152(9): Equivalence of Electrodynamic Metric and Minimal Prescription

In spherically symmetric spacetime the electrodynamic metric is:

$$ds^2 = c^2 d\tau^2 = e^{-r_0/r} c^2 dt^2 - e^{r_0/r} dr^2 - r^2 d\phi^2 \quad -(1)$$

in the XY plane in cylindrical polar coordinates. Here:

$$r_0 = 2 \frac{e_1}{m} \frac{e_2}{4\pi\epsilon_0 c^2} \quad -(2)$$

If $\mathcal{L}$ is the Lagrangian, H the Hamiltonian, and T the kinetic energy, then:

$$\mathcal{L} = H = T = \frac{1}{2} mc^2 = \frac{1}{2} m \left(e^{-r_0/r} c^2 \left(\frac{dt}{d\tau}\right)^2 - e^{r_0/r} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 \right) \quad -(3)$$

The Lagrange equation is:

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \quad -(4)$$

Therefore the constants of motion are the conserved quantities:

$$E = mc^2 e^{-r_0/r} \frac{dt}{d\tau} = \text{constant} \quad -(5)$$

$$p = m e^{r_0/r} \frac{dr}{d\tau} = \text{constant} \quad -(6)$$

$$L = mr^2 \frac{d\phi}{d\tau} = \text{constant} \quad -(7)$$

These are the total energy (E), linear momentum (p) and angular momentum (L). So:

$$\boxed{H = \frac{1}{2} mc^2 = \frac{1}{2} \left(e^{r_0/r} \frac{E^2}{mc^2} - e^{-r_0/r} \frac{p^2}{m} - \frac{L^2}{mr^2} \right)} \quad -(8)$$

2)
In the limit:

$$\frac{r_0}{r} \to 0 \quad -(9)$$

$$H = \frac{1}{2} mc^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - \frac{p^2}{m} - \frac{L^2}{mr^2}\right) \quad -(10)$$

If there is no angular momentum:

$$L = 0 \quad -(11)$$

then

$$H = \frac{1}{2} mc^2 = \frac{1}{2}\left(\frac{E^2}{mc^2} - \frac{p^2}{m}\right) \quad -(12)$$

i.e.

$$E^2 = c^2 p^2 + m^2 c^4 \quad -(13)$$

which is the Einstein energy equation of special relativity. The hamiltonian is:

$$H = \frac{1}{2m} p^\mu p_\mu \quad -(14)$$

where

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad -(15)$$

$$p_\mu = \left(\frac{E}{c}, -\underline{p}\right) \quad -(16)$$

For finite r_0/r eqn. (12) is:

$$H = \frac{1}{2m}\left(\left(\exp\left(\frac{r_0}{2r}\right)\frac{E}{c}\right)^2 - \left(\exp\left(-\frac{r_0}{2r}\right)p\right)^2\right) \quad -(17)$$

Consider the case:

$$\exp\left(\frac{r_0}{2r}\right) \sim 1 + \frac{r_0}{2r} \quad -(18)$$

$$\exp\left(-\frac{r_0}{2r}\right) \sim 1 - \frac{r_0}{2r} \quad -(19)$$

Then the change in the hamiltonian due to $r_0/(2r)$ can be represented by:

$$E \rightarrow E + \left(\frac{r_0}{2r}\right) E \quad -(20)$$

$$p \rightarrow p - \left(\frac{r_0}{2r}\right) p \quad -(21)$$

This representation is a minimal prescription:

$$p^\mu \rightarrow p^\mu - e_1 A^\mu \quad -(22)$$

where

$$A^\mu = \left(\frac{\phi}{c}, \underline{A}\right) \quad -(23)$$

if

$$\frac{r_0 E}{2r} = -e_1 \phi \quad -(24)$$

$$\frac{r_0 p}{2r} = e_1 A \quad -(25)$$

i.e.

$$\boxed{\begin{aligned} \phi &= -\frac{e_2}{4\pi\epsilon_0 r}\left(\frac{E}{mc^2}\right) \\ A &= \frac{e_2}{4\pi\epsilon_0 r c}\left(\frac{p}{mc}\right) \end{aligned}} \quad -(26)$$

For a particle at rest:

$$\phi = -\frac{e_2}{4\pi\epsilon_0 r} \quad -(27)$$

$$A = 0 \quad -(28)$$

and self-consistently, the problem is one of electrostatics.
...of the definition (26) the metric gives the relativistic
...milton Jacobi equation:

$$(p^\mu - eA^\mu)(p_\mu - eA_\mu) = m^2c^2 \quad -(29)$$

and the correct Dirac equation of an electron in a

4) potential A^μ:

$$\left(i\gamma^\mu \left(\partial_\mu - \frac{ie}{\hbar} A_\mu \right) - \frac{mc}{\hbar} \right) \psi = 0 \quad -(30)$$

or

$$\left(\gamma^\mu \left(p_\mu - eA_\mu \right) - mc \right) \psi = 0 \quad -(31)$$

This gives the g factor of the electron, the precessional effect of Thomas in spin orbit coupling, and ESR and NMR.

<u>Conclusion</u>

The metric (1) correctly reduces to the Dirac equation (31). For electrostatics:

$$A_\mu = \left(\frac{\phi}{c}, \underline{0} \right) \quad -(32)$$

In this case, using well known methods:

$$H \doteq \frac{p^2}{2m} + e\phi \quad -(33)$$

which is the non-relativistic limit of the RHS of eq. (26) of note 152(8).

<u>The most general</u> <u>Lagrangian</u> is <u>eq. (8)</u>, and this <u>generalizes</u> the <u>Dirac equation</u> <u>(31)</u>, giving <u>new</u> spectral <u>effects.</u>