158(9): Exact Analytical Solution of the Compton Effect with Photon Mass.

Conservation of Energy

$$\gamma mc^2 + Mc^2 = \gamma' mc^2 + \left(M^2c^4 + c^2 p_1^2\right)^{1/2} \quad -(1)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad \gamma' = \left(1 - \frac{v'^2}{c^2}\right)^{-1/2} \quad -(2)$$

m = photon mass, M = electron mass

p_1 = magnitude of final momentum of electron

The electron is initially at rest.

From eq. (1), + plus sign *

$$c^2 p_1^2 = \left(mc^2(\gamma - \gamma') + Mc^2\right)^2 - M^2c^4 \quad -(3)$$

Conservation of Momentum

$$p_1 = \hbar(\kappa - \kappa') \quad -(4)$$

because the initial electron momentum is zero. Therefore:

$$p_1^2 = \hbar^2(\underline{\kappa} - \underline{\kappa}') \cdot (\underline{\kappa} - \underline{\kappa}') = \hbar^2(\kappa^2 + \kappa'^2 - 2\kappa\kappa'\cos\theta) \quad -(5)$$

from vector analysis.

The de Broglie Einstein Equations of 1922-1924

These are:

$$\left.\begin{aligned} \hbar\omega &= \gamma mc^2, & \hbar\kappa &= \gamma mv \\ \hbar\omega' &= \gamma' mc^2, & \hbar\kappa' &= \gamma' mv' \end{aligned}\right\} -(6)$$

2) These have not been used in the theory of the Compton effect to date, and provide a rigorous experimental test of the foundations of physics. Here v is the photon velocity at angular frequency ω, and v' that at ω'. So:

$$\frac{\kappa}{\omega} = \frac{v}{c^2}, \quad \kappa = \frac{\omega v}{c^2} \quad -(7)$$

$$\frac{\kappa'}{\omega'} = \frac{v'}{c^2}, \quad \kappa' = \frac{\omega' v'}{c^2} \quad -(8)$$

and

$$\frac{\omega}{\omega'} = \frac{\gamma}{\gamma'}, \quad \gamma' = \frac{\omega'}{\omega}\gamma \quad -(9)$$

Note carefully that in the standard Compton effect theory:

$$v = v' = c, \quad -(10)$$

$$m = 0. \quad -(11)$$

and

Therefore from eqs (5) to (8):

$$c^2 p_1^2 = \frac{\hbar^2}{c^2}\left(\omega^2 v^2 + \omega'^2 v'^2 - 2\omega\omega' v v' \cos\theta\right) \quad -(12)$$

The electron momentum can now be eliminated between eqs. (3) and (12):

$$\left(mc^2(\gamma - \gamma') + Mc^2\right)^2 - M^2c^4 = \frac{\hbar^2}{c^2}\left(\omega^2 v^2 + \omega'^2 v'^2 - 2\omega\omega' v v' \cos\theta\right) \quad -(13)$$

3) Using eq. (9):

$$\gamma - \gamma' = \Omega \gamma \quad -(14)$$

where $$\Omega = 1 - \frac{\omega'}{\omega} \quad -(15)$$

and eq. (13) reduces to:

$$m\gamma\Omega c^4 \left(m\gamma\Omega + 2M \right) = \frac{\hbar^2}{c^2}\left(\omega^2 v^2 + \omega'^2 v'^2 - 2\omega\omega' v v' \cos\theta \right) \quad -(16)$$

where $$\gamma = \left(1 - \frac{v^2}{c^2} \right)^{-1/2} \quad -(17)$$

Eq. (9) is:

$$1 - \frac{v'^2}{c^2} = \left(\frac{\omega}{\omega'} \right)^2 \left(1 - \frac{v^2}{c^2} \right) \quad -(18)$$

so $$v'^2 = c^2 - \left(\frac{\omega}{\omega'} \right)^2 \left(c^2 - v^2 \right). \quad -(19)$$

Also: $$\frac{\hbar\omega'}{mc^2} = \left(1 - \frac{v'^2}{c^2} \right)^{-1/2} \quad -(20)$$

so $$v' = c\left(1 - \left(\frac{mc^2}{\hbar\omega'} \right)^2 \right)^{1/2} \quad -(21)$$

Eqs. (16) to (21) give a quadratic for v in terms of the photon mass m and experimental observables of the Compton effect.

4) Eq. (16) is:

$$m^2\Omega^2c^4 + 2mM\Omega c^4\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

$$= \frac{\hbar^2}{c^2}\left(1-\frac{v^2}{c^2}\right)\left(\omega^2v^2+\omega'^2v'^2-2\omega\omega'vv'\cos\theta\right) \quad -(22)$$

Now use:

$$\hbar\omega = mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} \quad -(23)$$

i.e

$$\left(1-\frac{v^2}{c^2}\right)^{1/2} = \frac{mc^2}{\hbar\omega} \quad -(24)$$

to find:

$$m^2c^4\Omega^2\left(1+\frac{2Mc^2}{\hbar\omega\Omega}\right) \quad -(25)$$

$$= \frac{\hbar^2}{c^2}\left(\frac{mc^2}{\hbar\omega}\right)^2\left(\omega^2v^2+\omega'^2v'^2-2\omega\omega'vv'\cos\theta\right)$$

i.e.

$$\boxed{\omega^2v^2+\omega'^2v'^2-2\omega\omega'vv'\cos\theta = A} \quad -(26)$$

where

$$A = \Omega^2\omega^2c^2\left(1+\frac{2Mc^2}{\hbar\omega\Omega}\right) \quad -(27)$$

$$\Omega = 1-\frac{\omega'}{\omega} \quad -(28)$$

5) The quantities A and Ω are made up of experimental observables of the Compton effect. Therefore:

$$\omega^2 v^2 + \omega'^2 v'^2 - 2\omega\omega' v v' \cos\theta = A \quad -(26)$$

$$\left(1 - \frac{v'^2}{c^2}\right) = \left(\frac{\omega}{\omega'}\right)^2 \left(1 - \frac{v^2}{c^2}\right) \quad -(27)$$

$$v' = c\left(1 - \left(\frac{mc^2}{\hbar\omega'}\right)^2\right)^{1/2} \quad -(28)$$

Eqs. (26) and (27) are two equations in two unknowns, v and v'. Eq. (28) contains an expression for v' in terms of m. The most direct procedure is to use computer algebra to find an expression for v from eqs. (26) and (27) while eliminating v' between the two equations. The result is that v is expressed in terms of the experimental variables.

Finally an analytical expression for the mass of the photon is obtained from:

$$\hbar\omega = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(29)$$

i.e.

$$m = \frac{\hbar\omega}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \quad -(30)$$