1)

153(5): The Equation of Motion of Dynamics and Electrodynamics in General Relativity.

In a spherical spacetime this is:

$$\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - e^{-r_0/r}\left(mc^2 + \frac{L^2}{mr^2}\right)\right) \quad -(1)$$

Various approximations of $exp(-r_0/r)$ introduce various new forces into dynamics and electrodynamics. These are hitherto unknown, but one of them occurs in a binary pulsar, others occur in a whirlpool galaxy.

The usual (mis-named "Schwarzschild") approx. is:

$$e^{-r_0/r} \sim 1 - \frac{r_0}{r} \quad -(2)$$

so eq. (1) becomes: $\quad -(3)$

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 - \frac{mMG}{r} - \frac{MGL^2}{mc^2 r^3} + \frac{L^2}{2mr^2}$$

$$= \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + V \quad -(4)$$

where "the effective potential" is, in joules:

$$V = -\frac{mMG}{r} + \frac{L^2}{2mr^2} - \frac{MGL^2}{mc^2 r^3}, \quad -(5)$$

$$F = -\partial V / \partial r. \quad -(6)$$

So in this approximation there are three fundamental forces in dynamics and electrodynamics. The total force between m and M in dynamics is:

2)

$$\underline{F} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3 \quad -(7)$$

where:

$$|\underline{F}_1| = F_1 = -\frac{m\underline{M}G}{r^2} \quad -(8)$$

$$|\underline{F}_2| = F_2 = \frac{L^2}{mr^3} \quad -(9)$$

$$|\underline{F}_3| = F_3 = -\frac{3\underline{M}GL^2}{mc^2r^4} \quad -(10)$$

These forces also occur in electrodynamics between e_1 and e_2. The force F_1 is negative and attractive and is the inverse square force of attraction. The force F_2 is positive valued and repulsive, and is the centrifugal force. The force F_3 is negative valued and attractive, and produces the precession of orbits.

In eq. (3), the infinitesimal $d\tau$ of proper time is defined by:

$$c^2 d\tau^2 = c^2 dt^2\left(1 - \frac{r_0}{r}\right) - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \quad -(11)$$

where by definition:

$$d\underline{r} \cdot d\underline{r} := \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2 d\phi^2. \quad -(12)$$

The velocity $\underline{v}$ is defined by:

$$v^2 dt^2 := d\underline{r} \cdot d\underline{r} \quad -(13)$$

In the frame in which m is at rest:

$$v = 0 \quad -(14)$$

3)

so

$$\frac{d\tau}{dt} = \left(1 - \frac{r_0}{r}\right)^{1/2} \text{ if } v = 0. \quad -(15)$$

otherwise:

$$d\tau^2 = \left(1 - \frac{r_0}{r}\right) dt^2 - \frac{v^2}{c^2} dt^2 \quad -(16)$$

$$\boxed{\frac{d\tau}{dt} = \left(1 - \frac{r_0}{r} - \frac{v^2}{c^2}\right)^{1/2}} \quad -(17)$$

Therefore:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{dr}{dt}\right)^2 \left(\frac{dt}{d\tau}\right)^2 = \left(1 - \frac{r_0}{r} - \frac{v^2}{c^2}\right)^{-1} \left(\frac{dr}{dt}\right)^2 \quad -(18)$$

The equation of motion (4) is therefore:

$$\boxed{\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2} m \left(1 - \frac{r_0}{r} - \frac{v^2}{c^2}\right)^{-1} \left(\frac{dr}{dt}\right)^2 + V} \quad -(19)$$

Here:

$$E = mc^2 \left(1 - \frac{r_0}{r}\right) \frac{dt}{d\tau} = mc^2 \left(1 - \frac{r_0}{r}\right) \left(1 - \frac{r_0}{r} - \frac{v^2}{c^2}\right)^{-1/2}, \quad -(20)$$

$$p = m \left(1 - \frac{r_0}{r}\right) \left(1 - \frac{r_0}{r} - \frac{v^2}{c^2}\right)^{-1/2} \frac{dr}{dt}, \quad -(21)$$

and

$$L = mr^2 \left(1 - \frac{r_0}{r} - \frac{v^2}{c^2}\right)^{-1/2} \frac{d\phi}{dt} \quad -(22)$$

are constants of motion and conserved.

Limit of Special Relativity

In this limit, the metric approaches the Minkowski metric:

4)

$$c^2 d\tau^2 = c^2 dt^2 - d\underline{r} \cdot d\underline{r} \quad -(23)$$

so

$$r \to \infty \quad -(24)$$

or

$$r_0 \to 0. \quad -(25)$$

Therefore:

$$\boxed{E = \gamma mc^2, \; p = \gamma m \frac{dr}{dt}, \; L = \gamma m r^2 \frac{d\phi}{dt}} \quad (26)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \quad -(27)$$

In special relativity, E is the relativistic total energy, p is the relativistic momentum and L the relativistic angular momentum. The angular velocity is defined as:

$$\omega := \frac{d\phi}{dt}. \quad -(28)$$

So eq. (19) reduces to:

$$\frac{1}{2}\left(\gamma^2 - 1\right) mc^2 = \frac{1}{2} m \gamma^2 \left(\frac{dr}{dt}\right)^2 + V \quad -(29)$$

where

$$V = -\frac{mMG}{r} + \frac{1}{2}\gamma^2 m r^2 \left(\frac{d\phi}{dt}\right)^2 \quad -(30)$$

Therefore:

$$\boxed{\frac{1}{2}\left(\gamma^2 - 1\right) mc^2 = -\frac{mMG}{r} + \frac{1}{2}\gamma^2 m v^2} \quad -(31)$$

where

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\phi}{dt}\right)^2 \quad -(32)$$

In the Newtonian limit the total energy

5) For Kepler orbits is defined as:

$$E = T + V \quad -(33)$$

$$E = \frac{1}{2}mv^2 - \frac{mMG}{r} \quad -(34)$$

where v is defined by eq. (32). In the limit:

$$r \rightarrow \infty \quad -(33)$$

eq. (31) becomes:

$$\frac{1}{2}(\gamma^2 - 1)mc^2 = \frac{1}{2}\gamma^2 mv^2 = \frac{1}{2}\frac{p^2}{m} \quad -(34)$$

which is the Einstein energy equation:

$$E^2 = p^2c^2 + m^2c^4 \quad -(35)$$

for a free particle. The relativistic kinetic energy in special relativity is:

$$T = mc^2(\gamma - 1) \quad -(36)$$

If

$$v << c \quad -(37)$$

$$\gamma \rightarrow 1 + \frac{1}{2}\frac{v^2}{c^2} \quad -(38)$$

so

$$T \rightarrow \frac{1}{2}mv^2 \quad -(39)$$

Also in the limit (37), $\gamma \rightarrow 1 \quad -(40)$

so eq. (34) becomes:

$$\frac{1}{2}mv^2 = \frac{1}{2}mv^2 \quad -(41)$$

More generally,

$$e^{-r_0/r} = 1 - \frac{r_0}{r} + \frac{1}{2!}\left(\frac{r_0}{r}\right)^2 - \frac{1}{3!}\left(\frac{r_0}{r}\right)^3 + \dots$$

and these are more forces.