167(7): The Hodge Dual of the Tetrad Postulate

The tetrad postulate is defined as:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu\nu} - \Gamma^a_{\mu\nu} = 0 \quad -(1)$$

the covariant derivative of q^a_ν transforms as a tensor, and it is known that:

$$\Gamma^a_{\mu\nu} = -\Gamma^a_{\nu\mu} \quad -(2)$$

so

$$\partial_\mu q^a_\nu + \omega^a_{\mu\nu} = -\left(\partial_\nu q^a_\mu + \omega^a_{\nu\mu}\right). \quad -(3)$$

The Hodge dual of the tensor $D_\mu q^a_\nu$ is:

$$\left(D_\mu q^a_\nu\right)_{HD} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu}{}^{\alpha\beta} D_\alpha q^a_\beta = 0, \quad -(4)$$

where:

$$\left(\partial_\mu q^a_\nu + \omega^a_{\mu\nu}\right)_{HD} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu}{}^{\alpha\beta}\left(\partial_\alpha q^a_\beta + \omega^a_{\alpha\beta}\right) -(5)$$

$$\Lambda^a_{\mu\nu} = \tilde{\Gamma}^a_{\mu\nu} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu}{}^{\alpha\beta}\Gamma^a_{\mu\nu} \quad -(6)$$

Write:

$$\partial_\mu Q^a_\nu + \Omega^a_{\mu\nu} := \left(\partial_\mu q^a_\nu + \omega^a_{\mu\nu}\right)_{HD} \quad -(7)$$

then

$$\boxed{\partial_\mu Q^a_\nu + \Omega^a_{\mu\nu} - \Lambda^a_{\mu\nu} = 0} \quad -(8)$$

which is the Hodge dual tetrad postulate.

The Hodge dual postulate is used in the proof of

$$D \wedge \tilde{T} := \tilde{R} \wedge q. \quad -(9)$$

The Hodge dual torsion is:

$$\tilde{T}^a_{\mu\nu} = \tilde{T}^\kappa_{\mu\nu}\, q^a_\kappa \quad -(10)$$

In tensor notation eq. (9) is:

$$\boxed{D_\mu T^{a\mu\nu} = R^a{}_\mu{}^{\mu\nu}} \quad -(11)$$

The antisymmetric connection is defined by:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\kappa\mu\nu} V^\kappa - T^\kappa_{\mu\nu} D_\kappa V^\rho \quad -(12)$$

where

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu}. \quad -(13)$$

Thus:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\kappa\mu\nu} V^\kappa - \Gamma^\kappa_{\mu\nu} D_\kappa V^\rho + \Gamma^\kappa_{\nu\mu} D_\kappa V^\rho. \quad -(14)$$

By definition:

$$R^\rho{}_{\kappa\mu\nu} = - R^\rho{}_{\kappa\nu\mu} \quad -(15)$$

$$[D_\mu, D_\nu] V^\rho = - [D_\nu, D_\mu] V^\rho \quad -(16)$$

It is seen that there is a one to one correspondence between $[D_\mu, D_\nu]$ and $R^\rho{}_{\kappa\mu\nu}$. This means that the curvature tensor takes the antisymmetry of the commutator by definition.

3)

Therefore the action of the commutator on the vector V^ρ isolates the curvature tensor. In exactly the same way the connection is isolated as follows:

$$[D_\mu, D_\nu] V^\rho = - \Gamma^\kappa_{\mu\nu} D_\kappa V^\rho + \dots \quad -(17)$$

and there is a one to one correspondence between the commutator and connection.

$$\text{If} \quad \mu = \nu \quad -(18)$$

then $$[D_\mu, D_\nu] V^\rho = 0 \quad -(19)$$

and by definition $$R^\rho{}_{\kappa\mu\nu} = 0. \quad -(20)$$

In exactly the same way:

$$\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu} = 0 \quad -(21)$$

if $\mu = \nu$. So: $$\boxed{\Gamma^\kappa_{\mu\nu} = - \Gamma^\kappa_{\nu\mu}} \quad -(22)$$

The basic quantity is the connection, because it defines the covariant derivative:

$$D_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\lambda} V^\lambda \quad -(23)$$

and the torsion is built from the connection:

$$T^\kappa_{\mu\nu} := \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu} \quad -(24)$$

If $\mu = \nu$ the curvature is zero, eq. (20)

so:

4)

$$D_\mu V^\rho = \partial_\mu V^\rho \quad -(25)$$

by definition. So if $\mu = \nu$:

$$\boxed{\Gamma^{\kappa}_{\mu\nu} = 0} \quad -(26)$$

Some authors perpetrate the error that $\Gamma^{\kappa}_{\mu\nu}$ can be symmetric :

$$\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu} \neq ? \; 0. \quad -(27)$$

This is incorrect because when $\mu = \nu$, the curvature is zero and eq. (25) follows. A spacetime with zero curvature is defined by eq. (25).

In eq. (3) on the other hand, the antisymmetry is in the sum, and occurs in the following way:

$$\partial_\mu q^a_\nu + \omega^a_{\mu\nu} = -\left(\partial_\nu q^a_\mu + \omega^a_{\nu\mu}\right) \quad -(28)$$

i.e.

$$\Gamma^a_{\mu\nu} = -\Gamma^a_{\nu\mu} \quad -(29)$$

so the Hodge dual operator applies to the marked indices.
