1) Notes 140(1): The Continuity Equation of the Navier Stokes Equations

This is:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \quad -(1)$$

("Vector Analysis Problem Solver", p. 463). It may be written as:

$$\boxed{\partial_\mu U^\mu = 0} \quad -(2)$$

where

$$U^\mu = (c\rho, \underline{v}\rho) \quad -(3)$$

Here:

$$\rho = \rho(X, Y, Z, t) \quad -(4)$$

is density and

$$\underline{v} = \underline{v}(X, Y, Z, t) \quad -(5)$$

is velocity.

Eq. (1) is

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\rho \cdot \underline{v} + \rho \underline{\nabla} \cdot \underline{v} = 0 \quad -(6)$$

i.e.

$$\frac{D\rho}{Dt} + \rho \underline{\nabla} \cdot \underline{v} = 0 \quad -(7)$$

where

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\underline{\nabla}\rho \cdot \underline{v}) \quad -(8)$$

is the Stokes derivative. This is a measure of the rate of change of density at a point moving with the fluid. It is the derivative

2) along a path moving with velocity $\underline{v}$. The Stokes derivative is therefore a covariant derivative of general relativity.

Therefore the continuity equation of the Navier Stokes system of equations is a tetrad postulate.

Eq. (6) may be written as

$$\frac{\partial \rho}{\partial t} + \left(\underline{v} \cdot \underline{\nabla} + \underline{\nabla} \cdot \underline{v} \right) \rho = 0 \quad -(9)$$

so the covariant derivative is

$$\frac{D}{\partial t} = \frac{\partial}{\partial t} + \left(\underline{v} \cdot \underline{\nabla} + \underline{\nabla} \cdot \underline{v} \right) \quad -(10)$$

the connection is a simple scalar connection

$$\boxed{\Gamma = \underline{v} \cdot \underline{\nabla} + \underline{\nabla} \cdot \underline{v}} \quad -(11)$$

So eq. (9) is

$$\boxed{\frac{\partial \rho}{\partial t} + \Gamma \rho = 0} \quad -(12)$$

which is an equation of general relativity.

The following is a check on the vector

3)

algebra.

$$\underline{\nabla} \rho \cdot \underline{v} = \frac{\partial \rho}{\partial x} \frac{dx}{dt} + \ldots \quad - (13)$$

$$(\underline{v} \cdot \underline{\nabla}) \rho = \frac{dx}{dt} \frac{\partial \rho}{\partial x} + \ldots \quad - (14)$$

So:

$$\underline{\nabla} \rho \cdot \underline{v} = (\underline{v} \cdot \underline{\nabla}) \rho \quad - (15)$$

Q.E.D.

The Stokes derivative of a velocity $\underline{v}$ is:

$$\boxed{\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla}) \underline{v}} \quad - (16)$$

and this is used in the other equation of the Navier Stokes system, the conservation of momentum and energy.

The tetrad postulate is

$$D_\mu V^a{}_\nu = 0 \quad - (17)$$

and in the next note it will be shown that eq. (17) is a generalization of eq. (16).

Note 140(2): Derivation of the Continuity Equation from the Tetrad postulate.

The tetrad postulate is:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda = 0 \quad -(1)$$

$$= \partial_\mu q^a_\nu + \omega^a_{\mu\nu} - \Gamma^a_{\mu\nu} \quad -(2)$$

So:

$$\Gamma^a_{\mu\nu} = -\Gamma^a_{\nu\mu} = \partial_\mu q^a_\nu + \omega^a_{\mu\nu} \quad -(2)$$

$$= -\left(\partial_\nu q^a_\mu + \omega^a_{\nu\mu}\right)$$

The continuity equation is

$$\frac{\partial \rho}{\partial t} + \Gamma \rho = 0 \quad -(3)$$

i. e.

$$\partial_0 \rho + \frac{\Gamma}{c} \rho = 0 \quad -(4)$$

Let

$$\mu = \nu = 0 \quad -(5)$$

in eq. (1) and use the special case:

$$q^0_0 = q^1_1 = q^2_2 = q^3_3 = 1 \quad -(6)$$

of a unit diagonal tetrad square matrix.

The density is:

$$\rho = \rho q^0_0 = \rho q^1_1 = \rho q^2_2 = \rho q^3_3 \quad -(7)$$

So without loss of generality:

$$\partial_0 q^0_0 + \omega^0_{00} - \Gamma^0_{00} = 0 \quad -(8)$$

However: $\Gamma^0_{00} = 0$ — (9)

So: $\partial_0 q^0 + \omega^0_{00} = 0$ — (10)

i.e. $\partial_0 \rho + \omega^0_{00}\rho = 0$ — (11)

From eqs. (4) and (11):

$$\boxed{\Gamma^0 = c\,\omega^0_{00} = \underline{v}\cdot\underline{\nabla} + \underline{\nabla}\cdot\underline{v}} \quad \text{— (12)}$$

Notes 140(3): Inhomogeneous Gravitational Field Equations, the Density Tetrad.

The basic ECE hypothesis for gravitation is:

$$g^a_{\mu\nu} = \Phi \, T^a_{\mu\nu} \quad -(1)$$

$$\Phi^a_\mu = \Phi \, q^a_\mu \quad -(2)$$

where $g^a_{\mu\nu}$ is the gravitational field and Φ^a_μ is the gravitational potential. These are exactly analogous to the electromagnetic:

$$F^a_{\mu\nu} = A^{(0)} T^a_{\mu\nu} \quad -(3)$$

$$A^a_\mu = A^{(0)} q^a_\mu \quad -(4)$$

where $A^{(0)}$ is the value of the electromagnetic potential. Here q^a_μ is the Cartan tetrad and $T^a_{\mu\nu}$ is the Cartan torsion. The tetrad is defined as

$$V^a = q^a_\mu V^\mu \quad -(5)$$

as the matrix relating the complex circular basis

$$a = (0), (1), (2), (3) \quad -(6)$$

and the Cartesian basis

$$\mu = 0, X, Y, Z. \quad -(7)$$

The a basis is that of Cartan's tangential spacetime at point P to the base manifold,

2) represented in the Cartesian basis.

The Cartan identity is:

$$d \wedge T^a := R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b$$
$$:= J^a \quad -(8)$$

and the Evans identity is:

$$d \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b - \Omega^a{}_b \wedge \tilde{T}^b$$
$$:= \tilde{J}^a \quad -(9)$$

in the usual notation of Cartan's differential geometry.

The homogeneous gravitational field equation is:

$$d \wedge g^a = \Phi J^a = G \mathcal{T}^a \quad -(10)$$

and the inhomogeneous gravitational field equation is:

$$d \wedge \tilde{g}^a = \Phi \tilde{J}^a = G \tilde{\mathcal{T}}^a \quad -(11)$$

Here G is Newton's constant, $\mathcal{T}^a$ is the mass density three-form, and $\tilde{\mathcal{T}}^a$ is its Hodge dual. The generalization of Newton's law is contained in eq. (11), with:

$$\tilde{\mathcal{T}}^a = \frac{\Phi}{G}\left(\tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b\right)$$
$$-(12)$$

In tensor notation, eqn. (12) is:

3) $\tilde{\Phi}^{a}_{\mu\nu\rho} + \tilde{\Phi}^{a}_{\rho\mu\nu} + \tilde{\Phi}^{a}_{\nu\rho\mu} = \tilde{X}^{a}_{\mu\nu\rho} + \tilde{X}^{a}_{\rho\mu\nu} + \tilde{X}^{a}_{\nu\rho\mu}$ — (13)

where: $\tilde{X}^{a}_{\mu\nu\rho} = \tilde{R}^{a}{}_{b\mu\nu}\, q^{b}_{\rho} - \omega^{a}_{\mu b}\, \tilde{T}^{b}_{\nu\rho}$ — (14)

and so on.

Define the Hodge duals of the three-form ($p = 3$) in four dimensions ($n = 4$) using the general definition given by Carroll in eq. (C1.87):

$$A^{\mu_1 \dots \mu_{n-p}} = \frac{1}{p!}\, \epsilon^{\nu_1 \dots \nu_p \mu_1 \dots \mu_{n-p}}\, \tilde{A}_{\nu_1 \dots \nu_p} \quad (15)$$

so: $$\boxed{\Phi^{a\alpha} = \frac{1}{6}\, \epsilon^{\mu\nu\rho\alpha}\, \tilde{\Phi}^{a}_{\mu\nu\rho}} \quad (16)$$

and so on.

So the ~~vector~~ tensor format of eq. (11)

is: $$\boxed{\partial_\mu g^{a\mu\nu} = \Phi\, j^{a\nu} = G\rho^{a\nu}} \quad (17)$$

where $$j^{a\nu} = \frac{1}{2} J^{a\nu} \quad (18)$$

$$\rho^{a\nu} = \frac{1}{2} \Phi^{a\nu} \quad (19)$$

so $$\boxed{\partial_\mu g^{a\mu\nu} = G\rho^{a\nu}} \quad (20)$$

4)

Eq. (20) splits into two vector equations:

$$\boxed{\underline{\nabla} \cdot \underline{g}^{a} = G \rho^{a0}} \quad -(21)$$

and

$$\underline{\nabla} \times \underline{g}^{a} - \frac{1}{c} \frac{\partial \underline{g}^{a}}{\partial t} = G \underline{\rho}^{a} \quad -(22)$$

i.e.

$$\boxed{\underline{\nabla} \times \underline{g}^{a} = \frac{1}{c} \frac{\partial \underline{g}^{a}}{\partial t} + G \underline{\rho}^{a}} \quad -(23)$$

Eqs. (21) and (23) are the inhomogeneous equations of gravitation.

