1) 167(4): Helmholtz Wave Equation and Metric.

The Helmholtz wave equation is obtained for each a from:

$$\underline{\nabla} \cdot \underline{B} = 0, \quad \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = \underline{0} \quad -(1)$$

$$\underline{\nabla} \cdot \underline{D} = 0, \quad \underline{\nabla} \times \underline{H} - \frac{\partial \underline{D}}{\partial t} = \underline{0} \quad -(2)$$

i.e. for a dielectric with no charge current density. The time dependence is assumed to be $\exp(-i\omega t)$, so:

$$\underline{\nabla} \cdot \underline{B} = 0, \quad \underline{\nabla} \times \underline{E} - i\omega \underline{B} = 0 \quad -(3)$$

$$\underline{\nabla} \cdot \underline{D} = 0, \quad \underline{\nabla} \times \underline{H} + i\omega \underline{D} = \underline{0} \quad -(4)$$

The dielectric is represented by:

$$\underline{D} = \epsilon \underline{E}, \quad \underline{B} = \mu \underline{H} \quad -(5)$$

so the Helmholtz wave equation is:

$$(\nabla^2 + \mu\epsilon\omega^2)\underline{E} = \underline{0} \quad -(6)$$

$$(\nabla^2 + \mu\epsilon\omega^2)\underline{B} = \underline{0} \quad -(7)$$

The plane wave solution gives:

$$\kappa = (\epsilon\mu)^{1/2}\omega \quad -(8)$$

The phase velocity is:

$$v_p = \frac{\omega}{\kappa} = \frac{1}{(\mu\epsilon)^{1/2}} = \frac{c}{n} \quad -(9)$$

where the refractive index is:

$$n = \left(\frac{\mu\epsilon}{\mu_0\epsilon_0}\right)^{1/2} \quad -(10)$$

In terms of metric elements the refractive

2) index is:

$$\left.\begin{aligned} n &= \left(g^{00} g^{11} g^{22} g^{33}\right)^{1/2} \\ &= \left(g^{00} g^{22} g^{11} g^{33}\right)^{1/2} \\ &= \ldots \quad \text{etc} \end{aligned}\right\} \quad -(11)$$

Therefore the refractive index of any substance such as glass or water can be expressed in terms of metric elements, with:

$$\left.\begin{aligned} \frac{\epsilon}{\epsilon_0} &= g^{00} g^{11} = g^{00} g^{22} = g^{00} g^{33} \\ \frac{\mu}{\mu_0} &= g^{22} g^{33} = g^{11} g^{33} = g^{22} g^{11} \end{aligned}\right\} \quad -(12)$$

This is the simplest example of the use of the metric in electrodynamics. For free space:

$$\frac{\epsilon}{\epsilon_0} = \frac{\mu}{\mu_0} = 1. \quad -(13)$$

These equations have been obtained from:

$$\begin{aligned} D_x &= \epsilon_0 g^{00} g^{11} E_x, & H_x &= \frac{1}{\mu_0} g^{22} g^{33} B_x, \\ D_y &= \epsilon_0 g^{00} g^{22} E_y, & H_y &= \frac{1}{\mu_0} g^{11} g^{33} B_y, \\ D_z &= \epsilon_0 g^{00} g^{33} E_z, & H_z &= \frac{1}{\mu_0} g^{22} g^{11} B_z. \end{aligned} \quad -(14)$$

In free space:

$$\left.\begin{aligned} D_x &= \epsilon_0 E_x \quad \text{etc.} \\ H_x &= \frac{1}{\mu_0} B_x \quad \text{etc.} \end{aligned}\right\} \quad -(15)$$

according to the usual definition of constitutive relation in nineteenth century electrodynamics. However, in free space:

$$g_{\mu\nu} = g^{\mu\nu} = (1, -1, -1, -1) \quad -(16)$$

so eqns. (14) give:

$$\left. \begin{aligned} D_x &= -\epsilon_0 E_x \quad \text{etc.}, \\ H_x &= \frac{1}{\mu_0} B_x \quad \text{etc.} \end{aligned} \right\} \quad -(17)$$

Therefore in order to obtain consistency with the nineteenth century convention, it is necessary to use the convention:

$$\frac{\epsilon}{\epsilon_0} := -g^{00}g^{11} = -g^{00}g^{22} = -g^{00}g^{33} \quad -(18)$$

$$\frac{\mu}{\mu_0} = g^{22}g^{33} = g^{11}g^{33} = g^{22}g^{11} \quad -(19)$$

The refractive index is then:

$$n := \left(-g^{00}g^{11}g^{22}g^{33}\right)^{1/2} \quad -(20)$$

For the metric (16):

$$n = 1 \quad -(21)$$

because

$$\mu = \mu_0, \quad \epsilon = \epsilon_0 \quad -(22)$$

With this convention, it is possible to define gravitation in terms of a change in refractive index. For example, if the

4) gravitational metric is:

$$g_{\mu\nu} = \mathrm{diag}\left(1 - \frac{r_0}{r}, -\left(1 - \frac{r_0}{r}\right)^{-1}, -1, -1\right) \quad -(23)$$

the inverse metric is:

$$g^{\mu\nu} = \mathrm{diag}\left(\left(1 - \frac{r_0}{r}\right)^{-1}, -\left(1 - \frac{r_0}{r}\right), -1, -1\right) \quad -(24)$$

so the refractive index is defined in a different way. The permittivity and permeability from eqns. (18) and (19) become <u>anisotropic</u>, as does the refractive index.

In ECE theory there is an analogous set of dynamical and gravitational equations to eqns. (1) and (2) of electrodynamics. The combined gravitational and electromagnetic metric must be used in general, and this must be found from experimental data. Knowledge of the combined metric would lead to knowledge of how an electromagnetic field can be used to engineer gravitation.