153(11): Tetrad Elements and Tetrad in Some Coordinate Systems.

Theorem

The tetrad elements are the elements of any unit vector in any coordinate system when those unit vectors are expressed in terms of Cartesian unit vectors.

The following are examples of this theorem.

Cartesian Coordinates

$$\underline{e}_1 = \underline{e}^{(1)} = \underline{i} = q^{(1)}_1 \underline{i} \quad -(1)$$
$$\underline{e}_2 = \underline{e}^{(2)} = \underline{j} = q^{(2)}_2 \underline{j} \quad -(2)$$
$$\underline{e}_3 = \underline{e}^{(3)} = \underline{k} = q^{(3)}_3 \underline{k} \quad -(3)$$

$$\boxed{\begin{aligned} q^{(1)}_1 &= q^{(2)}_2 = q^{(3)}_3 = 1 \\ \underline{q}^{(1)} &= \underline{i},\ \underline{q}^{(2)} = \underline{j},\ \underline{q}^{(3)} = \underline{k} \end{aligned}} \quad -(4)$$

Complex Circular Coordinates

$$\underline{q}^{(1)} = \underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) \quad -(5)$$
$$\underline{q}^{(2)} = \underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) \quad -(6)$$
$$\underline{q}^{(3)} = \underline{e}^{(3)} = \underline{k} \quad -(7)$$

$$\boxed{\begin{aligned} q^{(1)}_1 &= \frac{1}{\sqrt{2}}, & q^{(1)}_2 &= -\frac{i}{\sqrt{2}}, & q^{(1)}_3 &= 0, \\ q^{(2)}_1 &= \frac{1}{\sqrt{2}}, & q^{(2)}_2 &= \frac{i}{\sqrt{2}}, & q^{(2)}_3 &= 0, \\ q^{(3)}_1 &= 0, & q^{(3)}_2 &= 0, & q^{(3)}_3 &= 1. \end{aligned}} \quad -(8)$$

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Cylindrical Polar Coordinates

$$\underline{v}^{(1)} = \underline{e}^{(1)} = \underline{e}_r = \underline{e}_1 = \cos\phi\,\underline{i} + \sin\phi\,\underline{j} \quad -(9)$$

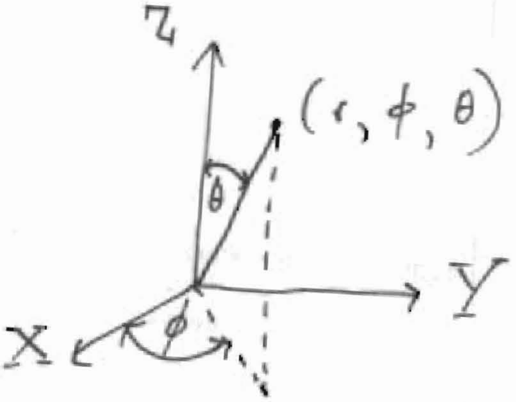
$$\underline{v}^{(2)} = \underline{e}^{(2)} = \underline{e}_\phi = \underline{e}_2 = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j} \quad -(10)$$

$$\underline{v}^{(3)} = \underline{e}^{(3)} = \underline{e}_z = \underline{e}_3 = \underline{k} \quad -(11)$$

$$\boxed{\begin{array}{lll} v_1^{(1)} = \cos\phi, & v_2^{(1)} = \sin\phi, & v_3^{(1)} = 0 \\ v_1^{(2)} = -\sin\phi, & v_2^{(2)} = \cos\phi, & v_3^{(2)} = 0 \\ v_1^{(3)} = 0, & v_2^{(3)} = 0, & v_3^{(3)} = 1 \end{array}} \quad -(12)$$

Spherical Polar Coordinates

$$\left.\begin{array}{l} X = r\sin\theta\cos\phi \\ Y = r\sin\theta\sin\phi \\ Z = r\cos\theta \end{array}\right\} \quad -(13)$$



$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad -(14)$$

$$\underset{\sim}{r} = r\left(\sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}\right) \quad -(15)$$

$$\underline{e}_1 = \underline{e}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k} \quad -(16)$$
$$= \partial\underline{r}/\partial r \Big/ |\partial\underline{r}/\partial r|$$

$$\underline{e}_2 = \underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k} \quad -(17)$$
$$= \partial\underline{r}/\partial\theta \Big/ |\partial\underline{r}/\partial\theta|$$

$$\underline{e}_3 = \underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j} \quad -(18)$$
$$= \partial\underline{r}/\partial\phi \Big/ |\partial\underline{r}/\partial\phi|$$

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$$\left.\begin{aligned} q^{(1)}_1 &= \sin\theta\cos\phi, & q^{(1)}_2 &= \sin\theta\sin\phi, & q^{(1)}_3 &= \cos\theta, \\ q^{(2)}_1 &= \cos\theta\cos\phi, & q^{(2)}_2 &= \cos\theta\sin\phi, & q^{(2)}_3 &= -\sin\theta, \\ q^{(3)}_1 &= -\sin\phi, & q^{(3)}_2 &= \cos\phi, & q^{(3)}_3 &= 0 \end{aligned}\right\} - (19)$$

Summary Table

Coordinate System	Tetrad q^a_μ
Cartesian	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Complex Circular	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$
Cylindrical Polar	$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Spherical Polar	$\begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}$