Eq. (10) splits into: $\quad -(24)$

$$\boxed{\begin{array}{l} \underline{\nabla} \times \underline{\tilde{g}}^{a} = G \tilde{\rho}^{a0} \\ \text{and} \quad \underline{\nabla} \times \underline{\tilde{g}}^{a} + \frac{1}{c} \frac{\partial \underline{\tilde{g}}^{a}}{\partial t} = G \underline{\tilde{\rho}}^{a} \end{array}}$$

$-(25)$

These are the homogeneous equations of gravitation.

The Newtonian law is a special case of eq. (21).

In general therefore the mass density

5) is a tetrad:

$$\rho_\mu^{\ a} = \left(\rho_0^{\ a}, \rho_1^{\ a}, \rho_2^{\ a}, \rho_3^{\ a} \right) \quad -(26)$$

$$= \left(\rho_0^{\ a}, \underline{\rho}^{\ a} \right) \quad -(27)$$

where $a = (0), (1), (2), (3)$.

By reference to paper 134, eq. (119):

$$\rho_\mu^{\ (0)} = \left(\rho_0^{\ (0)}, \underline{0} \right), \quad -(28)$$

so

$$\underline{\rho}^{\ (0)} = \underline{0} \quad -(29)$$

The other vectors of the density tetrad are:

$$\underline{\rho}^{\ (1)} = \rho_X^{\ (1)} \underline{i} + \rho_Y^{\ (1)} \underline{j} + \rho_Z^{\ (1)} \underline{k} \quad -(30)$$

and so on. So:

$$\rho_\mu^{\ (1)} = \left(\rho_0^{\ (1)}, \underline{\rho}^{\ (1)} \right) \quad -(31)$$

and so on.

The complex circular basis is:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) \quad -(32)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}} \left(\underline{i} + i \underline{j} \right) \quad -(33)$$

$$\underline{e}^{(3)} = \underline{k} \quad -(34)$$

So:

$$\boxed{\begin{aligned} \underline{\rho}^{(1)} &= \rho_x^{(1)}\,\underline{i} + \rho_y^{(1)}\,\underline{j} \\ \underline{\rho}^{(2)} &= \rho_x^{(2)}\,\underline{i} + \rho_y^{(2)}\,\underline{j} \\ \underline{\rho}^{(3)} &= \rho_z^{(3)}\,\underline{k} \end{aligned}} \quad -(35)$$

are the currents of mass density.

From paper 134, eqs. (110) and (111):

$$\underline{g}^a = -\underline{\nabla}\,\Phi_0^a - \frac{1}{c}\frac{\partial \underline{\Phi}^a}{\partial t} - \omega_{0b}^a\,\underline{\Phi}^b + \Phi_0^b\,\underline{\omega}^a_b \quad -(36)$$

which comes from:

$$g^a = d \wedge \Phi^a + \omega^a_b \wedge \Phi^b \quad -(37)$$

and the gravitomagnetic field is defined as having magnitude:

$$\Omega = g / c \quad -(38)$$

so:

$$\underline{\Omega}^a = \frac{1}{c}\left(\nabla \times \underline{\Phi}^a - \underline{\omega}^a_b \times \underline{\Phi}^b\right) \quad -(39)$$

The c factor enters because of the exact analogy with electromagnetism. The

7) $\underline{g}^a$ field is analogous with the electric field strength $\underline{E}^a$ and the $\underline{\Omega}^a$ field is analogous with the magnetic flux density $\underline{B}^a$. We have

$$E = cB \quad \text{— (40)}$$

in magnitude in S.I. units.

Newtonian Limit

This is:

$$\underline{g}^a = -\underline{\nabla}\,\Phi_0^a \quad \text{— (41)}$$

$$\underline{\nabla}\cdot\underline{g}^a = G\,\rho^{a0} \quad \text{— (42)}$$

where:

$$a = (1), (2), (3) \quad \text{— (43)}$$

Eq. (43) corresponds to the Moses decomposition. If we define:

$$\underline{g} := \underline{g}^{(1)} + \underline{g}^{(2)} + \underline{g}^{(3)} \quad \text{— (44)}$$

and

$$\rho = \rho^{(1)0} + \rho^{(2)0} + \rho^{(3)0} \quad \text{— (45)}$$

$$\Phi = \Phi_0^{(1)} + \Phi_0^{(2)} + \Phi_0^{(3)} \quad \text{— (46)}$$

then we obtain the familiar Newton theory:

8)

$$\underline{\nabla} \cdot \underline{g} = G\rho \quad -(47)$$

$$\underline{g} = -\underline{\nabla}\,\underline{\Phi} \quad -(48)$$

and:

$$\nabla^2 \underline{\Phi} = -G\rho \quad -(49)$$

which is the Poisson Equation

Static Gravitomagnetic Field Equation.

Similarly:

$$\boxed{\underline{\nabla} \times \underline{\Omega} = \frac{G}{c}\rho} \quad -(50)$$

where:

$$\underline{\Omega} = \underline{\Omega}^{(1)} + \underline{\Omega}^{(2)} + \underline{\Omega}^{(3)} \quad -(51)$$

$$\rho = \rho^{(1)} + \rho^{(2)} + \rho^{(3)} \quad -(52)$$

as used in papers 117 and 119.

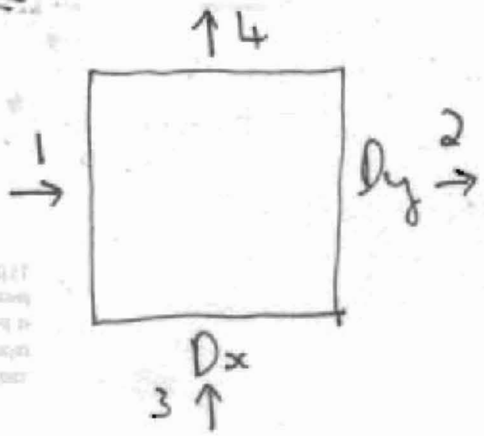
The familiar every day density is now understood to be a sum of time like components of the density tetrad, ρ^{a}_{μ}

$$\boxed{\rho = \rho^{(1)}_{0} + \rho^{(2)}_{0} + \rho^{(3)}_{0}} \quad -(53)$$

with

$$\boxed{D_\mu \rho^{a}_{\mu} = 0} \quad -(54)$$

) 140(4): Derivation of the Continuity Equation.



This is conservation of mass. Let u be the velocity entering 1, v be velocity entering 3, ρ be density within a control volume of unit side perpendicular to the plane. The mass within the control volume is $\rho \Delta x \Delta y$ and the rate of change of mass within the control volume is $\Delta x \Delta y \frac{\partial \rho}{\partial t}$.

The rate at which mass enters through #1 $= \rho u \Delta y$

" " " " #2 $= -\rho u \Delta y$ (leaves)

" " " " #3 $= \rho v \Delta x$ (enters)

" " " " #4 $= -\rho v \Delta x$ (leaves)

So:

$$\Delta x \Delta y \frac{\partial \rho}{\partial t} = (\rho u \Delta y)_1 - (\rho u \Delta y)_2 + (\rho v \Delta x)_3 - (\rho v \Delta x)_4 \quad -(1)$$

i.e.

$$\frac{\partial \rho}{\partial t} = \frac{(\rho u)_1 - (\rho u)_2}{\Delta x} + \frac{(\rho v)_3 - (\rho v)_4}{\Delta y} \quad -(2)$$

By definition:

$$\frac{\partial f}{\partial x} = \mathcal{L}_{\Delta x \to 0} \left(f(x + \Delta x, y) - f(x, y) \right) / \Delta x \quad -(3)$$

$$\frac{\partial f}{\partial y} = \mathcal{L}_{\Delta y \to 0} \left(f(x, y + \Delta y) - f(x, y) \right) / \Delta y \quad -(4)$$

Therefore:

2)
$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} = 0 \quad -(5)$$

In vector notation:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0 \quad -(6)$$

where

$$\underline{v} = u\underline{i} + v\underline{j} \quad -(7)$$

is the velocity vector.

Eq. (6) may be written as:

$$\frac{\partial \rho}{\partial t} + \Gamma \rho = 0 \quad -(8)$$

where

$$\Gamma = \underline{v} \cdot \underline{\nabla} + \underline{\nabla} \cdot \underline{v}. \quad -(9)$$

In order to derive eq. (8) from a unified structure for field theory its basic geometrical to be found. This is given by the tetrad postulate:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda = 0 \quad -(10)$$

$$= \partial_\mu q^a_\nu + \omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}. \quad -(11)$$

Therefore:

$$\boxed{\Gamma^a_{\mu\nu} = \partial_\mu q^a_\nu + \omega^a_{\mu\nu}} \quad -(12)$$

3)

Using the fundamental equation:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} D_\lambda V^\rho \quad -(13)$$

the symmetry of the connection is:

$$\Gamma^\lambda_{\mu\nu} = -\Gamma^\lambda_{\nu\mu} \quad -(14)$$

By definition:

$$\Gamma^a_{\mu\nu} = q^a_\lambda \Gamma^\lambda_{\mu\nu} \quad -(15)$$

so

$$\Gamma^a_{\mu\nu} = -\Gamma^a_{\nu\mu}. \quad -(16)$$

Therefore:

$$\left(\partial_\mu q^a_\nu + \omega^a_{\mu\nu}\right) = -\left(\partial_\nu q^a_\mu + \omega^a_{\nu\mu}\right) \quad -(17)$$

