158(5): General Theory of Compton Scattering of Electron and Massless Photon.

Conservation of Energy

$$\hbar\omega_1 + E_1 = \hbar\omega_2 + E_2 \quad (1)$$

where E_1 is the total energy of the electron before a collision with the photon. We have:

$$E_1 = \gamma_1 mc^2, \quad E_2 = \gamma_2 mc^2 \quad (2)$$

So:

$$\boxed{\hbar(\omega_1 - \omega_2) = mc^2(\gamma_2 - \gamma_1)} \quad (3)$$

Conservation of Momentum

$$\hbar \underline{\kappa}_1 + \gamma_1 m \underline{v}_1 = \hbar \underline{\kappa}_2 + \gamma_2 m \underline{v}_2 \quad (4)$$

Therefore

$$\omega_1 - \omega_2 = \frac{mc^2}{\hbar}(\gamma_2 - \gamma_1) \quad (5)$$

$$\boxed{\underline{\kappa}_1 - \underline{\kappa}_2 = \frac{m}{\hbar}(\gamma_2 \underline{v}_2 - \gamma_1 \underline{v}_1)} \quad (6)$$

Now use:

$$\omega_1 = c\kappa_1, \quad \omega_2 = c\kappa_2 \quad (7)$$

so

$$\boxed{\kappa_1 - \kappa_2 = \frac{mc}{\hbar}(\gamma_2 - \gamma_1)} \quad (8)$$

Define:

$$\Delta\underline{\kappa} = \underline{\kappa}_1 - \underline{\kappa}_2 \quad (9)$$

$$\Delta\kappa = \kappa_1 - \kappa_2 \quad (10)$$

2) ~~Alter~~:

$$\boxed{\begin{aligned} \Delta \kappa &= \frac{mc}{\hbar}\left(\gamma_2 - \gamma_1\right) \quad &-(11) \\ \Delta \underline{\kappa} &= \frac{m}{\hbar}\left(\gamma_2 \underline{v}_2 - \gamma_1 \underline{v}_1\right) \quad &-(12) \end{aligned}}$$

These are generally valid expressions for change of photon wavenumber.

Let the electron energy before and after collision be E and E', and the electron ~~energy~~ momentum before and after collision be $\underline{p}$ and $\underline{p}'$. In this first development the photon has no mass, and the photon energy before and after collision is $\hbar\omega$ and $\hbar\omega'$ respectively, the photon momentum before and after collision is $\hbar\underline{\kappa}$ and $\hbar\underline{\kappa}'$ respectively. From conservation of energy:

$$\hbar\omega + E = \hbar\omega' + E' \quad -(13)$$

and from conservation of momentum:

$$\hbar\underline{\kappa} + \underline{p} = \hbar\underline{\kappa}' + \underline{p}' \quad -(14)$$

Therefore

$$E' = E + \hbar(\omega - \omega') = \left(c^2 p'^2 + m^2 c^4\right)^{1/2} \quad -(15)$$

so

$$c^2 p'^2 + m^2 c^4 = \left(\hbar\omega - \hbar\omega' + mc^2\right)^2 \quad -(16)$$

3)

From eq. (14):

$$\underline{p}' - \underline{p} = \hbar(\underline{\kappa} - \underline{\kappa}') \quad -(17)$$

Electron Initially at Rest

In this case: $\underline{p} = \underline{0}$ $\quad -(18)$

so

$$\underline{p}' = \hbar(\underline{\kappa} - \underline{\kappa}') \quad -(19)$$

and

$$p'^2 = \hbar^2(\underline{\kappa} - \underline{\kappa}') \cdot (\underline{\kappa} - \underline{\kappa}') \quad -(20)$$

$$= \hbar^2(\kappa^2 + \kappa'^2 - 2\kappa\kappa'\cos\theta)$$

where we have used

$$\underline{\kappa} \cdot \underline{\kappa}' = \kappa\kappa'\cos\theta \quad -(21)$$

Now use:

$$\kappa = \frac{\omega}{c}, \; \kappa' = \frac{\omega'}{c} \quad -(22)$$

in eq. (20) to obtain:

$$c^2 p'^2 = (\hbar\omega)^2 + (\hbar\omega')^2 - 2\hbar\omega\omega'\cos\theta \quad -(23)$$

