158(2): A New Derivation of the Heisenberg Commutator Equation

In note 158(1), the motion of a particle along a straight line trajectory was considered in quantum mechanics.

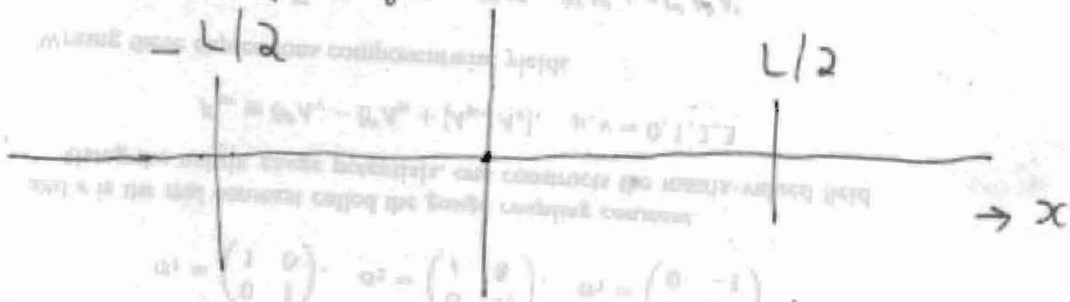

Fig. (1)

The wavefunction was found to be:

$$\psi = \frac{1}{L^{1/2}} \exp(i\kappa x) \quad -(1)$$

The following expectation values were found with a Born normalization:

$$\int_{-L/2}^{L/2} \psi^* \psi \, dx = 1 \quad -(2)$$

(In the usual interpretation of quantum mechanics (P.W. Atkins, "Molecular Quantum Mechanics", Oxford Univ. Press, 1983, 2nd edition & subsequent eds.):

$$L \rightarrow \infty \quad -(3)$$).

Expectation Values

$$\left.\begin{aligned} \langle \hat{x} \rangle &= 0, & \langle \hat{p} \rangle &= \hbar\kappa, \\ \langle \hat{x}^2 \rangle &= \frac{L^2}{12}, & \langle \hat{p}^2 \rangle &= \hbar^2\kappa^2 \end{aligned}\right] -(4)$$

In the usual interpretation of quantum mechanics

2) The expectation values are the observable value of the operators. The latter are denoted by ^. The Born normalization is:

$$\int \psi^* \psi \, d\Omega = 1 \qquad -(5)$$

where $d\Omega$ is defined over all space. In the very simplest case, the normalization (5) fails. This is because

$$\psi = e^{ikx}, \quad \psi^* = e^{-ikx} \qquad -(6)$$

and

$$\int \psi^* \psi \, d\Omega = \int_{-\infty}^{\infty} \psi^* \psi \, dx = \int_{-\infty}^{\infty} dx \rightarrow ? \ \infty \qquad -(7)$$

as discussed by Atkins.

The Heisenberg commutator equation (almost always incorrectly known as "the Heisenberg uncertainty principle") depends directly on the Born normalization (5). In the Copenhagen interpretation, eq. (5) means ~~that~~ the probability of finding the particle between x and $x + dx$ is $\psi^* \psi \, dx$. Therefore in the Copenhagen interpretation $\psi^* \psi$ is a probability density (Born 1926).

In the de Broglie interpretation, the wave-function ψ is that of a matter wave. The

3)

Born interpretation does not apply. In the Born interpretation, the probability of finding it somewhere in the entire universe is unity. Mathematically this is expressed as eq. (5). In the de Broglie interpretation, the true position and true momentum of a particle exist prior to any measurement procedure. In the Copenhagen interpretation, there are limits imposed on knowledge, this is known as indeterminacy. The weak point of the Copenhagen interpretation is that there are no particles before measurement. This is contrary to Baconian physics. In the causal and local interpretation of de Broglie, it is not possible to measure x and p simultaneously without some degree of uncertainty.

In the Copenhagen interpretation, particles are created out of an infinite number of possibilities by the measurement operation. Before that, particles are absolutely unknowable. In the de Broglie interpretation, particles exist before the measurement process. Louis de Broglie started work on his interpretation of quantum mechanics after Bohm refuted the von Neumann theorem in 1952. The de Broglie interpretation is the double solution. This retains objective physical reality.

The de Broglie interpretation is as valid as the Copenhagen one.

4)

For our present purposes in this note we use the de Broglie interpretation to invalidate the Born normalization. This means that L can be assumed to be any length. Position and momentum can be simultaneously measurable.

If L is defined to be the de Broglie wavelength:

$$L = \lambda = 2\pi\hbar / p \quad -(8)$$

then:

$$\langle \hat{x}^2 \rangle = \frac{4\pi^2\hbar^2}{p^2} = \frac{4\pi^2\hbar^2}{\langle \hat{p}^2 \rangle} \quad -(9)$$

so

$$\langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle = 4\pi^2\hbar^2 = h^2 \quad -(10)$$

i.e

$$\boxed{\langle \hat{x}^2 \rangle \langle \hat{p}^2 \rangle = h^2} \quad -(11)$$

Therefore:

$$\boxed{\langle \hat{x}^2 \rangle^{1/2} \langle \hat{p}^2 \rangle^{1/2} = h} \quad -(12)$$

As given by J. R. Croca, "Towards a Nonlinear Quantum Physics" (WS, 2001), page 27, eq. (2.4.7) eq. (12) is the Heisenberg commutator equation for conjugate variables x and p.

5) In Croca's notation:

$$\Delta x = \langle \hat{x}^2 \rangle^{1/2} \quad -(13)$$

$$\Delta p = \langle \hat{p}^2 \rangle^{1/2} \quad -(14)$$

Eq. (12) was first derived by Bohr, not by Heisenberg, during Bohr's summer holidays of 1927. It was Bohr who first gave what is now known as "the Copenhagen interpretation".

Conclusion

The so called "Heisenberg uncertainty principle" is a consequence of restricting the Born normalization and identifying L with the de Broglie wavelength λ. Eq. (12) means that the wavelength of the particle measures the size of the particle. The latter is not spread out over $-\infty$ to ∞.

Eq. (9) means that:

$$\langle \hat{x}^2 \rangle^{1/2} = \frac{2\pi}{\kappa} \quad -(15)$$

In some circumstances, Croca et al. have shown that:

$$\Delta x \, \Delta p \ll \hbar \quad -(16)$$

so eq. (12) is only a special case.