The quantity that is generated by the commutator is $\Gamma^a_{\mu\nu}$, which may be developed as the sum $\partial_\mu q^a_\nu + \omega^a_{\mu\nu}$. Eq. (17) states that this sum is antisymmetric:

$$[D_\mu, D_\nu] \longleftrightarrow \partial_\mu q^a_\nu + \omega^a_{\mu\nu}. \quad -(18)$$

To make this clearer the right hand side can be written as:

$$\left(\partial q^a + \omega^a\right)_{[\mu,\nu]} \quad -(19)$$

4) In eq. (13):

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma[\mu,\nu]} V^\sigma - T^\lambda_{[\mu,\nu]} D_\lambda V^\rho \quad -(20)$$

This notation can be refined even further:

$$D_{[\mu,\nu]} V^\rho = -\Gamma^\lambda_{[\mu,\nu]} + \dots \quad -(21)$$

$$= -\left(\partial q^a + \omega^a\right)_{[\mu,\nu]} q^\lambda_a + \dots$$

The $[\mu,\nu]$ subscript emphasizes the antisymmetry:

$$[\mu,\nu] = -[\nu,\mu]. \quad -(22)$$

In note 140(3) the mass / current density tetrad was introduced and the continuity equation of flow dynamics identified as:

$$D_\mu \rho^a_\nu = 0 \quad -(23)$$

i.e.

$$\partial_\mu \rho^a_\nu + \omega^a_{\mu\nu} - \Gamma^a_{\mu\nu} = 0 \quad -(24)$$

In note 140(1) the continuity equation was written as:

$$\frac{\partial \rho}{\partial t} + \Gamma \rho = 0 \quad -(25)$$

where

$$\Gamma = \underline{v} \cdot \underline{\nabla} + \underline{\nabla} \cdot \underline{v} \quad -(26)$$

Eq. (24) is also:

5)

$$\partial_\mu \rho^a_\nu + \omega^a_{\mu b}\rho^b_\nu - \Gamma^\lambda_{\mu\nu}\rho^a_\lambda = 0 \quad -(27)$$

Therefore eqs. (24) and (27) are generalizations of eq. (25). In note 140(3)(14) the Newtonian density was defined as:

$$\rho = \rho_0^{(1)} + \rho_0^{(2)} + \rho_0^{(3)} \quad -(28)$$

therefore eq. (25) is:

$$\frac{\partial\left(\rho_0^{(1)} + \rho_0^{(2)} + \rho_0^{(3)}\right)}{\partial t} + \Gamma\left(\rho_0^{(1)} + \rho_0^{(2)} + \rho_0^{(3)}\right) = 0 \quad -(29)$$

It is plausible to assume that:

$$\frac{\partial \rho_0^{(1)}}{\partial t} + \Gamma \rho_0^{(1)} = 0 \quad -(30)$$

$$\frac{\partial \rho_0^{(2)}}{\partial t} + \Gamma \rho_0^{(2)} = 0 \quad -(31)$$

$$\frac{\partial \rho_0^{(3)}}{\partial t} + \Gamma \rho_0^{(3)} = 0 \quad -(32)$$

So the indices in eq. (24) are determined by eqs. (30) to (32), i.e.

$$\frac{1}{c}\frac{\partial \rho_0^{(1)}}{\partial t} + \omega^{(1)}_{00} - \Gamma^{(1)}_{00} = 0 \quad -(33)$$

$$\frac{1}{c}\frac{\partial \rho_0^{(2)}}{\partial t} + \omega^{(2)}_{00} - \Gamma^{(2)}_{00} = 0 \quad -(34)$$

$$\frac{1}{c}\frac{\partial \rho_0^{(3)}}{\partial t} + \omega^{(3)}_{00} - \Gamma^{(3)}_{00} = 0 \quad -(35)$$

However:

$$\Gamma^{(1)}_{00} = \Gamma^{(2)}_{00} = \Gamma^{(3)}_{00} = 0 \quad -(36)$$

So:

$$\frac{\partial(\rho^{(1)}_0 + \rho^{(2)}_0 + \rho^{(3)}_0)}{\partial t} + c\left(\omega^{(1)}_{00} + \omega^{(2)}_{00} + \omega^{(3)}_{00}\right) = 0 \quad -(37)$$

i.e.

$$\frac{\partial \rho}{\partial t} + c\left(\omega^{(1)}_{00} + \omega^{(2)}_{00} + \omega^{(3)}_{00}\right) = 0 \quad -(38)$$

Comparing eqs. (35) and (38):

$$\boxed{\Gamma = c\left(\omega^{(1)}_{00} + \omega^{(2)}_{00} + \omega^{(3)}_{00}\right) = \underline{v}\cdot\underline{\nabla} + \underline{\nabla}\cdot\underline{v}} \quad -(39)$$

Therefore:

$$\boxed{\omega^{(1)}_{00} + \omega^{(2)}_{00} + \omega^{(3)}_{00} = \frac{1}{c}\left(\underline{v}\cdot\underline{\nabla} + \underline{\nabla}\cdot\underline{v}\right)} \quad -(40)$$

The convection of the Stokes derivative is given by (39) of spin connections of the condition Cartan geometry.

140(5): Fiel[d]

Electromagnetism

$$A^a_\mu = A^{(0)} q^a_\mu \quad -(1)$$

$$F^a_{\mu\nu} = A^{(0)} T^a_{\mu\nu} \quad -(2)$$

$$d \wedge F^a = j^a \quad -(3)$$

$$d \wedge \tilde{F}^a = \tilde{j}^a \quad -(4)$$

where

$$j^a = A^{(0)} \left(R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \right)$$

$$\tilde{j}^a = A^{(0)} \left(\tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b \right).$$

Eq. (3) translates into the tensor equation:

$$\partial_\mu \tilde{F}^{a\mu\nu} = \tilde{j}^{a\nu} \quad -(5)$$

which is the homogeneous field equation. Eq. (4) translates into:

$$\partial_\mu F^{a\mu\nu} = j^{a\nu} \quad -(6)$$

which is the inhomogeneous field equation.

The field tensors are:

$$\tilde{F}^{a\mu\nu} = \begin{bmatrix} 0 & -cB^a_x & -cB^a_y & -cB^a_z \\ cB^a_x & 0 & E^a_z & -E^a_y \\ cB^a_y & -E^a_z & 0 & E^a_x \\ cB^a_z & E^a_y & -E^a_x & 0 \end{bmatrix} \quad -(7)$$

and:

2)

$$F^{a\mu\nu} = \begin{bmatrix} 0 & -E_x^a & -E_y^a & -E_z^a \\ E_x^a & 0 & -cB_z^a & cB_y^a \\ E_y^a & cB_z^a & 0 & -cB_x^a \\ E_z^a & -cB_y^a & cB_x^a & 0 \end{bmatrix} \quad -(8)$$

So eq. (5) is developed into two vector laws of electrodynamics:

$$\underline{\nabla} \cdot \underline{B}^a = \tilde{j}^{a0} \quad -(9)$$

and

$$\underline{\nabla} \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = \tilde{\underline{j}}^a \quad -(10)$$

Eq. (6) is developed into:

$$\underline{\nabla} \cdot \underline{E}^a = \rho^a / \epsilon_0 \quad -(11)$$

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2}\frac{\partial \underline{E}^a}{\partial t} = \mu_0 \underline{J}^a \quad -(12)$$

For each a, eq. (9) is the magnetic law with magnetic monopole $\tilde{j}^{a0}$, eq. (10) is the Faraday law of induction with magnetic current $\tilde{\underline{j}}^a$, eq. (11) is the Coulomb law, eq. (12) is the Ampère Maxwell law. The field and potential are related by the first structure equation of Cartan and Maurer:

$$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b \quad -(13)$$

where $\omega^a{}_b$ is the spin connection.

3) Eq. (13) in vector notation is:

$$\underline{E}^a = -c\underline{\nabla} A_0^a - \frac{\partial \underline{A}^a}{\partial t} - c\,\omega^a_{0b}\underline{A}^b + c A_0^b\,\underline{\omega}^a_b \quad -(14)$$

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a_b \times \underline{A}^b \quad -(15)$$

Here

$$A^a_\mu = (A_0^a, -\underline{A}^a) \quad -(16)$$

$$\omega^a_{\mu b} = (\omega^a_{0b}, -\underline{\omega}^a_b) \quad -(17)$$

In eq. (14):

$$\left.\begin{array}{l} a = (1), (2), (3), \\ b = (0), (1), (2), (3) \end{array}\right\} \quad -(18)$$

In eq. (15):

$$\left.\begin{array}{l} a = (1), (2), (3), \\ b = (1), (2), (3). \end{array}\right\} \quad -(19)$$

The potential is defined by:

$$\left.\begin{array}{l} A^a_\mu = \left(\frac{\phi^a}{c}, \underline{A}^a \right) \\ a = (1), (2), (3) \end{array}\right\} \quad -(20)$$

and

$$A^{(0)}_\mu = \left(\frac{\phi^{(0)}}{c}, \frac{0}{c} \right) \quad -(21)$$

These equations are subject to antisymmetry constraints.

