1) 152(7): Relativistic Corrections from the Electrodynamic Metric

: The electrodynamic metric for a spherically symmetric spacetime is:

$$ds^2 = c^2 d\tau^2 = e^{-r_0/r} c^2 dt^2 - e^{r_0/r} dr^2 - r^2 d\phi^2 \quad -(1)$$

where

$$r_0 = 2\left(\frac{e_1}{m}\right)\left(\frac{e_2}{4\pi\epsilon_0 c^2}\right) \quad -(2)$$

Here e_1 and m are the charge and mass of an electron orbiting an object of charge e_2 and mass M. In the approximation:

$$e^{-r_0/r} \sim 1 - \frac{r_0}{r} + \cdots \quad -(3)$$

the equation of motion from metric (1) is:

$$\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - \left(1 - \frac{r_0}{r}\right)\left(mc^2 + \frac{L^2}{mr^2}\right)\right) \quad -(4)$$

where E is the total energy and L the angular momentum, E and L are constants of motion as in previous work. Both E and L are constants of motion.

Eq. (4) is:

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{L^2}{mr^2} - \frac{1}{2}\frac{r_0}{r}\left(mc^2 + \frac{L^2}{mr^2}\right) \quad -(5)$$

In the limit:

$$\frac{r_0}{r} \to 0 \quad -(6)$$

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{L^2}{mr^2} \quad -(7)$$

This is the free particle equation

2)

$$p^\mu p_\mu = m^2 c^2 \quad -(8)$$

i.e.

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{p^2}{2m} = \frac{1}{2} m \gamma^2 v^2 \quad -(9)$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(10)$$

Thus:

$$\gamma^2 v^2 = \left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{mr^2} \quad -(11)$$

In the non relativistic limit:

$$v \ll c \quad -(12)$$

the classical Newtonian result is obtained:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{mr^2} \quad -(13)$$

Thus eqn. (7) is:

$$\frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{1}{2} m \gamma^2 v^2 = \frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{L^2}{mr^2} \quad -(14)$$

Eq. (14) is eq. (8) in cylindrical polar coordinates. Eq. (8) quantizes to:

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right)\psi = 0, \quad -(15)$$

from which the Dirac equation is obtained for the Dirac electron.

The free particle Hamiltonian is:

$$H = \frac{p^2}{2m} = \frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) \quad -(16)$$

and the free particle is the free electron.

When the electron interacts with the charge e_2 the Hamiltonian is changed to:

$$H = \frac{p^2}{2m} - \frac{1}{2}\frac{r_0}{r}\left(mc^2 + \frac{L^2}{mr^2}\right) \quad -(17)$$

$$\boxed{H = \frac{p^2}{2m} - \frac{e_1 e_2}{4\pi\epsilon_0 r}\left(1 + \left(\frac{L}{mc}\right)^2 \frac{1}{r^2}\right)} \quad -(18)$$

where $L/(mc)$ is a constant. This Hamiltonian can be thought of for convenience as:

$$H = T + V \quad -(19)$$

where

$$T = \frac{p^2}{2m}, \quad -(20)$$

$$V = -\frac{e_1 e_2}{4\pi\epsilon_0 r}\left(1 + \left(\frac{L}{mc}\right)^2 \frac{1}{r^2}\right) \quad -(21)$$

Eq. (21) gives the relativistic correction to the Coulomb Law.

If there is no angular momentum, i.e.

$$L = 0 \quad -(22)$$

4) The Coulomb law is unchanged. So if a charge is attracted or repelled along the line joining the charges, there is no relativistic correction.

However when an electron orbits a proton as in the hydrogen atom, there is a relativistic correction. The usual effective potential of the H atom is:

$$V = -\frac{e^2}{4\pi\epsilon_0 r} + \frac{l(l+1)\hbar^2}{2\mu r^2} \quad -(23)$$

where the L^2 is quantized to $l(l+1)\hbar^2$. This potential is changed to: $\quad -(24)$

$$V = -\frac{e^2}{4\pi\epsilon_0 r}\left(1 + l(l+1)\left(\frac{\hbar}{mc}\right)^2 \frac{1}{r^2}\right) + \frac{l(l+1)\hbar^2}{2mr^2}$$

where μ is the mass of the electron (m) to a good approximation.

Here:

$$\hbar = 1.05459 \times 10^{-34}\ Js$$
$$m = 9.10953 \times 10^{-31}\ kgm$$
$$c = 2.997925 \times 10^{8}\ ms^{-1}$$

so

$$\left(\frac{\hbar}{mc}\right)^2 = 1.4912 \times 10^{-25}\ m^2 \quad -(25)$$

The relativistic correction is small but dominates as $r \to 0$. It causes a precession of the orbitals of H, and affects the spectrum of H.