153(9): Vector Potentials from the Metric.

Consider the cylindrical polar coordinates in three dimensions:

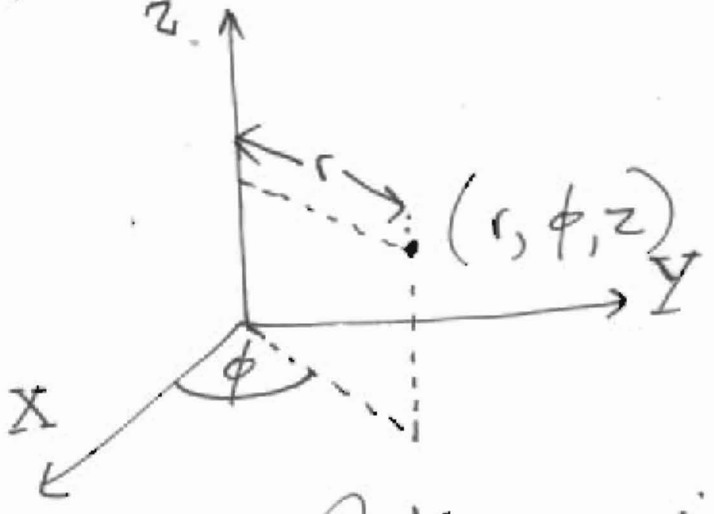
$x = r\cos\phi$

$Y = r\sin\phi$

$Z = Z$

$r = (X^2 + Y^2)^{1/2}$

$\phi = \tan^{-1}\frac{Y}{X}$



The position and velocity in Cartesian coordinates are:

$$\underset{\sim}{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(1)$$

$$\underline{v} = \frac{dX}{dt}\underline{i} + \frac{dY}{dt}\underline{j} + \frac{dZ}{dt}\underline{k} \quad -(2)$$

The unit vectors of the cylindrical polar system are (VAPS 21-14):

$$\underline{e}_r = \underline{i}\cos\phi + \underline{j}\sin\phi \quad -(3)$$

$$\underline{e}_\phi = -\underline{i}\sin\phi + \underline{j}\cos\phi \quad -(4)$$

$$\underline{e}_Z = \underline{k} \quad -(5)$$

so

$$\underline{r} = \underline{i}\, r\cos\phi + \underline{j}\, r\sin\phi + Z\underline{k} \quad -(6)$$

The position infinitesimal is:

$$d\underline{r} = \frac{\partial \underline{r}}{\partial r}dr + \frac{\partial \underline{r}}{\partial \phi}d\phi + \frac{\partial \underline{r}}{\partial Z}dZ \quad -(7)$$

where:

2) $$\frac{\partial \underline{r}}{\partial r} = \frac{\partial}{\partial r}\left(r\cos\phi\, \underline{i} + r\sin\phi\, \underline{j} + Z\, \underline{k} \right) \quad -(8)$$

$$= \underline{i}\cos\phi + \underline{j}\sin\phi$$

$$\frac{\partial \underline{r}}{\partial \phi} = -\underline{i}\, r\sin\phi + \underline{j}\, r\cos\phi \quad -(9)$$

$$\frac{\partial \underline{r}}{\partial Z} = \underline{k} \quad -(10)$$

So:

$$d\underline{r} = \left(\cos\phi\, dr - r\sin\phi\, d\phi\right)\underline{i} + \left(\sin\phi\, dr + r\cos\phi\, d\phi\right)\underline{j} + dZ\, \underline{k} \quad -(11)$$

$$d\underline{r} \cdot d\underline{r} = dr^2 + r^2 d\phi^2 + dZ^2 \quad -(12)$$

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(13)$$

The element of arc length is:

$$ds^2 = \left|\frac{\partial \underline{r}}{\partial r}\right|^2 dr^2 + \left|\frac{\partial \underline{r}}{\partial \phi}\right|^2 d\phi^2 + \left|\frac{\partial \underline{r}}{\partial Z}\right|^2 dZ^2 \quad -(14)$$

The metric elements are:

$$g_{00} = \left|\frac{\partial \underline{r}}{\partial r}\right|^2 = 1 \quad -(15)$$

$$g_{11} = \left|\frac{\partial \underline{r}}{\partial \phi}\right|^2 = r^2 \quad -(16)$$

$$g_{22} = \left|\frac{\partial \underline{r}}{\partial Z}\right|^2 = 1 \quad -(17)$$

3) The metric elements are also known as metric coefficients and scale factors:

$$h_1 = h_r = \left| \frac{\partial \underline{r}}{\partial r} \right| = \left(\cos^2\phi + \sin^2\phi \right)^{1/2} = 1 \quad -(18)$$

$$h_2 = h_\phi = \left| \frac{\partial \underline{r}}{\partial \phi} \right| = \left(\left(-r\sin\phi \right)^2 + \left(r\cos\phi \right)^2 \right)^{1/2} = r \quad -(19)$$

$$h_3 = h_Z = \left| \frac{\partial \underline{r}}{\partial Z} \right| = 1 \quad -(20)$$

The diagonal tetrad elements are:

$$q^1_1 = h_1 = 1 \quad -(21)$$

$$q^2_2 = h_2 = r \quad -(22)$$

$$q^3_3 = h_3 = 1 \quad -(23)$$

The metric form in curvilinear coordinates is defined by:

$$g_{ij} = g_{ji} = \frac{\partial \underline{r}}{\partial u_i} \cdot \frac{\partial \underline{r}}{\partial u_j} \quad -(24)$$