Gravitation

$$\Phi^a_\mu = \Phi q^a_\mu \qquad -(22)$$

$$g^a_{\mu\nu} = \Phi T^a_{\mu\nu} \qquad -(23)$$

$$d \wedge g^a = j^a \qquad -(24)$$

$$d \wedge \tilde{g}^a = \tilde{j}^a \qquad -(25)$$

$$j^a = \Phi \left(R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \right)$$

$$\tilde{j}^a = \Phi \left(\tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b \right).$$

Therefore the homogeneous field equation is

$$\partial_\mu \tilde{g}^{a\mu\nu} = \tilde{j}^{a\nu} \qquad -(26)$$

and the inhomogeneous field equation is:

$$\partial_\mu g^{a\mu\nu} = j^{a\nu} \qquad -(27)$$

The field tensors are:

$$\tilde{g}^{a\mu\nu} = \begin{bmatrix} 0 & -c\Omega^a_x & -c\Omega^a_y & -c\Omega^a_z \\ c\Omega^a_x & 0 & g^a_z & -g^a_y \\ c\Omega^a_y & -g^a_z & 0 & g^a_x \\ c\Omega^a_z & g^a_y & -g^a_x & 0 \end{bmatrix} \qquad -(28)$$

5)

$$g^{a\mu\nu} = \begin{bmatrix} 0 & -g^a_X & -g^a_Y & -g^a_Z \\ g^a_X & 0 & -c\Omega^a_Z & c\Omega^a_Y \\ g^a_Y & c\Omega^a_Z & 0 & -c\Omega^a_X \\ g^a_Z & -c\Omega^a_Y & c\Omega^a_X & 0 \end{bmatrix} \quad -(29)$$

where $\underline{g}^a$ is the gravitational field and $\underline{\Omega}^a$ is the gravitomagnetic field.

The four laws of gravitation are:

$$\underline{\nabla} \cdot \underline{\Omega}^a = \tilde{j}^{a0} \quad -(30)$$

$$\underline{\nabla} \times \underline{g}^a + \frac{\partial \underline{\Omega}^a}{\partial t} = \underline{\tilde{j}}^a \quad -(31)$$

$$\underline{\nabla} \cdot \underline{g}^a = 6\,\rho^a \quad -(32)$$

$$\underline{\nabla} \times \underline{\Omega}^a - \frac{1}{c^2}\frac{\partial \underline{g}^a}{\partial t} = \frac{6}{c}\,\underline{J}^a \quad -(33)$$

The mass / current density is:

$$J^a_\mu = \left(\rho^a, -\underline{J}^a\right) \quad -(34)$$

Here, the magnitude of g is c times greater than the magnitude of Ω.

6)

In Newtonian gravitation eq. (32) is used in the form:

$$\underline{\nabla} \cdot \underline{g} = G\rho. \qquad -(35)$$

The field and potential are related by:

$$g^a = d \wedge \Phi^a + \omega^a{}_b \wedge \Phi^b \qquad -(36)$$

In vector notation:

$$\underline{g}^a = -c\underline{\nabla}\Phi^a_0 - \frac{\partial \underline{\Phi}^a}{\partial t} - c\,\omega^a{}_{0b}\underline{\Phi}^b + c\,\Phi^b_0\,\underline{\omega}^a{}_b \qquad -(37)$$

$$\underline{\Omega}^a = \underline{\nabla} \times \underline{\Phi}^a - \underline{\omega}^a{}_b \times \underline{\Phi}^b \qquad -(38)$$

The gravitational field equations are also subjected to antisymmetry constraints.

<u>Free</u> <u>Fields</u>

$$\underline{\nabla} \cdot \underline{B}^a = 0 \qquad \underline{\nabla} \cdot \underline{\Omega}^a = 0$$

$$\underline{\nabla} \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = \underline{0} \qquad \underline{\nabla} \times \underline{g}^a + \frac{\partial \underline{\Omega}^a}{\partial t} = \underline{0}$$

$$\underline{\nabla} \cdot \underline{E}^a = 0 \qquad \underline{\nabla} \cdot \underline{g}^a = 0$$

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2}\frac{\partial \underline{E}^a}{\partial t} = \underline{0} \qquad \underline{\nabla} \times \underline{\Omega}^a - \frac{1}{c^2}\frac{\partial \underline{g}^a}{\partial t} = \underline{0}$$

1) 140(6): Generalization of Newton's Law

The gravitational field is defined as:

$$g^a_{\mu\nu} = \Phi T^a_{\mu\nu} \quad \text{— (1)}$$

and the gravitational potential as:

$$\Phi^a_\mu = \Phi q^a_\mu . \quad \text{— (2)}$$

Therefore, by the first Cartan structure equation:

$$g^a_{\mu\nu} = \partial_\mu \Phi^a_\nu - \partial_\nu \Phi^a_\mu + \omega^a_{\mu b}\Phi^b_\nu - \omega^a_{\nu b}\Phi^b_\mu . \quad \text{— (3)}$$

The following tetrad has the units of velocity:

$$\boxed{V^a_\mu = \frac{1}{c}\Phi^a_\mu} \quad \text{— (4)}$$

Therefore: — (5)

$$\underline{g}^a = -\frac{\partial \underline{v}^a}{\partial t} - c\nabla v^a_0 - \omega^a_{0b}\, c\,\underline{v}^b + c\, v^b_0\, \underline{\omega}^a_b$$

$$\underline{\Omega}^a = \underline{\nabla} \times \underline{v}^a - \omega^a{}_b \times \underline{v}^b \quad \text{— (6)}$$

The Newton law is therefore generalized to:

$$\underline{F}^a = m g^a \quad \text{— (7)}$$

where $\underline{g}^a$ is defined by eq. (5).

Furthermore

$$D_\mu V^a_\nu = 0 \quad \text{— (8)}$$

2) So:

$$\partial_\mu V^a_\nu = \frac{\Phi}{c}\left(\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}\right) \quad -(9)$$

and

$$\frac{\partial V^a_\mu}{\partial t} = \Phi\left(\Gamma^a_{0\mu} - \omega^a_{0\mu}\right) \quad -(10)$$

In vector notation:

$$\boxed{\frac{\partial \underline{V}^a}{\partial t} = \Phi\left(\underline{\Gamma}^a_0 - \underline{\omega}^a_0\right)} \quad -(11)$$

Eqs. (5) and (11) give two expressions for acceleration due to gravity.

In Newtonian dynamics:

$$\underline{F} = m\underline{g} = -m\frac{\Phi}{r}\underline{k} \quad -(12)$$

where

$$\Phi = -\frac{GM}{r}, \quad -(13)$$

so

$$g = -\frac{\Phi}{r} \quad -(14)$$

Taking the limit of eq. (5) where:

$$\underline{g}^a \rightarrow -\frac{\partial \underline{V}^a}{\partial t} \quad -(15)$$

then from eq. (11):

3) $$\underline{g}^a \to \Phi\left(\underline{\omega}^a_0 - \underline{\Gamma}^a_0\right). \quad \text{— (16)}$$

Define: $$g^a = \Phi\left|\underline{\omega}^a_0 - \underline{\Gamma}^a_0\right|. \quad \text{— (17)}$$

then for each a:

$$\boxed{\left|\underline{\omega}_0 - \underline{\Gamma}_0\right| = \frac{1}{r}} \quad \text{— (18)}$$

<u>Conclusions</u>

The equivalence principle (12) has been derived from the tetrad postulate (8), and the Newtonian law has been generalized using the first Cartan structure equation.

140(7): General Expression for Acceleration Derived from First Principles.

In note 140(6) an expression for acceleration due to gravity was derived

$$g^a = -\frac{\partial v^a}{\partial t} - c\,\nabla v_0^a - \omega^a{}_{0b} v^b + c v_0^b\, \omega^a{}_b \quad -(1)$$

and also a general expression for angular velocity:

$$\Omega^a = \nabla \times v^a - \omega^a{}_b \times v^b. \quad -(2)$$

Here, velocity was identified as the tetrad:

$$v^a{}_\mu = \frac{1}{c}\, \Phi^a{}_\mu \quad -(3)$$

where

$$\Phi^a{}_\mu = \Phi\, q^a{}_\mu \quad -(4)$$

is the gravitational potential.

In the non-relativistic limit a general expression for acceleration may be derived as follows, following "Vector Analysis Problem Solver", pp 472 ff. The velocity is:

$$\mathbf{v} = v_X \mathbf{i} + v_Y \mathbf{j} + v_Z \mathbf{k} \quad -(5)$$

where

$$v_X = \frac{\Delta X}{\Delta t},\ v_Y = \frac{\Delta Y}{\Delta t},\ v_Z = \frac{\Delta Z}{\Delta t} \quad -(6)$$

Assuming that $\mathbf{v}$ is differentiable, and neglecting higher order terms:

$$\mathbf{v}(X + \Delta X,\ Y + \Delta Y,\ Z + \Delta Z,\ t + \Delta t)$$

$$= \mathbf{v}(X + v_X \Delta t,\ Y + v_Y \Delta t,\ Z + v_Z \Delta t,\ t + \Delta t)$$

$$= \mathbf{v}(X, Y, Z, t) + \frac{\partial \mathbf{v}}{\partial X} v_X \Delta t + \frac{\partial \mathbf{v}}{\partial Y} v_Y \Delta t$$

$$+ \frac{\partial \underline{v}}{\partial z} v_z \Delta t + \frac{\partial \underline{v}}{\partial t} \Delta t$$

$$= \underline{v}(x,y,z,t) + (\underline{\nabla} \cdot \underline{v}) \underline{v} \Delta t + \frac{\partial \underline{v}}{\partial t} \Delta t \quad -(7)$$

The acceleration is given by:

$$\underline{a} = \frac{1}{\Delta t}\left(\underline{v}(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - \underline{v}(x,y,z,t)\right) \quad -(8)$$

$$= \frac{1}{\Delta t}\left(\underline{v}(x,y,z,t) + (\underline{\nabla}\cdot\underline{v})\underline{v}\Delta t + \frac{\partial \underline{v}}{\partial t}\Delta t - \underline{v}(x,y,z,t)\right)$$

$$\boxed{\underline{a} = (\underline{\nabla}\cdot\underline{v})\underline{v} + \frac{\partial \underline{v}}{\partial t}} \quad -(9)$$

The vorticity is defined by:

$$\boxed{\underline{\Omega} = \underline{\nabla} \times \underline{v}} \quad -(10)$$

The formalism of eq. (1) means that acceleration in general relativity is

$$\underline{a}^a = \frac{\partial \underline{v}^a}{\partial t} + \omega^a{}_b \underline{v}^b + c \underline{\nabla} v_0^a - c v_0^b \underline{\omega}^a{}_b \quad -(11)$$

and that vorticity is

$$\underline{\Omega}^a = \underline{\nabla} \times \underline{v}^a - \underline{\omega}^a{}_b \times \underline{v}^b \quad -(12)$$

Eq. (11) comes from the orbital part of torsion and

3) eq. (12) for the spin part of torsion.