From eq. (16):

$$c^2 p'^2 = (\hbar\omega - \hbar\omega' + mc^2)^2 - m^2c^4 \quad -(24)$$

From eqs. (23) and (24):

$$(\hbar\omega)^2 + (\hbar\omega')^2 - 2\hbar\omega\hbar\omega' + m^2c^4 - m^2c^4$$
$$+ 2\hbar(\omega - \omega')mc^2 = (\hbar\omega)^2 + (\hbar\omega')^2 - 2\hbar\omega\omega'\cos\theta$$
$$-(25)$$

4)

i.e. $$\hbar(\omega-\omega')mc^2 = \hbar\omega\hbar\omega'(1-\cos\theta) \quad -(26)$$

or $$\omega-\omega' = \frac{\hbar\omega\omega'}{mc^2}(1-\cos\theta) \quad -(27)$$

i.e $$\frac{1}{\omega'}-\frac{1}{\omega} = \frac{\hbar}{mc^2}(1-\cos\theta) \quad -(28)$$

or $$\lambda'-\lambda = \frac{h}{mc}(1-\cos\theta) \quad -(29)$$

where $$\lambda = \frac{2\pi}{\omega} \quad -(30)$$

<u>General Case where $\underline{p} \neq \underline{0}$</u>

In this case:

$$(E'-E)^2 = \hbar^2\omega^2+\hbar^2\omega'^2-2\hbar^2\omega\omega', \quad -(31)$$

$$c^2(\underline{p}'-\underline{p})\cdot(\underline{p}'-\underline{p}) = \hbar^2\omega^2+\hbar^2\omega'^2-2\hbar^2\omega\omega'\cos\theta \quad -(32)$$

So:

$$\boxed{c^2(\underline{p}'-\underline{p})\cdot(\underline{p}'-\underline{p})-(E'-E)^2 = 2\hbar^2\omega\omega'(1-\cos\theta)} \quad -(33)$$

This is the <u>generalization</u> of the well known <u>eq. (29)</u>.

5) To check this formula in the special case of

$$\underline{p} = 0 \quad -(34)$$

we have:

$$c^2 p'^2 - (E' - E)^2 = 2\hbar^2 \omega\omega' (1 - \cos\theta) \quad -(35)$$

i.e.

$$c^2 p'^2 - E'^2 - E^2 + 2EE' = 2\hbar^2 \omega\omega' (1 - \cos\theta) \quad -(36)$$

Use:

$$E'^2 = c^2 p'^2 + m^2 c^4 \quad -(37)$$

to find:

$$EE' = m^2 c^4 + \hbar^2 \omega\omega' (1 - \cos\theta) \quad -(38)$$

Use eq. (15):

$$E(E + \hbar(\omega - \omega')) = m^2 c^4 + \hbar^2 \omega\omega' (1 - \cos\theta),$$

i.e.

$$\omega - \omega' = \frac{\omega\omega'}{mc^2} \hbar (1 - \cos\theta) \quad -(39)$$

which is eq. (26), Q.E.D.

The general equation (33) may be rewritten as:

$$c^2(p'^2 + p^2 - 2pp'\cos\theta') - E'^2 - E^2 + 2EE' = 2\hbar^2 \omega\omega' (1 - \cos\theta) \quad -(40)$$

where

$$\underline{p} \cdot \underline{p}' = pp'\cos\theta'. \quad -(41)$$

Eq. (40) reduces to:

6)

$$\boxed{EE' - c^2 pp' \cos\theta' = m^2c^4 + \hbar^2\omega\omega'(1-\cos\theta)} \quad -(42)$$

Eq. (42) can be checked again in the case $\underline{p} = \underline{0}$, when

$$E' = \hbar\omega - \hbar\omega' + mc^2 \quad -(43)$$

$$E = mc^2 \quad -(44)$$

so

$$mc^2(mc^2 + \hbar\omega - \hbar\omega') = m^2c^4 + \hbar^2\omega\omega'(1-\cos\theta) \quad -(45)$$

i.e.

$$\omega - \omega' = \frac{\hbar\omega'\omega}{mc^2}(1-\cos\theta) \quad -(46)$$

which is Eq. (39), Q.E.D.

