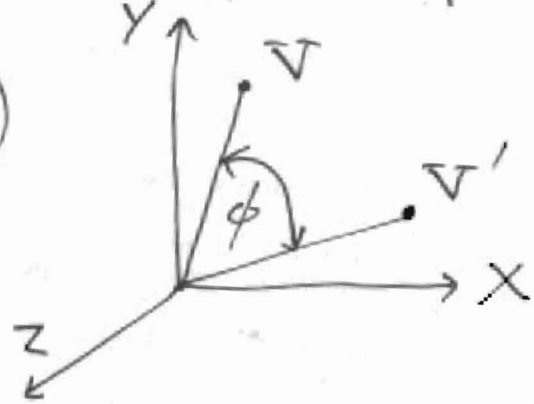
1) 153(12): Connection Matrix and Rotation Generator.

It was shown in note 153(11) that the tetrad matrix is the transformation matrix. Consider the rotation of a vector in 3-D in the XY plane about the Z axis (Ryder p. 30):

$$\begin{bmatrix} V_X' \\ V_Y' \\ V_Z' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_X \\ V_Y \\ V_Z \end{bmatrix} \quad -(1)$$



The passive rotation is defined as the rotation of a vector about the Z axis by rotating the axes anticlockwise (see Ryder p. 30). The rotation of the axes introduces a connection. The rotation generator is defined by:

$$J_Z = \frac{1}{i} \left. \frac{dR_Z(\phi)}{d\phi} \right|_{\phi=0} = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad -(2)$$

where

$$R_Z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(3)$$

so, as in Ryder:

$$[J_X, J_Y] = iJ_Z \quad -(4)$$

et cyclicum

The angular momentum operators of quantum mechanics are

$$[J_X, J_Y] = i\hbar J_Z \quad -(5)$$

et cyclicum.

These operators originate in the tetrad:

2)

$$q^a_\mu = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} q^{(1)}_1 & q^{(1)}_2 & 0 \\ q^{(2)}_1 & q^{(2)}_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(6)$$

where, in the cylindrical polar system:

$$\underline{e}_r = \underline{e}_1 = \underline{e}^{(1)} = \underline{q}^{(1)} = q^{(1)}_1 \underline{i} + q^{(1)}_2 \underline{j} \quad -(7)$$

$$\underline{e}_\phi = \underline{e}_2 = \underline{e}^{(2)} = \underline{q}^{(2)} = q^{(2)}_1 \underline{i} + q^{(2)}_2 \underline{j} \quad -(8)$$

$$\underline{e}_Z = \underline{e}_3 = \underline{e}^{(3)} = \underline{q}^{(3)} = \underline{k} \quad -(9)$$

In Cartan geometry this process is governed by the tetrad postulate, because rotation conserves the vector field, so:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda = 0 \quad -(10)$$

$$= \partial_\mu q^a_\nu + \omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}$$

Therefore

$$\boxed{\partial_\mu q^a_\nu = \alpha^a_{\mu\nu}} \quad -(11)$$

where:

$$\alpha^a_{\mu\nu} = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}. \quad -(12)$$

So the connection matrix $\alpha^a_{\mu\nu}$ is a rotation generator matrix and an angular momentum operator.

In the notation of eq. (6) the lower indices without brackets are indices of the Cartesian system and the upper indices with brackets are indices of the cylindrical polar system.

