1)

153(2): The Wave equation of the Spherical Spacetime.

The metric of spherical spacetime is considered to be:

$$ds^2 = c^2 d\tau^2 = e^{-r_0/r} c^2 dt^2 - e^{r_0/r} dr^2 - r^2 d\phi^2 - dZ^2 \quad -(1)$$

in cylindrical polar coordinates. So the metric tensor is:

$$g_{\mu\nu} = \begin{bmatrix} e^{-r_0/r} & 0 & 0 & 0 \\ 0 & -e^{r_0/r} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(2)$$

The Minkowski metric in cylindrical polar coordinates is:

$$\eta_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(3)$$

By definition in Cartan geometry:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(4)$$

where q^a_μ and q^b_ν are tetrads. So:

$$g_{00} = q^0_0 q^0_0 \eta_{00} \quad -(5)$$

$$g_{11} = q^1_1 q^1_1 \eta_{11} \quad -(6)$$

$$g_{22} = q^2_2 q^2_2 \eta_{22} \quad -(7)$$

$$g_{33} = q^3_3 q^3_3 \eta_{33} \quad -(8)$$

Thus:

$$q^0_0 = \exp\left(-\frac{r_0}{2r}\right), \; q^1_1 = \exp\left(\frac{r_0}{2r}\right), \; q^2_2 = q^3_3 = 1 \quad -(9)$$

Consider the tetrad postulate of Cartan geometry:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda = 0. \quad -(10)$$

This can be written as:

$$\partial_\mu q^a_\nu + \omega^a_{\mu\nu} - \Gamma^a_{\mu\nu} = 0. \quad -(11)$$

Thus

$$\Box q^a_\nu + \partial^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right) = 0, \quad -(12)$$

where

$$\Box := \partial^\mu \partial_\mu. \quad -(13)$$

Write eq. (12) as the ECE Lemma:

$$\boxed{(\Box + R)\, q^a_\nu = 0} \quad -(14)$$

where

$$R := q^\nu_a \partial^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right) \quad -(15)$$

and

$$q^a_\nu q^\nu_a := 1. \quad -(16)$$

Eq. (14) is the wave equation of any spacetime. All the wave equations of physics derive from it.

For the spherical spacetime of eq. (1) the relevant tetrad postulates are:

$$\partial_1 q^0_0 + \omega^0_{10} - \Gamma^0_{10} = 0 \quad -(17)$$

$$\partial_1 q^1_1 + \omega^1_{11} - \Gamma^1_{11} = 0 \quad -(18)$$

3) where
$$\partial_1 = \frac{\partial}{\partial r}, \quad \partial^1 = -\frac{\partial}{\partial r} \quad -(19)$$

so
$$\partial_1 q^0_0 = \frac{\partial}{\partial r} \exp\left(-\frac{r_0}{2r}\right) = \frac{r_0}{2r^2} \exp\left(-\frac{r_0}{2r}\right) \quad -(20)$$

i.e.
$$\boxed{\begin{aligned} \Gamma^0_{10} - \omega^0_{10} &= \frac{r_0}{2r^2} \exp\left(-\frac{r_0}{2r}\right) \quad -(21) \\ \Gamma^1_{11} - \omega^1_{11} &= -\frac{r_0}{2r^2} \exp\left(\frac{r_0}{2r}\right) \quad -(22) \end{aligned}}$$

i.e.
$$\Gamma^0_{10} - \omega^0_{10} = \frac{r_0}{2r^2} q^0_0 \quad -(23)$$
$$\Gamma^1_{11} - \omega^1_{11} = -\frac{r_0}{2r^2} q^1_1 \quad -(24)$$

The relevant wave equations are:

1)
$$\partial^1 \partial_1 q^0_0 - \partial^1\left(\frac{r_0}{2r^2} q^0_0\right) = 0 \quad -(25)$$

i.e.
$$R_0 = -\left(q^0_0\right)^{-1} \partial^1\left(\frac{r_0}{2r^2} q^0_0\right) \quad -(26)$$

$$R_0 = \exp\left(\frac{r_0}{2r}\right) \frac{\partial}{\partial r}\left(\frac{r_0}{2r^2} q^0_0\right). \quad -(27)$$

i.e.
$$\boxed{R_0 = \frac{r_0}{r^3}\left(\frac{r_0}{4r} - 1\right)} \quad -(28)$$

2)
$$\partial^1 \partial_1 q^1_1 + \partial^1\left(\omega^1_{11} - \Gamma^1_{11}\right) = 0$$
$$-(29)$$

4)

i.e $R_1 = -\left(q^1_1\right)^{-1}\frac{\partial}{\partial r}\left(\omega^1_{11} - \Gamma^1_{11}\right)$

$$= -\exp\left(-\frac{r_0}{2r}\right)\frac{\partial}{\partial r}\left(\frac{r_0}{2r^2}q^1_1\right)$$

$$= \frac{r_0}{r^3}\left(1 + \frac{r_0}{4r}\right). \quad -(30)$$

Therefore:

$$\left(\Box + R_0\right) q^0_0 = 0 \quad -(31)$$
$$\left(\Box + R_1\right) q^1_1 = 0 \quad -(32)$$
$$\left(\Box + R_2\right) q^2_2 = 0 \quad -(33)$$
$$\left(\Box + R_3\right) q^3_3 = 0 \quad -(34)$$

$q^0_0 = \exp\left(-\frac{r_0}{2r}\right)$ $\qquad R_0 = \frac{r_0}{r^3}\left(\frac{r_0}{4r} - 1\right)$

$q^1_1 = \exp\left(\frac{r_0}{2r}\right)$ $\qquad R_1 = \frac{r_0}{r^3}\left(\frac{r_0}{4r} + 1\right)$

$q^2_2 = 1$ $\qquad R_2 = 0$

$q^3_3 = 1$ $\qquad R_3 = 0$

$$R = R_0 + R_1 + R_2 + R_3$$
$$= \left(\frac{r_0}{2r^2}\right)^2$$