In the general case:

$$\underline{p} \neq \underline{0} \quad -(47)$$

then

$$E' = \hbar\omega - \hbar\omega' + E \quad -(48)$$

$$E = (c^2p^2 + m^2c^4)^{1/2} \quad -(49)$$

and

$$\hbar^2\omega\omega'(1-\cos\theta) = EE' - c^2pp'\cos\theta' - m^2c^4 \quad -(50)$$

where

$$EE' = E(E + \hbar(\omega - \omega')) \quad -(51)$$

$$\underline{p}\cdot\underline{p}' = pp'\cos\theta' = \underline{p}\cdot(\underline{p} + \hbar(\underline{\kappa} - \underline{\kappa}')) \quad -(52)$$

7)

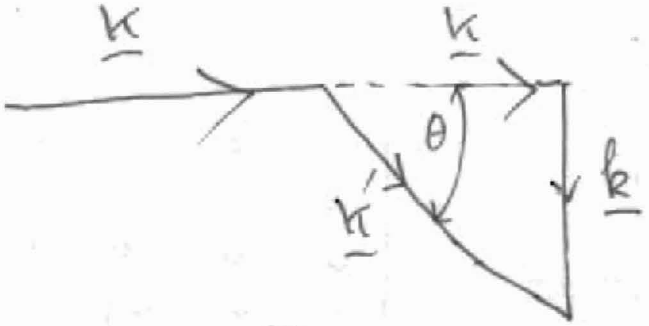

Fig. (1)

W.r.t reference to Fig. (1):

$$\underline{\kappa}' = \underline{\kappa} + \underline{k} \quad -(53)$$

So

$$\underline{p} \cdot \left(\underline{p} + \hbar \left(\underline{\kappa} - \underline{\kappa}' \right) \right) = p^2 - \hbar \, \underline{p} \cdot \underline{k} \quad -(54)$$

Let

$$\underline{p} \cdot \underline{k} = p k \cos\phi \quad -(55)$$

and use:

$$k = \left(\kappa'^2 + \kappa^2 - 2 \kappa \kappa' \cos\theta \right)^{1/2} \quad -(56)$$

$$= \frac{1}{c} \left(\omega'^2 + \omega^2 - 2 \omega \omega' \cos\theta \right)^{1/2}$$

So in eq. (50):

$$\hbar^2 \omega \omega' \left(1 - \cos\theta \right) = E^2 + \hbar E \left(\omega - \omega' \right) - c^2 \left(p^2 - \hbar p k \cos\phi \right) - m^2 c^4 \quad -(57)$$

so

$$\hbar^2 \omega \omega' \left(1 - \cos\theta \right) = \hbar E \left(\omega - \omega' \right) + \hbar c^2 p k \cos\phi \quad -(58)$$

i.e.

$$\boxed{\omega - \omega' = \frac{\hbar}{E} \omega \omega' \left(1 - \cos\theta \right) + c^2 \frac{p}{E} k \cos\phi} \quad -(59)$$

Limits of Eq. (59)

1) In the limit: $p \to 0$ — (60)

then $\omega - \omega' \to \frac{\hbar\omega\omega'}{mc^2}(1-\cos\theta)$ — (61)

which is eq. (46), QED.

2) In the ultra relativistic limit

$E \to cp$ — (61)

then $\omega - \omega' = \frac{\hbar\omega\omega'}{E}(1-\cos\theta) + ck\cos\phi$ — (62)

and if $E \gg \omega\omega'(1-\cos\theta)$ — (63)

then $\boxed{\omega - \omega' \to ck\cos\phi}$ — (64)

which describes a very high energy electron colliding with an electromagnetic beam as follows:

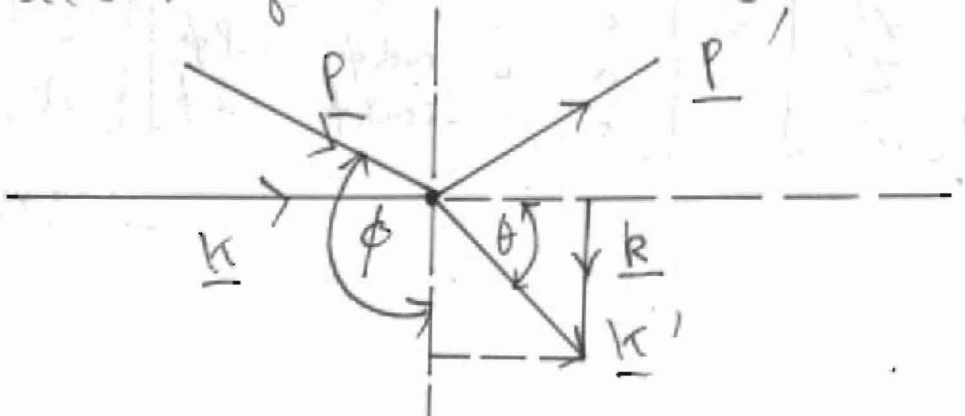

Fig. (2)