The four velocity is:

$$v^a_\mu = \left(v^a_0 , - \underline{v}^a \right) \quad -(13)$$

$$= \frac{1}{c} \left(\Phi^a_0 , - \underline{\Phi} \right) \quad -(14)$$

In situations where there is no scalar potential

$$\Phi^a_0 = 0 \quad -(15)$$

eq. (11) reduces to

$$\underline{a}^a = \frac{\partial \underline{v}^a}{\partial t} + \omega^a{}_{ob} \underline{v}^b \quad -(16)$$

Therefore eq. (a) may be derived from eq. (16) in the limit

$$\omega^a{}_{ob} \underline{v}^b \rightarrow \left(\underline{\nabla} \cdot \underline{v} \right) \underline{v}^a \quad -(17)$$

for each a. The index a is part of the fundamental topology of three dimensional space.

So in eq. (16):

$$\underline{a}^{(1)} = \frac{\partial \underline{v}^{(1)}}{\partial t} + \omega^{(1)}{}_{ob} \underline{v}^b \quad -(18)$$

$$= \frac{\partial \underline{v}^{(1)}}{\partial t} + \left(\underline{\nabla} \cdot \underline{v} \right) \underline{v}^{(1)}$$

and also for $\underline{a}^{(2)}$ and $\underline{a}^{(3)}$.

4) In situations where there is no vector potential:

$$\underline{\Phi}^a = \underline{0} \qquad -(19)$$

then:

$$\underline{a}^a = c\left(\underline{\nabla} V_0^a - V_0^b \underline{\omega}^a_b\right) \qquad -(20)$$

$$\underline{a}^a = \underline{\nabla}\, \Phi_0^a - \Phi_0^b \underline{\omega}^a_b \qquad -(20a)$$

In an inviscid fluid (VAPS Problem 11-19)

$$\underline{a} = -\frac{1}{m}\left(\underline{\nabla} p + \underline{\nabla} \phi\right) \qquad -(21)$$

where p is pressure, m is mass and ϕ is the total potential energy per unit mass due to all body forces.

So in an inviscid fluid:

$$\frac{\partial \underline{v}}{\partial t} + (\underline{\nabla} \cdot \underline{v})\underline{v} = -\frac{1}{m}\left(\underline{\nabla} p + \underline{\nabla} \phi\right) \qquad -(22)$$

or:

$$\boxed{\underline{a} = \frac{\partial \underline{v}}{\partial t} + (\underline{\nabla} \cdot \underline{v})\underline{v} + \frac{1}{m}\underline{\nabla} p + \frac{1}{m}\underline{\nabla}\phi = \underline{0}} \qquad -(23)$$

and the net acceleration is zero.

5) For zero acceleration, eq. (11) is:

$$\frac{\partial \underline{v}^a}{\partial t} + \omega^a{}_{ob}\, \underline{v}^b + c \underline{\nabla} v^a{}_o - c v^b{}_o\, \underline{\omega}^a{}_b$$

$$= \frac{1}{c} \frac{\partial \underline{\Phi}^a}{\partial t} + \frac{\omega^a{}_{ob} \underline{\Phi}^b}{c} + \underline{\nabla} \Phi^a{}_o - \Phi^b{}_o\, \underline{\omega}^a{}_b \qquad -(24)$$

$$= \underline{0}$$

It is seen that eq. (23) is a limit of the generally covariant equation (24).

Now use:

$$\omega^a{}_{\mu b} \Phi^b{}_\nu = \Phi\, \omega^a{}_{\mu b}\, q^b{}_\nu \qquad -(25)$$

$$= \Phi\, \omega^a{}_{\mu\nu} \qquad -(26)$$

So

$$\omega^a{}_{ib} \Phi^b{}_o = \Phi\, \omega^a{}_{io} \qquad -(27)$$

$$\omega^a{}_{ob} \Phi^b{}_i = \Phi\, \omega^a{}_{oi}$$

Define:

$$\underline{\omega}^a{}_b = -\left(\omega^a{}_{xb}\, \underline{i} + \omega^a{}_{yb}\, \underline{j} + \omega^a{}_{zb}\, \underline{k}\right)$$

$$= -\left(\omega^a{}_{1b}\, \underline{i} + \omega^a{}_{2b}\, \underline{j} + \omega^a{}_{3b}\, \underline{k}\right)$$

etc. $\quad -(28)$

In vector notation, eqs. (26) and (27) are:

$$\Phi^b{}_o\, \underline{\omega}^a{}_b = \Phi\, \underline{\omega}^a \qquad -(29)$$

$$\omega^a{}_{ob} \Phi^b{}_i = \Phi\, \underline{\Omega}^a \qquad -(30)$$

6)

where:

$$\underline{\omega}^a = \omega^a_{10}\,\underline{i} + \omega^a_{20}\,\underline{j} + \omega^a_{30}\,\underline{k}, \quad -(31)$$

$$\underline{\Omega}^a = \omega^a_{01}\,\underline{i} + \omega^a_{02}\,\underline{j} + \omega^a_{03}\,\underline{k}. \quad -(32)$$

Therefore eq. (11) becomes:

$$\underline{a}^a = \frac{1}{c}\frac{\partial \underline{\Phi}^a}{\partial t} + \frac{\Phi}{c}\underline{\Omega}^a + \underline{\nabla}\,\Phi^a_0 - \Phi\,\underline{\omega}^a \quad -(33)$$

and for each a:

$$\boxed{\underline{a} = \frac{1}{c}\frac{\partial \underline{\Phi}}{\partial t} + \frac{\Phi}{c}\underline{\Omega} + \underline{\nabla}\,\Phi_0 - \Phi\,\underline{\omega}} \quad -(34)$$

Comparing eqs. (23) and (34) term by term:

$$\boxed{\begin{aligned} \underline{v} &= \frac{1}{c}\underline{\Phi} && -(35)\\ \frac{\Phi}{c}\underline{\Omega} &= (\underline{\nabla}\cdot\underline{v})\,\underline{v} && -(36)\\ \Phi_0 &= \phi/m && -(37)\\ \frac{1}{m}\underline{\nabla}\,p &= -\Phi\,\underline{\omega} && -(38) \end{aligned}}$$

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1) 140(8): Acceleration from Velocity.

The tetrad q^a_μ introduces a higher topology into the vector analysis of Heaviside and Gibbs. So the velocity vector is generalized to the velocity tetrad:

$$v^a_\mu = v\, q^a_\mu \quad -(1)$$

where v is a scalar magnitude. The acceleration is defined in analogy to the field of force, so:

$$a^a_{\mu\nu} = c\, v\, T^a_{\mu\nu} \quad -(2)$$

in units of $m\, s^{-2}$. In eq. (2), $T^a_{\mu\nu}$ is the Cartan torsion. From eq. (2):

$$\underline{a}^a = -\frac{\partial \underline{v}^a}{\partial t} - c\nabla v^a_0 - c\,\omega^a_{0b}\,\underline{v}^b + c\, v^b_0\, \underline{\omega}^a_b \quad -(3)$$

$$\underline{\Omega}^a = \underline{\nabla} \times \underline{v}^a - \underline{\omega}^a_b \times \underline{v}^b \quad -(4)$$

In eq. (3), the potential is:

$$\Phi^a_0 = c\, v^a_0 \quad -(5)$$

In tensor notation, the relation between acceleration and velocity is:

$$a^a_{\mu\nu} = c\left(\partial_\mu v^a_\nu - \partial_\nu v^a_\mu + \omega^a_{\mu b} v^b_\nu - \omega^a_{\nu b} v^b_\mu\right)$$

$$= c\left(\partial_\mu v^a_\nu - \partial_\nu v^a_\mu + v\left(\omega^a_{\mu\nu} - \omega^a_{\nu\mu}\right)\right) \quad -(6)$$

So in general relativity:

2)
$$\underline{a}^{a} = -\frac{\partial \underline{v}^{a}}{\partial t} - \underline{\nabla} \Phi^{a} + c v \underline{\omega}^{a}_{\text{orbital}} \quad -(7)$$

$$\underline{\Omega}^{a} = \underline{\nabla} \times \underline{v}^{a} + v \underline{\omega}^{a}_{\text{spin}} \quad -(8)$$

Here:

$$\underline{\omega}^{a}_{\text{orbital}} = \left(\omega^{a}_{01} - \omega^{a}_{10}\right)\underline{i} + \left(\omega^{a}_{02} - \omega^{a}_{20}\right)\underline{j} + \left(\omega^{a}_{03} - \omega^{a}_{30}\right)\underline{k} \quad -(9)$$

$$\underline{\omega}^{a}_{\text{spin}} = \left(\omega^{a}_{32} - \omega^{a}_{23}\right)\underline{i} + \left(\omega^{a}_{13} - \omega^{a}_{31}\right)\underline{j} + \left(\omega^{a}_{21} - \omega^{a}_{12}\right)\underline{k} \quad -(10)$$

and

$$v \underline{\omega}^{a}_{\text{orbital}} = -\omega^{a}_{0b} \underline{v}^{b} + v_{0}^{b} \underline{\omega}^{a}_{b} \quad -(11)$$

$$v \underline{\omega}^{a}_{\text{spin}} = -\underline{\omega}^{a}_{b} \times \underline{v}^{b} \quad -(12)$$

1) Eq. (11) is a new kind of Coriolis acceleration due to orbital torsion.

