52(8): Sommerfeld and Dirac Theories of the H Atom from the Electrodynamical Metric.

The electrodynamical metric is:

$$ds^2 = c^2 d\tau^2 = e^{-r_0/r} c^2 dt^2 - e^{r_0/r} dr^2 - r^2 d\phi^2 \quad -(1)$$

where:

$$r_0 = \left(\frac{e_1}{m}\right)\left(\frac{e_2}{4\pi\epsilon_0 c^2}\right) \quad -(2)$$

The Hamiltonian is conserved and is the total energy:

$$H = \mathcal{L} = \frac{1}{2} mc^2 = \frac{1}{2} m \left(c^2 \left(\frac{dt}{d\tau}\right)^2 e^{-r_0/r} - e^{r_0/r} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \right) \quad -(3)$$

The Hamiltonian is defined as half the rest energy mc^2:

$$H = \frac{1}{2} E_0 \quad -(4)$$

The Hamiltonian and Lagrangian are the same because there is no concept of potential energy in general relativity. From the Lagrange equation the total energy E is conserved and is:

$$E = mc^2 e^{-r_0/r} \frac{dt}{d\tau} = \text{constant} \quad -(5)$$

The angular momentum is conserved and is:

$$L = mr^2 \frac{d\phi}{d\tau} = \text{constant} \quad -(6)$$

Therefore:

1)

$$\frac{1}{2}mc^2 e^{-r_0/r} = \frac{1}{2}m\left(c^2\left(\frac{dt}{d\tau}\right)^2 e^{-2r_0/r} - \left(\frac{dr}{d\tau}\right)^2 - e^{-r_0/r} r^2\left(\frac{d\phi}{d\tau}\right)^2\right)$$

$$= \frac{1}{2}\left(\frac{E^2}{mc^2} - m\left(\frac{dr}{d\tau}\right)^2 - e^{-r_0/r}\frac{L^2}{mr^2}\right) \quad -(7)$$

i.e.

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2 + mc^2\frac{r_0}{r} - \frac{L^2}{mr^2} + \frac{r_0 L^2}{mr^3}\right)$$

$$= \frac{1}{2}\left(\frac{E^2}{mc^2} - \left(1 - \frac{r_0}{r}\right)\left(mc^2 + \frac{L^2}{mr^2}\right)\right) \quad -(8)$$

So:

$$H = \frac{1}{2}mc^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - \frac{p^2}{m} + mc^2\frac{r_0}{r} + \frac{r_0 L^2}{mr^3}\right) \quad -(9)$$

As:

$$r \to \infty \quad -(10)$$

$$\boxed{H = \frac{1}{2}mc^2 = \frac{1}{2m}p^\mu p_\mu + \frac{e_1 e_2}{4\pi\epsilon_0 r}} \quad -(10)$$

where

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 \quad -(11)$$

$$p^2 = m^2\left(\frac{dr}{d\tau}\right)^2 + \frac{L^2}{r^2} \quad -(12)$$

For gravitation:

$$\boxed{H = \frac{1}{2}mc^2 = \frac{1}{2m}p^\mu p_\mu + \frac{mG}{r}} \quad -(13)$$

3) Eq. (10) is the correct form of the hamiltonian used by Sommerfeld in the old quantum theory to show that the orbit of the electron around the proton precesses. In the minimal prescription:

$$H = \frac{1}{2m}\left(p^{\mu} - eA^{\mu}\right)\left(p_{\mu} - eA_{\mu}\right) = \frac{1}{2}mc^2 \quad -(14)$$

It is clear that H is conserved, and is deduced from spacetime. The latter has electrodynamical as well as gravitational energy.

In general relativity, H is kinetic energy T and so:

$$H = \mathcal{L} = T. \quad -(15)$$

The concept of "potential" energy in classical dynamics and electrodynamics is subsumed into spacetime. In classical electrodynamics the object:

$$V = \frac{e_1 e_2}{4\pi\epsilon_0 r} \quad -(16)$$

is known as "the Coulomb potential energy". It is e_1 multiplied by the Coulomb potential $e_2/(4\pi\epsilon_0 r)$. However, in metric (1) V is part of spacetime itself.

This means that electromagnetic energy can be obtained from spacetime with H conserved.

For an electron proton system:

4)

$$V = -\frac{e^2}{4\pi\epsilon_0 r} \quad -(17)$$

because:

$$e_1 = -e_2 = e \quad -(18)$$

In classical dynamics the object:

$$V = -\frac{mMG}{r} \quad -(19)$$

is "the gravitational potential energy" obtained by multiplying the gravitational potential by m. The minus sign is the convention, denoting attraction. In general eq. (19) is a convention, denoting attraction. In general relativity, the "gravitational potential" is part of the purely kinetic H.

From eq. (10) & (13) it is seen that:

$$H = \frac{1}{2m}\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right) = \frac{1}{2}mc^2 \quad -(20)$$

is an invariant. The "potential" A^μ originates in metric (1) and is part of spacetime itself.

The Dirac equation is obtained using

$$p^\mu p_\mu = -\hbar^2 \square \quad -(21)$$

in the limit:

$$r_0 / r \rightarrow 0 \quad -(22)$$

of asymptotically flat spacetime. From eq. (10):

$$p^\mu p_\mu = m^2 c^2 - \frac{e_1 e_2 m}{2\pi \epsilon_0 r} \quad -(23)$$

so

$$-\hbar^2 \Box \phi = \left(m^2 c^2 - \frac{e_1 e_2 m}{2\pi \epsilon_0 r} \right) \phi \quad -(24)$$

i.e.

$$\boxed{\left(\Box + \left(\frac{mc}{\hbar} \right)^2 - \frac{e_1 e_2 m}{2\pi \epsilon_0 r \hbar^2} \right) \phi = 0} \quad -(25)$$

The classical limit of eq. (8) is obviously:

$$\frac{p^2}{2m} = \frac{1}{2} \left(\frac{E^2}{mc^2} - mc^2 \right) + \frac{e_1 e_2}{4\pi \epsilon_0 r} \quad -(26).$$

In the limit: $r \to \infty, \; v \ll c \quad -(27)$

$$\frac{p^2}{2m} \to \frac{1}{2} m v^2 + \frac{e_1 e_2}{4\pi \epsilon_0 r} \quad -(28)$$

The hamiltonian is defined in the non-relativistic limit as

$$H_{class} = \frac{p_{class}^2}{2m} + \frac{e_1 e_2}{4\pi \epsilon_0 r} \quad -(29)$$

where

$$p_{class} = mv \quad -(30)$$

Eq. (28) may be written as:

6)

$$T \rightarrow T + e_1 \phi \quad -(31)$$

where
$$\phi = \frac{e_2}{4\pi\epsilon_0 r} \quad -(32)$$

and
$$T = \frac{p^2}{2m} \quad -(33)$$

Relativistically, T is generalized to:

$$T \rightarrow \frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) = \frac{\gamma^2 m v^2}{2}$$

$$= \frac{p^2}{2m} \quad -(34)$$

Summary

The metric (1) gives:

$$\frac{p^2}{2m} = \frac{1}{2}\left(\frac{E^2}{mc^2} - mc^2\right) + e_1\phi$$

which is the minimal prescription:

$$T \rightarrow T + e_1\phi$$