1)

158(10): Solution of Note 158(9) for Photon Mass

The two equations are:

$$\omega^2 v^2 + \omega'^2 v'^2 - 2\omega\omega' v v' \cos\theta = A \quad -(1)$$

and:

$$1 - \frac{v'^2}{c^2} = \left(\frac{\omega}{\omega'}\right)^2 \left(1 - \frac{v^2}{c^2}\right) \quad -(2) \qquad -(3)$$

where

$$A = \left(1 - \frac{\omega'}{\omega}\right)^2 \omega^2 c^2 \left(1 - 2\frac{Mc^2}{\hbar\omega}\left(1 - \frac{\omega'}{\omega}\right)^{-1}\right)$$

From eq. (2):

$$\frac{v'^2}{c^2} = 1 - \left(\frac{\omega}{\omega'}\right)^2 \left(1 - \frac{v^2}{c^2}\right) \quad -(4)$$

In eq. (1):

$$\omega^2 v^2 + c^2\left(1 - \left(\frac{\omega}{\omega'}\right)^2\left(1 - \frac{v^2}{c^2}\right)\right)\omega'^2 - 2\omega\omega' v v' \cos\theta = A \quad -(5)$$

$$2\omega^2 v^2 + c^2\left(1 - \left(\frac{\omega}{\omega'}\right)^2\right)\omega'^2 - 2\omega\omega' v v' \cos\theta = A$$

$$\omega^2 v^2 - \omega\omega' v v' \cos\theta = B \quad -(6)$$

where

$$B = \frac{1}{2}\left(A - c^2\left(1 - \left(\frac{\omega}{\omega'}\right)^2\right)\omega'^2\right)$$

so

$$\boxed{v' = \frac{\omega^2 v^2 - B}{\omega\omega' v \cos\theta}} \quad -(7)$$

2) From eq. (2):

$$1 - \frac{v^2}{c^2} = \left(\frac{\omega'}{\omega}\right)^2 \left(1 - \frac{v'^2}{c^2}\right) \quad -(8)$$

So $$\frac{v^2}{c^2} = 1 - \left(\frac{\omega'}{\omega}\right)^2 \left(1 - \frac{v'^2}{c^2}\right) \quad -(9)$$

$$= 1 - \left(\frac{\omega'}{\omega}\right)^2 + \frac{1}{c^2}\left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega^2 v^2 - B}{\omega\omega' v\cos\theta}\right)^2$$

Multiply both sides by v^2:

$$\frac{v^4}{c^2} = v^2\left(1 - \left(\frac{\omega'}{\omega}\right)^2\right) + \frac{\omega^4 v^4 - 2B\omega^2 v^2 + B^2}{(c\omega^2\cos\theta)^2}$$

$$= v^2\left(1 - \left(\frac{\omega'}{\omega}\right)^2\right) + \frac{v^4}{c^2\cos^2\theta} - \frac{2Bv^2}{(\omega c\cos\theta)^2} + \left(\frac{B}{\omega^2 c\cos\theta}\right)^2$$

i.e.

$$\frac{v^4}{c^2}\left(\frac{1}{\cos^2\theta} - 1\right) + \left(1 - \left(\frac{\omega'}{\omega}\right)^2 - \frac{2B}{(c\omega\cos\theta)^2}\right)v^2 + \left(\frac{B}{\omega^2 c\cos\theta}\right)^2 = 0$$

Multiply through by $\cos^2\theta$:

$$\boxed{\frac{v^4}{c^2}\left(1 - \cos^2\theta\right) + \left(\left(1 - \left(\frac{\omega'}{\omega}\right)^2\right)\cos^2\theta - \frac{2B}{c^2\omega^2}\right)v^2 + \left(\frac{B}{c\omega^2}\right)^2 = 0} \quad -(10)$$

This is a quadratic:

$$ax^2 + bx + c' = 0 \quad -(11)$$

3) where:

$$x = v^2 \quad -(12)$$

$$a = \frac{1}{c^2}\left(1 - \cos^2\theta\right) \quad -(13)$$

$$b = \left(1 - \left(\frac{\omega'}{\omega}\right)^2\right)\cos^2\theta - \frac{2B}{c^2\omega^2} \quad -(14)$$

$$c' = \left(\frac{B}{c\omega^2}\right)^2 \quad -(15)$$

where:

$$B = \frac{1}{2}\left[\left(1 - \frac{\omega'}{\omega}\right)^2\omega^2c^2\left(1 - 2\frac{Mc^2}{\hbar\omega}\left(1 - \frac{\omega'}{\omega}\right)^{-1} - c^2\omega'^2\left(1 - \left(\frac{\omega'}{\omega}\right)^2\right)\right)\right] \quad -(16)$$

The solution of eqn. (11) is:

$$x = \frac{1}{2a}\left(-b \pm \left(b^2 - 4ac'\right)^{1/2}\right) \quad -(17)$$

$$= v^2$$

Finally the mass of the photon is:

$$m = \frac{\hbar\omega}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \quad -(18)$$