2) Eq. (12) is the Coriolis acceleration due to spin torsion.

3) Eq. (7) is an expression of the equivalence principle, in that acceleration can be due to rate of change of velocity and due to the gradient of potential.

) ; In "the inertial frame," the spin connection is not present, so:

$$\underline{a}^a \rightarrow -\frac{\partial \underline{v}^a}{\partial t} - \underline{\nabla} \underline{\Phi}^a \quad - (13)$$

$$\underline{\Omega}^a \rightarrow \underline{\nabla} \times \underline{v}^a \quad - (14)$$

The equivalence principle assumes that:

$$-\frac{\partial \underline{v}^a}{\partial t} = -\underline{\nabla} \underline{\Phi}^a \quad - (15)$$

which is the direct result of the antisymmetry law:

$$\boxed{\partial_\mu v_\nu^a = -\partial_\nu v_\mu^a} \quad - (16)$$

where:

$$\mu = 0, \; \nu = 1. \quad - (17)$$

If force is defined as mass multiplied by acceleration, then

$$\boxed{\frac{1}{m} \underline{F}^a = -\frac{\partial \underline{v}^a}{\partial t} = -\underline{\nabla} \underline{\Phi}^a} \quad - (18)$$

which is a generalization of

$$\underline{F} = m\underline{g} = -\frac{mMG}{r^2} \underline{k} \quad - (19)$$

140(9): Some Notes on Fluid Dynamics

The analysis starts by determining the total force from the external pressure on the cube of liquid. When the fluid is at rest there are no shear forces, so the stresses are normal to any surface inside the fluid. The pressure at any point is the same in all directions. The pressure is a function of position and not a function of time because the fluid is at rest.

$$p = p(x, y, z) \quad -(1)$$

Considering the x direction the pressure on face A is $p(x)$ while that on face B is $p(x + \Delta x)$. If p is a differentiable function, then by the Maclaurin theorem:

$$p(x + \Delta x) \approx p(x) + \frac{\partial p}{\partial x} \Delta x \quad -(2)$$

neglecting higher order terms.

The force acting on side A is $p \Delta y \Delta z$, and the force on side B is:

$$F = -\left(p + \frac{\partial p}{\partial x} \Delta x\right) \Delta y \Delta z \quad -(3)$$

The total force in the x direction is:

$$F_x = p \Delta y \Delta z - \left(p + \frac{\partial p}{\partial x} \Delta x\right) \Delta y \Delta z \quad -(4)$$

$$= -\frac{\partial p}{\partial x} \Delta x \Delta y \Delta z \quad -(5)$$

Similarly:

$$F_y = -\frac{\partial p}{\partial y} \Delta x \Delta y \Delta z$$

$$F_z = -\frac{\partial p}{\partial z} \Delta x \Delta y \Delta z \quad -(6)$$

The total force on the cube is:

2)

$$\underline{F} = F_x \underline{i} + F_y \underline{j} + F_z \underline{k} \quad -(7)$$

$$= -\left(\frac{\partial p}{\partial x} \underline{i} + \frac{\partial p}{\partial y} \underline{j} + \frac{\partial p}{\partial z} \underline{k} \right) dx\,dy\,dz$$

So the force per unit volume due to external pressure is

$$\frac{F}{dx\,dy\,dz} = \frac{1}{dV} \underline{F} = -\underline{\nabla} p \quad -(8)$$

This force is balanced by the internal or body forces. These are described by a general potential function ϕ ("Vector Analysis Problem Solver", p. 471), per unit mass. If ρ denotes the density of the fluid, the total force per unit volume of these body forces is $-\rho \underline{\nabla} \phi$. The total force per unit volume (m^3) is:

$$\boxed{\underline{f} = -\underline{\nabla} p - \rho \underline{\nabla} \phi} \quad -(9)$$

The cube is in equilibrium so:

$$\underline{f} = \underline{0} \quad (10)$$

In an incompressible fluid, ρ is constant so:

$$\underline{\nabla} (\rho \phi) = \rho \underline{\nabla} \phi \quad (11)$$

and

$$p + \rho \phi = \text{constant} \quad -(12)$$

If the velocity of the fluid is significantly

3) lower than the speed of sound the fluid is of constant density and incompressible. The continuity equation is:

$$\frac{\partial \rho}{\partial t} + \underline{\nabla}\cdot(\rho \underline{v}) = 0 \quad (13)$$

This is the principle of conservation of matter. In an incompressible fluid:

$$\underline{\nabla}\cdot \underline{v} = 0 \quad (14)$$

because:

$$\frac{\partial \rho}{\partial t} = 0 \quad (15)$$

Applying Newton's law to eq. (9):

$$\underline{f} = \rho \underline{a} \quad (16)$$

where $\underline{a}$ is the acceleration of the fluid.

From previous notes:

$$\underline{f} = \rho \underline{a} = \rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{\nabla}\cdot\underline{v})\underline{v}\right) \quad (17)$$

so:

$$\boxed{\frac{\partial \underline{v}}{\partial t} + (\underline{\nabla}\cdot\underline{v})\underline{v} + \frac{1}{\rho}\underline{\nabla} p - \underline{\nabla}\phi = 0} \quad (18)$$

The vorticity is defined by

$$\boxed{\underline{\Omega} = \underline{\nabla}\times\underline{v}} \quad (19)$$

Now use the vector identity:

4) $$(\underline{b}\cdot\underline{\nabla})\underline{b} = (\underline{\nabla}\times\underline{b})\times\underline{b} + \frac{1}{2}\underline{\nabla}(\underline{b}\cdot\underline{b}) \quad -(20)$$

to find the equation of motion of the fluid:

$$\frac{\partial \underline{v}}{\partial t} + \underline{\Omega}\times\underline{v} + \frac{1}{2}\underline{\nabla} v^2 = -\frac{\underline{\nabla} p}{\rho} - \underline{\nabla}\phi \quad -(21)$$

The pressure is eliminated using:

$$\underline{\nabla}\times(\underline{\nabla} p) = \underline{0} \quad -(22)$$

From eqs. (21) and (22):

$$\boxed{\frac{\partial\underline{\Omega}}{\partial t} + \underline{\nabla}\times(\underline{\Omega}\times\underline{v}) = \underline{0}} \quad -(23)$$

with:

$$\underline{\nabla}\cdot\underline{v} = 0 \quad -(24)$$

$$\underline{\Omega} = \underline{\nabla}\times\underline{v} \quad -(25)$$

Eqs. (23) – (25) completely describe the velocity field $\underline{v}$ of the fluid.

Analogy with Electrodynamics

Eq. (23) has the same structure as the Faraday law of induction

$$\frac{\partial \underline{B}}{\partial t} + \underline{\nabla}\times\underline{E} = \underline{0}. \quad -(26)$$

5) Eq. (25) has the structure:

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad -(27)$$

and eq. (24) is analagous to:

$$\underline{\nabla} \cdot \underline{A} = 0 \quad -(28)$$

Using the minimal prescription:

$$\underline{p} = m\underline{v} = e\underline{A} \quad -(29)$$

we have

$$\underline{E} = \frac{e}{m} \underline{B} \times \underline{A} \quad -(30)$$

$$= \underline{B} \times \underline{v}$$

The second half of this equation is <u>the Lorentz force law</u>.

<u>SUMMARY</u>

$$\frac{\partial \underline{\Omega}}{\partial t} + \underline{\nabla} \times (\underline{\Omega} \times \underline{v}) = 0 \Leftrightarrow \frac{\partial \underline{B}}{\partial t} + \underline{\nabla} \times \underline{E} = \underline{0}$$

$$\underline{\Omega} = \underline{\nabla} \times \underline{v} \Leftrightarrow \underline{B} = \underline{\nabla} \times \underline{A}$$

$$\underline{\nabla} \cdot \underline{v} = 0 \Leftrightarrow \underline{\nabla} \cdot \underline{A} = 0$$

The quantity $\underline{\Omega} \times \underline{v}$ is analagous to the Coriolis acceleration and to the electric field strength $\underline{E}$. The vorticity $\underline{\Omega}$ plays the role of magnetic flux density $\underline{B}$. The velocity $\underline{v}$ plays the role of vector potential $\underline{A}$.

