1) 153(10): Calculation and Meaning of Tetrads in the Cylindrical Polar Coordinates.

In curvilinear coordinates in three dimensions let:

$$\underset{\sim}{r} = \underset{\sim}{r}(u_1, u_2, u_3) \quad -(1)$$

then the scaling factors are defined by:

$$h_1 = \left|\frac{\partial r}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial r}{\partial u_2}\right|, \quad h_3 = \left|\frac{\partial r}{\partial u_3}\right| \quad -(2)$$

The unit tangent vectors are defined by:

$$\underline{e}_1 = \frac{1}{h_1}\frac{\partial r}{\partial u_1}, \quad \underline{e}_2 = \frac{1}{h_2}\frac{\partial r}{\partial u_2}, \quad \underline{e} = \frac{1}{h_3}\frac{\partial r}{\partial u_3} \quad -(3)$$

In this curvilinear representation the metric form is defined by

$$g_{ij} = \frac{\partial r}{\partial u_1} \cdot \frac{\partial r}{\partial u_2} \quad -(4)$$

so for a diagonal tetrad:

$$g_{11} = h_1^2, \quad g_{22} = h_2^2, \quad g_{33} = h_3^2. \quad -(5)$$

In Cartan geometry however the metric is generalized to:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(6)$$

where $q^a_\mu$ and $q^b_\nu$ are tetrads and where $\eta_{ab}$ is the Minkowski metric.

In this note the metrics and tetrads are worked out in both representations.

Specializing to cylindrical polar coordinates:

$$\underset{\sim}{r} = X\,\underline{i} + Y\,\underline{j} + Z\,\underline{k} = r\cos\phi\,\underline{i} + r\sin\phi\,\underline{j} + Z\,\underline{k} \quad -(7)$$

so

$$\partial \underline{r} / \partial r = \cos\phi\,\underline{i} + \sin\phi\,\underline{j} \quad -(8)$$

$$\partial \underline{r} / \partial \phi = r(-\sin\phi\,\underline{i} + \cos\phi\,\underline{j}) \quad -(9)$$

$$\partial \underline{r} / \partial Z = \underline{k} \quad -(10)$$

The scaling factors are:

2)

$$h_1 = |\partial \underline{r} / \partial r| = (\cos^2\phi + \sin^2\phi)^{1/2} = 1 \quad -(11)$$

$$h_2 = |\partial \underline{r} / \partial \phi| = r(\sin^2\phi + \cos^2\phi)^{1/2} = r \quad -(12)$$

$$h_3 = |\partial \underline{r} / \partial z| = 1 \quad -(13)$$

So:

$$\underline{e}_r = \underline{e}_1 = \cos\phi\, \underline{i} + \sin\phi\, \underline{j} \quad -(14)$$

$$\underline{e}_\phi = \underline{e}_2 = -\sin\phi\, \underline{i} + \cos\phi\, \underline{j} \quad -(15)$$

$$\underline{e}_z = \underline{e}_3 = \underline{k} \quad -(16)$$

So

$$g_{11} = h_1^2 = 1 \quad -(17)$$

$$g_{22} = h_2^2 = r^2 \quad -(18)$$

$$g_{33} = h_3^2 = 1 \quad -(19)$$

and

$$g_{ij} = g_{ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad -(20)$$

In Cartan geometry the tetrad $q^a_\mu$ is a matrix that is defined by:

$$e^a = q^a_\mu e^\mu \quad -(21)$$

where $e^a$ are unit elements of the basis labelled $a$ and $e^\mu$ are unit elements of the basis labelled $\mu$. Let $a$ denote the cylindrical polar basis and $\mu$ the Cartesian basis. Let the unit vectors of the cylindrical polar basis be denoted by:

$$\underline{e}_r = (e^{(1)}, 0, 0) \quad -(22)$$

$$\underline{e}_\phi = (0, e^{(2)}, 0) \quad -(23)$$

$$\underline{e}_z = (0, 0, e^{(3)}) \quad -(24)$$

In the Cartesian basis these unit vectors are:

3)

$$\underline{e}_r = (e_1, e_2, 0) = (\cos\phi, \sin\phi, 0) \quad -(25)$$

$$\underline{e}_\phi = (e_1, e_2, 0) = (-\sin\phi, \cos\phi, 0) \quad -(26)$$

$$\underline{e}_Z = (0, 0, e_3) = (0, 0, 1) \quad -(27)$$

We have

$$(e^{(1)}, 0, 0) = (1, 0, 0) \quad -(28)$$

$$(0, e^{(2)}, 0) = (0, 1, 0) \quad -(29)$$

$$(0, 0, e^{(3)}) = (0, 0, 1) \quad -(30)$$

Applying eq. (21):

$$e^{(1)} = q^{(1)}_1 e^1 + q^{(1)}_2 e^2 \quad -(31)$$

$$e^{(2)} = q^{(2)}_1 e^1 + q^{(2)}_2 e^2 \quad -(32)$$

i.e.

$$q^{(1)}_1 \cos\phi + q^{(1)}_2 \sin\phi = 1 \quad -(33)$$

$$-q^{(2)}_1 \sin\phi + q^{(2)}_2 \cos\phi = 1 \quad -(34)$$

Thus, a possible solution is:

$$q^{(1)}_1 = \cos\phi, \quad q^{(1)}_2 = \sin\phi, \quad -(35)$$

$$q^{(2)}_1 = -\sin\phi, \quad q^{(2)}_2 = \cos\phi \quad -(36)$$

$$q^{(3)}_3 = 1 \quad -(37)$$

$$\boxed{q^a_\mu = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad -(38)$$

The tetrad vectors are defined by the unit vector as follows:

4)

$$\boxed{\begin{aligned} \underline{q}^{(1)} &= \underline{e}^{(1)} = \underline{e}_1 = \underline{e}_r = q^{(1)}_1 \, \underline{i} + q^{(1)}_2 \, \underline{k} \quad &-(39)\\ \underline{q}^{(2)} &= \underline{e}^{(2)} = \underline{e}_2 = \underline{e}_\phi = q^{(2)}_1 \, \underline{i} + q^{(2)}_2 \, \underline{k} \quad &-(40)\\ \underline{q}^{(3)} &= \underline{e}^{(3)} = \underline{e}_3 = \underline{e}_Z = \underline{k} \quad &-(41) \end{aligned}}$$

So any vector can be defined by a tetrad vector, because the latter is a unit vector.

For example the electromagnetic potential tetrad is:

$$A^a_\mu = A^{(0)} q^a_\mu . \quad -(42)$$

This is the first ECE hypothesis.

Finally using:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab}, \quad -(43)$$

then

$$g_{11} = q^{(1)}_1 q^{(1)}_1 \eta_{(1)(1)} + q^{(2)}_1 q^{(2)}_1 \eta_{(2)(2)} \quad -(44)$$

$$g_{22} = q^{(1)}_2 q^{(1)}_2 \eta_{(1)(1)} + q^{(2)}_2 q^{(2)}_2 \eta_{(2)(2)} \quad -(45)$$

$$g_{33} = q^{(3)}_3 q^{(3)}_3 \eta_{(3)(3)} \quad -(46)$$

So:

$$g_{11} = \cos^2\phi \, \eta_{(1)(1)} + \sin^2\phi \, \eta_{(2)(2)} \quad -(47)$$

$$g_{22} = \sin^2\phi \, \eta_{(1)(1)} + \cos^2\phi \, \eta_{(2)(2)} \quad -(48)$$

$$g_{33} = \eta_{(3)(3)} = 1 \quad -(49)$$

So if

$$\eta_{(1)(1)} = \eta_{(2)(2)} = \underline{1} \quad -(50)$$

then

$$g_{11} = g_{22} = 1 \quad -(51)$$

2) In this representation:

$$g_{ij} = g_{ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(52)$$

$$\eta_{ij} = \eta_{ji} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(53)$$

This is the correct and more consistent Cartan representation, showing that both cylindrical polar and Cartesian coordinates represent the same flat space.

In the curvilinear definition (4), the result is eq. (20), which is different from the Cartan result (52). The latter is independent of coordinates. The line element is:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \quad -(54)$$

In order to obtain tetrads from metrics the coordinate independent Cartan representation should be used, i.e. eq. (6), with the tetrad defined by eq. (21). In four dimensions:

$$\eta_{ab} = \mathrm{diag}(1, -1, -1, -1) \quad -(55)$$

$$e^a = q^a_{\ \mu}\, e^\mu \quad -(56)$$

$$g_{\mu\nu} = q^a_{\ \mu}\, q^b_{\ \nu}\, \eta_{ab} \quad -(57)$$

The method of obtaining the tetrad from the metric must be precisely defined because the field equations are obtained from the tetrads.