53(6): On the Approach to the Newtonian Limit

The equation of motion of note 153(5) is:

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + V \quad -(1)$$

here:

$$V = -\frac{mMG}{r} + \frac{L^2}{2mr^2} - \frac{MGL^2}{mc^2r^3} \quad -(2)$$

The Newtonian limit is:

$$E_N = \frac{1}{2}m\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2mr^2} - \frac{mMG}{r} \quad -(3)$$

Marion and Thornton eq. (7.14)).

In eqns (1) and (2):

$$\frac{E^2}{mc^2} = mc^2\left(1-\frac{r_0}{r}\right)^2\left(1-\frac{r_0}{r}-\frac{v^2}{c^2}\right)^{-1} \quad -(4)$$

$$L = mr^2\left(1-\frac{r_0}{r}-\frac{v^2}{c^2}\right)^{-1/2}\frac{d\phi}{dt} \quad -(5)$$

The way in which eqns. (1) and (2) reduce to (3) is as follows, and is rarely if ever explained correctly in textbooks. On the left hand side of eq. (1):

$$\left(1-\frac{r_0}{r}-\frac{v^2}{c^2}\right) = \left(1-\frac{r_0}{r}\right)\left(1-\frac{v^2}{c^2}\right) - \frac{r_0}{r}\frac{v^2}{c^2} \quad -(6)$$

Now assume:

$$\frac{r_0}{r} << 1, \quad \frac{v}{c} << 1 \quad -(7)$$

so

$$\left(1-\frac{r_0}{r}-\frac{v^2}{c^2}\right) \sim \left(1-\frac{r_0}{r}\right)\left(1-\frac{v^2}{c^2}\right) \quad -(7)$$

so

$$\frac{E^2}{mc^2} \sim mc^2\left(1-\frac{r_0}{r}\right)\left(1-\frac{v^2}{c^2}\right)^{-1} \quad -(8)$$

2) Now use:

$$\left(1-\frac{v^2}{c^2}\right)^{-1} \sim 1+\frac{v^2}{c^2} \quad -(9)$$

So $$\frac{E^2}{mc^2} \sim mc^2\left(1-\frac{r_0}{r}\right)\left(1+\frac{v^2}{c^2}\right) \quad -(10)$$

and $$\frac{1}{2}\left(\frac{E^2}{mc^2}-mc^2\right) \sim \frac{1}{2}mv^2 - \frac{1}{2}mc^2\frac{r_0}{r}$$

$$= \frac{1}{2}mv^2 - \frac{mMG}{r} \quad -(11)$$

$$= E_N$$

Q.E.D.

On the right hand side of eq. (1), in the Newtonian limit the following is assumed:

$$V \rightarrow -\frac{mMG}{r} + \frac{L^2}{2mr^2} \quad -(12)$$

$$L \rightarrow mr^2\frac{d\phi}{dt} \quad -(13)$$

i.e eq. (7) is assumed again self-consistently.

So the ~~left~~ right hand side of eq. (1) becomes:

$$RHS \rightarrow \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}mr^2\frac{d\phi}{dt} - \frac{mMG}{r}$$

$$-(14)$$

Finally it is assumed that:

$$\left(\frac{dr}{d\tau}\right)^2 = \left(1-\frac{r_0}{r}-\frac{v^2}{c^2}\right)^{-1}\left(\frac{dr}{dt}\right)^2 \rightarrow \left(\frac{dr}{dt}\right)^2 \quad -(15)$$

Therefore:

3)

$$\text{RHS} \rightarrow \frac{1}{2} m \left(\left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 \right) - \frac{mMG}{r}$$

$$= \frac{1}{2} m v^2 - \frac{mMG}{r} \quad - (16)$$

$$v^2 = \left(\frac{dr}{dt} \right)^2 + r^2 \left(\frac{d\phi}{dt} \right)^2 \quad - (17)$$

So eq. (1) reduces to the Newtonian:

$$E_N = \frac{1}{2} m v^2 - \frac{mMG}{r} = T + V \quad - (18)$$

QED. Having checked that the equation (1) reduces correctly to the Newtonian (18), it is now possible to extend the analysis in many ways, e.g. to e/m and to spherical spacetime, to mixed metrics and resonance equations.