6) We note that these equations are special cases of the more general:

$$\underline{a}^a = -\frac{\partial \underline{v}^a}{\partial t} - c\underline{\nabla} v_0^a - c\omega^a{}_{0b}\, \underline{v}^b + c v_0^b\, \underline{\omega}^a{}_b \quad -(31)$$

$$\underline{\Omega}^a = \underline{\nabla} \times \underline{v}^a - \underline{\omega}^a{}_b \times \underline{v}^b \quad -(32)$$

The expression for acceleration used in eq. (18)

is

$$\underline{a} = \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \underline{\nabla})\underline{v} \quad -(33)$$

and is a special case of

$$\underline{a}_I^a = -\frac{\partial \underline{v}^a}{\partial t} - c\omega^a{}_{0b}\, \underline{v}^b \quad -(34)$$

Eq. (9) is a special case of

$$\underline{a}_{II}^a = -c\underline{\nabla} v_0^a + c v_0^b\, \underline{\omega}^a{}_b \quad -(35)$$

$$= -\underline{\nabla}\,\Phi^a + \Phi_0^b\, \underline{\omega}^a{}_b$$

and the inviscid, incompressible fluid is:

$$\boxed{\underline{a}_I^a + \underline{a}_{II}^a = \underline{0}} \quad -(36)$$

We see that general relativity is no longer a minor correction to Newton, but an intrinsic part of everyday dynamics

1) 140(10): General Development of Dynamics and Vector Analysis.

Helmholtz showed ~~that~~ any vector field can be written as:

$$\underline{V} = \underline{V}_S + \underline{V}_I \quad -(1)$$

where

$$\underline{\nabla} \cdot \underline{V}_S = 0 \quad -(2)$$

$$\underline{\nabla} \times \underline{V}_I = 0 \quad -(3)$$

This development can be extended as follows:

$$\underline{V}_S = \underline{V}^{(1)} + \underline{V}^{(2)} \quad -(4)$$

$$\underline{V}_I = \underline{V}^{(3)} \quad -(5)$$

so

$$\boxed{\underline{V} = \underline{V}^{(1)} + \underline{V}^{(2)} + \underline{V}^{(3)}} \quad (6)$$

where

$$a = (1), (2), (3) \quad -(7)$$

is the complex circular basis.

Now consider the vector

$$V^a_\mu = V_\mu^a \quad (8)$$

where

$$V_\mu = V_\mu^{(0)} + V_\mu^{(1)} + V_\mu^{(2)} + V_\mu^{(3)} \quad -(9)$$

where

$$\mu = 0, 1, 2, 3 \quad -(10)$$

$$(a) = (0), (1), (2), (3) \quad -(11)$$

Eq. (9) extends eq. (6) to four dimensions.

2) Eq. (6) is a general property of the vector field in three dimensions, and shows that in three dimensions:

$$V^a_i = V q^a_i \quad -(12)$$

where

$$\left.\begin{aligned} a &= (1), (2), (3), \\ i &= 1, 2, 3 \end{aligned}\right\} \quad -(13)$$

Now further define:

$$V^{(0)}_\mu = \left(V^{(0)}_0, \underline{0}\right) \quad -(14)$$

$$V^{(1)}_\mu = \left(V^{(1)}_0, -\underline{V}^{(1)}\right) \quad -(15)$$

$$V^{(2)}_\mu = \left(V^{(2)}_0, -\underline{V}^{(2)}\right) \quad -(16)$$

$$V^{(3)}_\mu = \left(V^{(3)}_0, -\underline{V}^{(3)}\right) \quad -(17)$$

Eq. (14) means that the space like components of $V^{(0)}_\mu$ are assumed to be zero, because superscript (0) denotes a pure timelike property. In general the matrix V^a_μ is defined by

$$x^a = V^a_\mu x^\mu \quad -(18)$$

and this is also the definition of the Cartan tetrad V^a_μ. Therefore vector analysis can be extended to Cartan's differential geometry. Any three vector can be written as eq. (6) and any four vector as eq. (9).

3) Position Vector

This is written as:

$$r^a_\mu = \left(r^a_0 , - \underline{r}^a \right) = r q^a_\mu , \quad (19)$$

with:

$$r^{(0)}_\mu = \left(ct, \underline{0} \right), \quad (20)$$

$$r^{(1)}_\mu = \left(r^{(1)}_0 , - \underline{r}^{(1)} \right) \quad (21)$$

$$r^{(2)}_\mu = \left(r^{(2)}_0 , - \underline{r}^{(2)} \right) \quad (22)$$

$$r^{(3)}_\mu = \left(r^{(3)}_0 , - \underline{r}^{(3)} \right) \quad (23)$$

with:

$$\underline{r} = \underline{r}^{(1)} + \underline{r}^{(2)} + \underline{r}^{(3)} \quad (24)$$

Velocity Vector

This is defined by applying the exterior covariant derivative, the $D \wedge$ operator,

$$v^a = D \wedge r^a \quad (25)$$

Acceleration Vector

This is defined by:

$$a^a = D \wedge v^a = D \wedge \left(D \wedge r^a \right) \quad (26)$$

From eq. (25):

$$\underline{v}^a = - \frac{\partial \underline{r}^a}{\partial t} - c \underline{\nabla} r^a_0 - c \omega^a_{0b} \underline{r}^b + c r^b_0 \underline{\omega}^a_b \quad (27)$$

and

$$\underline{w}^a = c \left(\underline{\nabla} \times \underline{r}^a - \underline{\omega}^a_b \times \underline{r}^b \right) \quad (28)$$

4) are two velocity vector fields. They are parts of the four vectors:

$$V^a_\mu = (V^a_0, -\underline{V}^a) \quad -(29)$$

and

$$W^a_\mu = (W^a_0, -\underline{W}^a) \quad -(30)$$

The acceleration vectors are:

$$\underline{a}^a = -\frac{\partial \underline{V}^a}{\partial t} - c\nabla V^a_0 - c\omega^a_{0b} \underline{V}^b + cV^b_0 \underline{\omega}^a_b \quad -(31)$$

and

$$\underline{\Omega}^a = \nabla \times \underline{V}^a - \underline{\omega}^a_b \times \underline{V}^b \quad -(32)$$

Eqs. (27), (28), (31) and (32) give all the information about the dynamics.

There are also acceleration vectors such as:

$$\underline{\alpha}^a = -\frac{\partial \underline{W}^a}{\partial t} - c\nabla W^a_0 - c\omega^a_{0b} \underline{W}^b + cW^b_0 \underline{\omega}^a_b \quad -(33)$$

$$\underline{\beta}^a = \nabla \times \underline{W}^a - \underline{\omega}^a_b \times \underline{W}^b \quad -(34)$$

1) Note 140(11): Complex Circular Basis and Cartan Geometry

Any three dimensional vector field $\underline{V}$ may be expressed as:

$$\underline{V} = \underline{V}^{(1)} + \underline{V}^{(2)} + \underline{V}^{(3)} \quad - (1)$$

in the complex circular basis:

$$a = (1), (2), (3) \quad - (2)$$

So in 4-D spacetime

$$V_\mu = (V_0, -\underline{V}) \quad - (3)$$

Similarly:

$$V_0 = V_0^{(1)} + V_0^{(2)} + V_0^{(3)} \quad - (4)$$

so

$$V_\mu = V_\mu^{(1)} + V_\mu^{(2)} + V_\mu^{(3)} \quad - (5)$$

Define

$$V^{(0)} = (V_0^{(0)}, \underline{0}) \quad - (6)$$

so

$$V^a = V^\mu q^a_{\ \mu} \quad - (7)$$

where:

$$a = (0), (1), (2), (3) \quad - (8)$$
$$\mu = 0, 1, 2, 3$$

Extend this reasoning to Cartan differential geometry. Then the a index is the complex circular representation of the Minkowski tangent spacetime at point P to the base manifold represented by μ. So $q^a_{\ \mu}$ is the Cartan tetrad. Note carefully however that $q^a_{\ \mu}$ in eq. (7) is also the matrix linking two frames. Also, $q^a_{\ \mu}$ may be the matrix

2) defined by:

$$W^a = q^a_\mu W^\mu \quad (9)$$

in the same spacetime. The latter may be represented both by a and μ.

In all cases the Cartan Maurer structure equations define the torsion and curvature:

$$T^a_{\mu\nu} = (D \wedge q^a)_{\mu\nu} \quad (10)$$

$$R^a{}_{b\mu\nu} = (D \wedge \omega^a{}_b)_{\mu\nu} \quad (11)$$

It is shown as follows that these definitions are equivalent to the definitions of torsion and curvature in Riemann geometry. The existence of the complex circular representation is necessary and sufficient to define torsion and curvature by use of eq. (9).

Proofs

Torsion

Eq. (10) is:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu\nu} - \omega^a_{\nu\mu} = \Gamma^a_{\mu\nu} - \Gamma^a_{\nu\mu} \quad (12)$$

using the tetrad postulate:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu\nu} - \Gamma^a_{\mu\nu} = 0 \quad (13)$$

so the Riemannian torsion is

3) $T^{\lambda}_{\mu\nu} = q^{\lambda}_{a} T^{a}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$ — (14)

Q.E.D. It has been shown that the fundamental property (1) creates a Riemannian torsion.

Curvature — (15)

Eq. (11) is:

$$R^{a}{}_{b\mu\nu} = \partial_\mu \omega^{a}_{\nu b} - \partial_\nu \omega^{a}_{\mu b} + \omega^{a}_{\mu c}\omega^{c}_{\nu b} - \omega^{a}_{\nu c}\omega^{c}_{\mu b} \quad — (16)$$

Here:

$$\omega^{a}_{\mu b} = q^{\nu}_{b}\,\omega^{a}_{\mu\nu} = q^{\nu}_{b}\left(\Gamma^{a}_{\mu\nu} - \partial_\mu q^{a}_{\nu}\right)$$

$$= \Gamma^{a}_{\mu b} - q^{\nu}_{b}\,\partial_\mu q^{a}_{\nu} \quad — (17)$$

$$\boxed{\omega^{a}_{\mu b} = \Gamma^{a}_{\mu b} - \partial_\mu q^{a}_{b}}$$

and so on.

