1)

158(7): Photon Mass from the Compton Effect

Consider a photon of mass m colliding with an electron of mass $\underline{M}$ initially at rest.

Conservation of Energy

$$\gamma m c^2 + \underline{M} c^2 = \gamma' m c^2 + \gamma_1 \underline{M} c^2 \quad -(1)$$

Conservation of Momentum

$$\gamma m \underline{v} = \gamma' m \underline{v}' + \gamma_1 \underline{M} \underline{v}_e \quad -(2)$$

Here $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad \gamma' = \left(1 - \frac{v'^2}{c^2}\right)^{-1/2}, \quad -(3)$

$$\gamma_1 = \left(1 - \frac{v_1^2}{c^2}\right)^{-1/2}$$

where v_1 = magnitude of final electron velocity,
v = magnitude of initial photon velocity,
v' = magnitude of final photon velocity.

The de Broglie / Einstein Equations

$$\hbar\omega = \gamma m c^2 \quad -(4)$$

$$\hbar\omega' = \gamma' m c^2 \quad -(5)$$

where ω = initial angular frequency of photon
ω' = final angular frequency of photon

So: $$\gamma' = \left(\frac{\omega'}{\omega}\right)\gamma \quad -(6)$$

Use eq. (6) in eq. (1):

$$\gamma m c^2 \left(1 - \frac{\omega'}{\omega}\right) = (\gamma_1 - 1)\, \underline{M} c^2 \quad -(7)$$

From eq. (2):

$$\gamma_1 \underline{M}\, \underline{v}_1 = m\left(\gamma \underline{v} - \gamma' \underline{v}'\right) \quad -(8)$$

Therefore:

$$\gamma_1^2 \underline{M}^2 v_1^2 = m^2\left(\gamma^2 v^2 + \gamma'^2 v'^2 - 2 v v' \gamma \gamma' \cos\theta\right)$$

$$= m^2 \gamma^2 \left(v^2 + \left(\frac{\omega'}{\omega}\right)^2 v'^2 - 2 v v' \frac{\omega'}{\omega} \cos\theta\right) \quad -(9)$$

Define: $$B = \gamma_1^2 \underline{M}^2 v_1^2 \quad -(10)$$

$$= v_1^2 \left(1 - \frac{v_1^2}{c^2}\right)^{-1} \underline{M}^2.$$

<u>This</u> is experimentally <u>observable</u> <u>through</u> <u>the electron</u> <u>velocity</u> <u>v_1</u> after <u>collision</u>.

3) From eq. (7):

$$\gamma m = M(\gamma_1 - 1)\left(1 - \frac{\omega'}{\omega}\right)^{-1} \quad -(11)$$

$$:= A$$

The quantity A is also observable experimentally.

So $$\gamma m = A \quad -(12)$$

$$\gamma^2 m^2\left(v^2 + \left(\frac{\omega'}{\omega}\right)^2 v'^2 - 2vv'\frac{\omega'}{\omega}\cos\theta\right) = B \quad -(13)$$

Divide (13) by (12):

$$v^2 + \left(\frac{\omega'}{\omega}\right)^2 v'^2 - 2vv'\frac{\omega'}{\omega}\cos\theta = \frac{B}{A^2} \quad -(14)$$

Now use eq. (6) to eliminate v':

$$\gamma' = \frac{\omega'}{\omega}\gamma, \quad \frac{1}{\gamma'} = \frac{\omega}{\omega'}\frac{1}{\gamma} \quad -(15)$$

So $$1 - \frac{v'^2}{c^2} = \left(\frac{\omega}{\omega'}\right)^2\left(1 - \frac{v^2}{c^2}\right). \quad -(16)$$

From eq. (16):

4)

$$1 - \left(\frac{\omega}{\omega'}\right)^2 = \frac{v'^2}{c^2} - \left(\frac{\omega}{\omega'}\right)^2 \frac{v^2}{c^2},$$

i.e. $$v'^2 - \left(\frac{\omega}{\omega'}\right)^2 v^2 = c^2\left(1 - \left(\frac{\omega}{\omega'}\right)^2\right),$$

so $$\left(\frac{\omega'}{\omega}\right)^2 v'^2 - v^2 = c^2 \left(\frac{\omega'}{\omega}\right)^2 \left(1 - \left(\frac{\omega}{\omega'}\right)^2\right)$$

$$= c^2\left(\left(\frac{\omega'}{\omega}\right)^2 - 1\right) \quad -(17)$$

$$:= D$$

Therefore:

$$v^2 + \left(\frac{\omega'}{\omega}\right)^2 v'^2 - 2vv'\frac{\omega'}{\omega}\cos\theta = \frac{B}{A^2} \quad -(14)$$

$$-(17)$$

$$\left(\frac{\omega'}{\omega}\right)^2 v'^2 - v^2 = D \quad -(18)$$

Subtract eq. (17) from eq. (14):

$$2v^2 - 2vv'\frac{\omega'}{\omega}\cos\theta = \frac{B}{A^2} - D := E$$

i.e. $$\boxed{2v^2 - 2vv'\frac{\omega'}{\omega}\cos\theta - E = 0} \quad -(19)$$

i.e. $$2vv'\frac{\omega'}{\omega}\cos\theta = 2v^2 - E$$

so $$v' = \frac{\omega}{2v\omega'\cos\theta}\left(2v^2 - E\right)$$

$$v' = xv - \frac{y}{v} \quad \text{— (20)}$$

where: $$x = \frac{\omega}{\omega'\cos\theta}, \quad y = \left(\frac{\omega}{2\omega'\cos\theta}\right)E$$

Use eq (20) in eq. (14):

$$\boxed{v^2 + \left(\frac{\omega'}{\omega}\right)^2\left(xv - \frac{y}{v}\right)^2 - 2v\left(xv - \frac{y}{v}\right)\frac{\omega'}{\omega}\cos\theta = \frac{B}{A} \quad \text{— (21)}}$$

This is a quartic equation for v in terms of experimental observables.

Use computer algebra to solve eq. (21) to give four roots for v. The

6) physically meaningful root is close to c.

Finally find the photon mass m from

$$\hbar\omega = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} mc^2 \quad -(22)$$

i.e.

$$m = \frac{\hbar\omega}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \quad -(23)$$

Conclusion

Given the final electron momentum and electron velocity, the photon mass can be found from the ordinary Compton effect, in which the electron is initially at rest.

Approximation for Electron Momentum

This can be found from:

$$\underline{p}_1 = \hbar(\underline{k} - \underline{k}') \quad -(24)$$

$$= \gamma_1 m \underline{v}_1$$

In these equations

7) $$p_1^2 = \hbar^2\left(\kappa^2 + \kappa'^2 - 2\kappa\kappa'\cos\theta\right) = \gamma_1^2 m^2 v_1^2 \quad -(25)$$

The approximation is:

$$\kappa \doteq \frac{\omega}{c}, \quad \kappa' \doteq \frac{\omega'}{c} \quad -(26)$$

This is an approximation because it assumes zero photon mass. This approximation is always used in the usual theory of the Compton effect. The photon mass is about 10^{-52} kilograms so it is an excellent approximation, but nevertheless an approximation.

So:

$$p_1^2 \doteq \left(1 - \frac{v_1^2}{c^2}\right)^{-1} m^2 v_1^2 \quad -(27)$$

$$= \frac{\hbar^2}{c^2}\left(\omega^2 + \omega'^2 - 2\omega\omega'\cos\theta\right)$$

This is a quadratic for v_1, the electron velocity, in terms of experimental observables.