153(8): Check on the Maclaurin Series.

In UFT 152, eq. (21), a Maclaurin series was used for $e^{-r_0/r}$:

$$e^{-r_0/r} = 1 - \frac{r_0}{r} + \frac{1}{2!}\left(\frac{r_0}{r}\right)^2 - \frac{1}{3!}\left(\frac{r_0}{r}\right)^3 + \ldots \quad -(1)$$

The Maclaurin series is defined by:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots \quad -(2)$$

and is a special case of the Taylor series:

$$f(a+x) = f(a) + x f'(a) + \frac{x^2}{2!} f''(a) + \ldots \quad -(3)$$

(G. Stephenson, "Mathematical Methods for Science Students" (Longmans, 1968)), chapter 6. The Maclaurin series for combinations of elementary functions is discussed by Stephenson on p. 92 ff. For example:

$$\exp(\sin x) = 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} \ldots \quad -(4)$$

is found from:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad -(5)$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad -(6)$$

so

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots \quad -(7)$$

$$y = \sin x \quad -(8)$$

so

$$e^y = 1 + \sin x + \frac{(\sin x)^2}{2!} + \ldots \quad -(9)$$

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$$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots\dots\right)^2 + \dots$$

$$= 1 + x + \frac{x^2}{2!} - \frac{3x^4}{4!} \dots\dots \quad -(10)$$

Q.E.D.

In order for this procedure to be valid (Stephenson p. 81), the two power series must converge for a common interval of convergence.

If we consider:

$$e^{-\alpha} = 1 - \alpha + \frac{\alpha^2}{2!} - \frac{\alpha^3}{3!} + \dots \quad -(11)$$

This series converges if: $\alpha < 1 \quad -(12)$

so if $\alpha = \frac{r_0}{r} \quad -(13)$

then eq. (11) converges if: $r_0 < r \quad -(14)$

The Maclaurin series (11) is obtained from:

$$e^{-(\alpha + a)} = f(a) + \alpha f'(a) + \frac{\alpha^2}{2!} f''(a) + \dots \quad -(15)$$

Here $f'(a) = \left(\frac{d}{d\alpha} e^{-\alpha}\right)_{\alpha = a} \quad -(16)$

$$= -e^{-a} \quad -(17)$$

Eq. (11) can be regarded as a power series

3) consisting of one term. The interval of convergence is:

$$0 < r_0 < r \quad - (18)$$

i.e.

$$0 < \alpha < 1 \quad - (19)$$

In this interval eq. (11) and (13) both converge, so:

$$\exp\left(-\frac{r_0}{r}\right) = 1 - \frac{r_0}{r} + \frac{1}{2!}\left(\frac{r_0}{r}\right)^2 - \frac{1}{3!}\left(\frac{r_0}{r}\right)^3 + \cdots \quad - (20)$$

In general, the Taylor series is:

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots \quad - (21)$$

where

$$f(x) = e^{-r_0/x} \quad - (22)$$

$$f'(x) = \frac{r_0}{x^2} e^{-r_0/x} \quad - (23)$$

$$f''(x) = \frac{r_0}{x^3}\left(\frac{r_0}{x} - 2\right) e^{-r_0/x} \quad - (24)$$

so:

$$f(x) = f(a) + \frac{r_0}{a^2}(x-a)e^{-r_0/a} + \frac{1}{2!}(x-a)^2 \frac{r_0}{a^3}\left(\frac{r_0}{a} - 2\right)e^{-r_0/a} + \cdots$$

$$= e^{-r_0/a}\left(1 + \frac{r_0}{a^2}(x-a) + \frac{1}{2!}(x-a)^2 \frac{r_0}{a^3}\left(\frac{r_0}{a} - 2\right) + \cdots\right) \quad - (25)$$

so:

$$e^{-r_0/r} = e^{-r_0/a}\left(1 + \frac{r_0}{a^2}(r-a) + \frac{1}{2!}(r-a)^2 \frac{r_0}{a^3}\left(\frac{r_0}{a} - 2\right) + \cdots\right)$$

$$= 1 - \frac{r_0}{r} + \frac{1}{2!}\left(\frac{r_0}{r}\right)^2 - \frac{1}{3!}\left(\frac{r_0}{r}\right)^3 + \cdots \quad - (26)$$