3) This system is defined by:

$$\cos\phi = \frac{X}{r}, \quad \sin\phi = \frac{Y}{r}, \quad Z = Z \quad -(13)$$

where

$$r = \left(X^2 + Y^2\right)^{1/2} \quad -(14)$$
$$= \text{constant}$$

Therefore:

$$\partial_2 q^a_\mu = \frac{\partial q^a_\mu}{\partial Y} = \frac{1}{r}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad -(15)$$

$$\partial_1 q^a_\mu = \frac{\partial q^a_\mu}{\partial X} = \frac{1}{r}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad -(16)$$

So:

$$\frac{1}{r}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \omega^{(1)}_{22} & 0 \\ \omega^{(2)}_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad -(17)$$

i.e.

$$\boxed{\omega^{(1)}_{22} = \frac{1}{r}, \quad \omega^{(2)}_{21} = -\frac{1}{r}} \quad -(18)$$

As in paper 63, the connections are inversely proportional to the radial component r.

$$\boxed{\begin{bmatrix} 0 & \omega^{(1)}_{22} & 0 \\ \omega^{(2)}_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{i}{r} J_Z} \quad -(19)$$

Q.E.D.

The connection matrix is the rotation generator matrix within i/r, and the angular momentum operator

4)

matrix with $r\phi$: $1/(2r)$.

These conclusions have been built up from consideration only of the cylindrical polar coordinate system and how it is related to the Cartesian coordinate system. The whole of quantum mechanics in molecular spectroscopy (see Atkins) depends on the angular momentum operators.

The metric is defined by:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta \qquad -(20)$$

so

$$g_{11} = q^{(1)}_1 q^{(1)}_1 \eta_{(1)(1)} + q^{(2)}_1 q^{(2)}_1 \eta_{(2)(2)} \qquad -(21)$$

$$= \cos^2\phi + \sin^2\phi = 1$$

if

$$\eta_{(1)(1)} = \eta_{(2)(2)} = 1 \qquad -(22)$$

so:

$$\boxed{g_{\mu\nu} = \eta_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad -(23)$$

The Cartan torsion is defined by:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu\nu} - \omega^a_{\nu\mu}$$

$$= \Gamma^a_{\mu\nu} - \Gamma^a_{\nu\mu} \qquad -(24)$$

and there are non-zero elements such as:

$$\boxed{\begin{aligned} T^{(2)}_{12} &= \partial_1 q^{(2)}_2 - \partial_2 q^{(2)}_1 + \omega^{(2)}_{12} - \omega^{(2)}_{21} \\ &= 2/r + \omega^{(2)}_{12} - \omega^{(2)}_{21} \end{aligned}} \qquad -(25)$$

5) The Riemann torsion is defined by:

$$T^{\lambda}_{\mu\nu} = q^{\lambda}_{a} T^{a}_{\mu\nu} \quad (26)$$

so elements of the Riemann torsion also exist. The Riemann torsion is:

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \quad (27)$$

so the $\Gamma^{\lambda}_{\mu\nu}$ connection also exists. The latter is defined by:

$$[D_\mu, D_\nu] V^{\sigma} = R^{\sigma}{}_{\lambda\mu\nu} V^{\lambda} - T^{\rho}_{\mu\nu} D_\rho V^{\sigma} \quad (28)$$

$$= - \Gamma^{\rho}_{\mu\nu} D_\rho V^{\sigma} + \ldots$$

so:

$$\boxed{\Gamma^{\lambda}_{\mu\nu} = - \Gamma^{\lambda}_{\nu\mu}} \quad (29)$$

because:

$$[D_\mu, D_\nu] = - [D_\nu, D_\mu] \quad (30)$$

The covariant derivative is:

$$D_\mu X^{\nu} = \partial_\mu X^{\nu} + \Gamma^{\nu}_{\mu\lambda} X^{\lambda} \quad (31)$$

so is different from $\partial_\mu X^{\nu}$ in general. This difference is due to the passive rotation of eq. (1).

6)

Finally, eq. (11) is the mathematical expression of the fact that a passive rotation is equivalent to the usual way in which a rotation is thought of – as an active rotation of a vector keeping coordinates fixed. The active rotation is $\partial_\mu q^a_\nu$ and the passive rotation is $\omega^a_{\mu\nu}$.

Obviously, in the usual way of dealing with rotations, the idea of connection is never considered in what is referred to as "three dimensional space", but the connection is inherent in the analysis. The very definition of the cylindrical polar coordinates is enough to generate a connection.