153(3): Convenient Derivation of Energy Equation from Metric.

Start for example from the gravitational metric:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_0}{r}\right) c^2 dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2 d\phi^2 \quad - (1)$$

then:

$$H = \frac{1}{2} m c^2 = \frac{1}{2} m \left(\left(1 - \frac{r_0}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{r_0}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \right) \quad - (2)$$

So:

$$\frac{1}{2}\left(1 - \frac{r_0}{r}\right) m c^2 = \frac{mc^2}{2}\left(1 - \frac{r_0}{r}\right)^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 - \left(1 - \frac{r_0}{r}\right)\frac{m r^2}{2}\left(\frac{d\phi}{d\tau}\right)^2 \quad - (3)$$

The constants of motion are:

$$E = mc^2 \left(1 - \frac{r_0}{r}\right)\left(\frac{dt}{d\tau}\right), \quad L = m r^2 \frac{d\phi}{d\tau} \quad - (4)$$

so the hamiltonian H and total energy E come from the metric, as does the angular momentum L. These are conserved.

So:

$$\boxed{\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - \left(1 - \frac{r_0}{r}\right)\left(mc^2 + \frac{L^2}{mr^2}\right)\right)} \quad - (5)$$

This is the equation of motion. The orbital equation is obtained from:

$$\frac{dr}{d\tau} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \left(\frac{L}{mr^2}\right)\frac{dr}{d\phi} \quad - (6)$$