda}$$

$$= q^{a}_{\lambda} T^{\lambda}_{\mu\nu} \quad -(26)$$

QED.

Similarly, if

$$\tilde{T}^{\lambda}_{\mu\nu} = \Lambda^{\lambda}_{\mu\nu} - \Lambda^{\lambda}_{\nu\mu} \quad -(27)$$

and $$\tilde{T}^{\lambda}_{\mu\nu} = q^{\lambda}_{a} \tilde{T}^{a}_{\mu\nu} \quad -(28)$$

then: $$\tilde{T}^{a}_{\mu\nu} = \partial_\mu q^{a}_{\nu} - \partial_\nu q^{a}_{\mu} + \tilde{\omega}^{a}_{\mu b} q^{b}_{\nu} - \tilde{\omega}^{a}_{\nu b} q^{b}_{\mu}$$

$$= \Lambda^{\lambda}_{\mu\nu} q^{a}_{\lambda} - \Lambda^{\lambda}_{\nu\mu} q^{a}_{\lambda}$$

$$= q^{a}_{\lambda} \tilde{T}^{\lambda}_{\mu\nu} \quad -(29)$$

QED.

Therefore:

$$\boxed{\begin{aligned} F^{a}_{\mu\nu} &= \partial_\mu A^{a}_{\nu} - \partial_\nu A^{a}_{\mu} + \omega^{a}_{\mu b} A^{b}_{\nu} - \omega^{a}_{\nu b} A^{b}_{\mu} \\ \partial_\mu F^{a\mu\nu} &= \mu_0 J^{a\nu} \end{aligned}} \quad -(30)$$

Similarly:

4)

$$\boxed{\begin{aligned}\tilde{F}^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \tilde{\omega}^a_{\mu b} A^b_\nu - \tilde{\omega}^a_{\nu b} A^b_\mu \\ \partial_\mu \tilde{F}^{a\mu\nu} &= 0\end{aligned}} \quad -(31)$$

In eq. (30):

$$F^{a\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F^a_{\rho\sigma} \quad -(32)$$

In vector notation:

$$\underline{E}^a = -c\underline{\nabla} A^a_0 - \frac{\partial \underline{A}^a}{\partial t} - c\omega^a_{0b}\underline{A}^b + c A^b_0 \underline{\omega}^a{}_b \quad -(33)$$

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad -(34)$$

and

$$\underline{\nabla} \cdot \underline{E}^a = \rho^a / \epsilon_0 \quad -(35)$$

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2}\frac{\partial \underline{E}^a}{\partial t} = \mu_0 \underline{J}^a \quad -(36)$$

In eqns. (35) and (36) the $\underline{E}^a$ and $\underline{B}^a$ fields are:

$$\underline{E}^a = \frac{1}{\epsilon_0}\left(\underline{D}^a - \underline{P}^a\right); \quad \underline{B}^a = \mu_0\left(\underline{H}^a + \underline{M}^a\right) \quad -(37)$$

and must be calculated from the $\underline{E}^a$ and $\underline{B}^a$ fields in eqs. (33) and (34) <u>by using the metric</u>. For example:

$$F^{a01} = g^{00} g^{11} F^a_{01} \quad -(38)$$

and so on. Therefore

$$\frac{1}{\epsilon_0}\left(D^a_X - P^a_X\right) = g^{00} g^{11} E^a_X \quad -(39)$$

$$\frac{1}{\epsilon_0}\left(D_y^a - P_y^a\right) = g^{00} g^{22} E_y^a \quad -(40)$$

$$\frac{1}{\epsilon_0}\left(D_z^a - P_z^a\right) = g^{00} g^{33} E_z^a \quad -(41)$$

$$\mu_0\left(H_x^a + M_x^a\right) = g^{22} g^{33} B_x^a \quad -(42)$$

$$\mu_0\left(H_y^a + M_y^a\right) = g^{11} g^{33} B_y^a \quad -(43)$$

$$\mu_0\left(H_z^a + M_z^a\right) = g^{22} g^{11} B_x^a \quad -(44)$$

If there is no polarization or magnetization:

$$\underline{D}^a = \epsilon_0 \underline{E}^a \quad -(45)$$

$$\underline{H}^a = \frac{1}{\mu_0} \underline{B}^a \quad -(46)$$

and

$$\underline{\nabla} \cdot \underline{D}^a = \rho^a \quad -(47)$$

$$\underline{\nabla} \times \underline{H}^a - \frac{\partial \underline{D}^a}{\partial t} = \underline{J}^a \quad -(48)$$

$$\underline{D}^a = \epsilon_0 \left( g^{00} g^{11} E_x^a \underline{i} + g^{00} g^{22} E_y^a \underline{j} + g^{00} g^{33} E_z^a \underline{k} \right) \quad -(49)$$

$$\underline{H}^a = \frac{1}{\mu_0} \left( g^{22} g^{33} B_x^a \underline{i} + g^{11} g^{33} B_y^a \underline{j} + g^{22} g^{11} B_z^a \underline{k} \right) \quad -(50)$$

with $\underline{E}^a$ and $\underline{B}^a$ defined by eqs. (33) and (34).