(VAPS 22-13(4)). Therefore for the cylindrical polar coordinates:

$$\boxed{\begin{aligned} q^1_1 &= e_r = |\underline{e}_r| = 1 \\ q^2_2 &= e_\phi = |\underline{e}_\phi| = r \\ q^3_3 &= e_Z = |\underline{e}_Z| = 1 \end{aligned}} \quad \begin{aligned} &-(25) \\ &-(26) \\ &-(27) \end{aligned}$$

For the Cartesian coordinates:

$$\boxed{\begin{aligned} q^1_1 &= i = |\underline{i}| = 1 \\ q^2_2 &= j = |\underline{j}| = 1 \\ q^3_3 &= k = |\underline{k}| = 1 \end{aligned}} \quad \begin{aligned} &-(28) \\ &-(29) \\ &-(30) \end{aligned}$$

4)

Therefore the tetrad elements are the magnitudes of the unit vectors.

Denote the metrics of the cylindrical polar and Cartesian coordinates by:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \eta_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(31)$$

then

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(32)$$

i.e.

$$g_{11} = (q^1_1)^2 \eta_{11} \quad -(33)$$

$$g_{22} = (q^2_2)^2 \eta_{22} \quad -(34)$$

$$g_{33} = (q^3_3)^2 \eta_{33} \quad -(35)$$

i.e.

$$q^1_1 = 1, \quad q^2_2 = r, \quad q^3_3 = 1 \quad -(36)$$

Q.E.D.

This is an illustration of the meaning of the Cartan tetrad.

With these preliminaries define the velocity:

$$\boxed{\begin{aligned} \underline{v} &= \left(\frac{dX}{dt}\right)\underline{i} + \left(\frac{dY}{dt}\right)\underline{j} + \left(\frac{dZ}{dt}\right)\underline{k} \\ &= \left(\frac{dr}{dt}\right)\underline{e}_r + r\frac{d\phi}{dt}\underline{e}_\phi + \frac{dZ}{dt}\underline{k} \end{aligned}} \quad -(37)$$

5)

and the vector potential :

$$e\underline{A} = m\underline{v} \,, \quad \underline{A} = \frac{m}{e}\underline{v} \quad -(38)$$

Restrict consideration to the XY plane and define :

$$e\underline{A} = \underline{p} = p_r \underline{e}_r + \frac{L}{r} e_\phi \quad -(39)$$

where :

$$\underline{p} = m\underline{v} \,, \quad -(40)$$

$$\underline{L} = mr^2 \frac{d\phi}{dt} \quad -(41)$$

$$p_r = m\frac{dr}{dt} \quad -(42)$$

Here $\underline{L}$ is the angular momentum. The angular velocity is

$$\omega = \frac{d\phi}{dt} \quad -(43)$$

Thus :

$$\boxed{\begin{aligned} \underline{A}_r &= \frac{m}{e}\frac{dr}{dt}\underline{e}_r \quad -(44) \\ \underline{A}_\phi &= \frac{m}{e} r \frac{d\phi}{dt}\underline{e}_\phi \quad -(45) \end{aligned}}$$

and

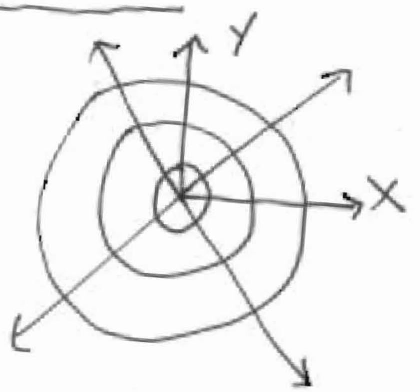
$$\underline{A} = \underline{A}_r + \underline{A}_\phi \quad -(46)$$

is the total vector potential.

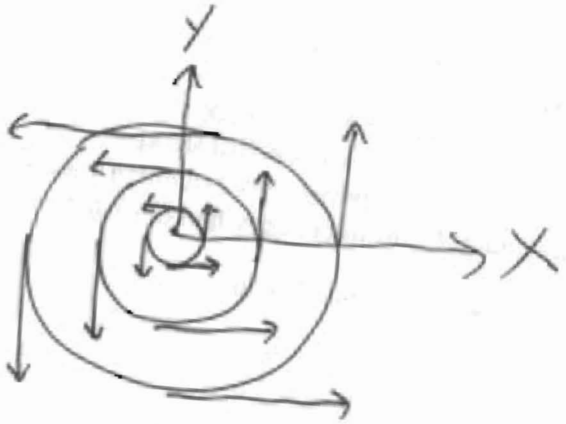
In Cartesian coordinates :

$$\boxed{\begin{aligned} \underline{A}_r &= \frac{m}{e}\frac{dr}{dt}\left(\frac{X\,\underline{i} + Y\,\underline{j}}{(X^2+Y^2)^{1/2}}\right) \quad -(47) \\ \underline{A}_\phi &= \frac{m}{e}\frac{d\phi}{dt}\left(-Y\,\underline{i} + X\,\underline{j}\right) \quad -(48) \end{aligned}}$$

Thus $\underline{A}_r$ has zero divergence and $\underline{A}_\phi$ has zero curl.



$\underline{A}_r$



$\underline{A}_\phi$

Eq. (46) is the Helmholtz Theorem. It can be extended to:

$$\underline{A} = \underline{A}^{(1)} + \underline{A}^{(2)} + \underline{A}^{(3)} \quad -(49)$$

where

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right)$$

$$\underline{e}^{(3)} = \underline{k}$$