Therefore:

$$R^{a}{}_{b\mu\nu} = \partial_\mu \Gamma^{a}_{\nu b} - \partial_\nu \Gamma^{a}_{\mu b} + \Gamma^{a}_{\mu c}\Gamma^{c}_{\nu b} - \Gamma^{a}_{\nu c}\Gamma^{c}_{\mu b}$$
$$- \partial_\mu \partial_\nu q^{a}_{b} + \partial_\nu \partial_\mu q^{a}_{b} \quad — (18)$$
$$- \Gamma^{c}_{cb}\,\partial_\mu q^{a}_{\nu} - \Gamma^{a}_{cb}\,\partial_\nu q^{c}_{\mu}$$
$$+ \Gamma^{c}_{cb}\,\partial_\mu q^{a}_{\nu} - \Gamma^{a}_{cb}\,\partial_\nu q^{c}_{\mu}$$

$$\boxed{R^{a}{}_{b\mu\nu} = \partial_\mu \Gamma^{a}_{\nu b} - \partial_\nu \Gamma^{a}_{\mu b} + \Gamma^{a}_{\mu c}\Gamma^{c}_{\nu b} - \Gamma^{a}_{\nu c}\Gamma^{c}_{\mu b}} \quad — (19)$$

Finally we:

4)

$$R^\rho{}_{\sigma\mu\nu} = q^\rho_a q^b_\sigma R^a{}_{b\mu\nu}$$

$$= \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu c}\Gamma^c_{\nu\sigma} - \Gamma^\rho_{\nu c}\Gamma^c_{\mu\sigma} \quad -(20)$$

and use:

$$\Gamma^\rho_{\mu c}\Gamma^c_{\nu\sigma} = q^c_\lambda q^\lambda_c \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}$$

$$= \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} \quad -(21)$$

to find the Riemannian curvature:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \quad -(22)$$

Q.E.D.

It has been shown that the fundamental property (1) creates a Riemannian curvature.

Conclusion

The complex circular representation of a 4-D and the Cartesian representation of the 4-D spacetime (or 3-D spacetime) are necessary and sufficient to define torsion and curvature tensors in that spacetime.

5) The connection is defined by the tetrad postulate, eq. (13):

$$\Gamma^{a}_{\mu\nu} = \partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu\nu} \quad -(23)$$

and the Riemannian connection by:

$$\Gamma^{\lambda}_{\mu\nu} = q^{\lambda}_{a} \Gamma^{a}_{\mu\nu} \quad -(24)$$

The complex circular basis is:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) \quad -(25)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}} \left(\underline{i} + i \underline{j} \right) \quad -(26)$$

$$\underline{e}^{(3)} = \underline{k} \quad -(27)$$

The vector field is in two dimensions:

$$\underline{V} = V^{(1)} \underline{e}^{(1)} + V^{(2)} \underline{e}^{(2)} \quad -(28)$$

$$= V_x \underline{i} + V_y \underline{j} \quad -(29)$$

where

$$V^{(1)} = \frac{1}{\sqrt{2}} \left(V_x + i V_y \right) \quad -(30)$$

$$V^{(2)} = \frac{1}{\sqrt{2}} \left(V_x - i V_y \right) \quad -(31)$$

A covariant derivative may be defined as:

$$D_\mu V^{(1)} = \partial_\mu V^{(1)} + \omega^{(1)}_{\mu(2)} V^{(2)} \quad -(32)$$

for example.

)

$$\frac{DV^{(1)}}{DX} = \frac{\partial V^{(1)}}{\partial X} + \omega^{(1)}_{\ \ 1(2)} V^{(2)} \quad -(33)$$

If $$\frac{\partial V^{(1)}}{\partial X} = 0 \quad -(34)$$

then $$\frac{\omega^{(1)}_{\ \ 1(2)}}{\sqrt{2}}\left(V_X - iV_Y\right) = \frac{DV^{(1)}_{\theta}}{DX} \quad -(35)$$

so $$\boxed{\begin{array}{c}\omega^{(1)}_{\ \ 1(2)} \neq 0 \\ \cancel{V_X = iV_Y}\end{array}} \quad -(36)$$

The existence of eq. (1) generates a spin connection

If $$\frac{DV^{(1)}}{DX} = \frac{\partial V^{(1)}}{\partial X} \quad -(37)$$

then $$\frac{\omega^{(1)}_{\ \ 1(2)}}{\sqrt{2}}\left(V_X - iV_Y\right) = 0 \quad -(38)$$

and a possible solution is

$$V_X = iV_Y = 0 \quad -(39)$$

$$\boxed{\omega^{(1)}_{\ \ 1(2)} \neq 0} \quad -(40)$$

140(12): Some Further Details of Proof 140(11)

We have the result leading to eq. (17):

$$\partial_\mu q^a{}_b = q^\nu{}_b \, \partial_\mu q^a{}_\nu \quad — (1)$$

because $\partial_\mu q^a{}_\nu$ is a mixed index rank three tensor.

Therefore:

$$\partial_\mu \omega^a{}_{\nu b} - \partial_\nu \omega^a{}_{\mu b} = \partial_\mu \Gamma^a{}_{\nu b} - \partial_\nu \Gamma^a{}_{\mu b} + \left(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu\right) q^a{}_b \quad — (2)$$

$$= \partial_\mu \Gamma^a{}_{\nu b} - \partial_\nu \Gamma^a{}_{\mu b}$$

In eq. (1):

$$q^a{}_b = q^a{}_\mu \, q^\mu{}_b \quad — (3)$$

So:

$$\partial_\nu q^a{}_b = \partial_\nu \left(q^a{}_\mu \, q^\mu{}_b \right)$$

$$= q^\mu{}_b \, \partial_\nu q^a{}_\mu + q^a{}_\mu \, \partial_\nu q^\mu{}_b$$

$$= q^\mu{}_b \, \partial_\nu q^a{}_\mu \quad — (4)$$

So

$$\boxed{q^a{}_\mu \, \partial_\nu q^\mu{}_b = 0} \quad — (5)$$

This is a general constraint on the Cartan tetrad.

2) Eq. (5) means:

$$q^a_0 \partial_\omega q^0_b + q^a_1 \partial_\omega q^1_b + q^a_2 \partial_\omega q^2_b + q^a_3 \partial_\omega q^3_b = 0 \quad - (6)$$

Electrodynamics

$$\boxed{A^a_\mu \partial_\omega A^\mu_b = 0} \quad - (7)$$

Gravitation

$$\boxed{\Phi^a_\mu \partial_\omega \Phi^\mu_b = 0} \quad - (8)$$

From eq. (5):

$$q^a_\mu \partial_\omega q^\mu_b = \partial_\omega q^a_b = 0 \quad - (9)$$

This is true because in a Minkowski spacetime:

$$q^a_b = 0 \quad - (10)$$

its metric being diagonal.

Therefore from eq. (17) of note 140(1):

$$\boxed{\omega^a_{\mu b} = \Gamma^a_{\mu b}} \quad - (11)$$

140(13): New Constants or Potentials

The metric is defined in general by:

$$x^\mu = g^{\mu\nu} x_\nu \ldots \quad -(1)$$

In Minkowski spacetime:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(2)$$

Eq. (1) can be expressed

$$x^\mu = g^\mu{}_\nu x^\nu \quad -(3)$$

where

$$g^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(4)$$

in Minkowski spacetime.

In Cartan's differential geometry the tangent spacetime at point P to a base manifold is labelled a, so:

$$x^a = g^a{}_b x^b \quad -(5)$$

and so

$$q^a{}_b = g^a{}_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(6)$$

where $q^a{}_b$ is the tetrad.

Therefore

$$\boxed{\partial_\mu q^a{}_b = 0} \quad -(7)$$

2) This result leads to:

$$\boxed{q^a_\mu \, \partial_\nu q^\mu_b = 0} \quad -(8)$$

which is a general constraint on the Cartan tetrad. The constraint (8) can be checked with the circularly polarized tetrad:

$$\underline{q}^{(1)} = \frac{1}{\sqrt{2}}\left(\mathbf{i} - i\mathbf{j}\right) e^{i(\omega t - \kappa Z)} \quad -(9)$$

and

$$\underline{q}^{(2)} = \frac{1}{\sqrt{2}}\left(\mathbf{i} + i\mathbf{j}\right) e^{-i(\omega t - \kappa Z)} \quad -(10)$$

Denote:

$$\phi = \omega t - \kappa Z, \quad -(11)$$

then:

$$\left. \begin{aligned} q^{(1)}_X &= \frac{1}{\sqrt{2}} e^{i\phi}, \quad q^{(1)}_Y = \frac{i}{\sqrt{2}} e^{i\phi}, \\ q^{(2)}_X &= \frac{1}{\sqrt{2}} e^{-i\phi}, \quad q^{(2)}_Y = \frac{i}{\sqrt{2}} e^{-i\phi}. \end{aligned} \right\} \quad -(12)$$

Now apply the rule:

$$q^a_\mu \, q^\mu_a = 1 \quad -(13)$$

to find that

$$q^{(1)}_X q^X_{(1)} + q^{(1)}_Y q^Y_{(1)} + q^{(2)}_X q^X_{(2)} + q^{(2)}_Y q^Y_{(2)} = 1. \quad -(14)$$

3) A solution of eq. (4) is:

$$\left.\begin{aligned} q^X_{(1)} &= \frac{1}{\sqrt{2}} e^{-i\phi}, \quad q^Y_{(1)} = \frac{i}{\sqrt{2}} e^{-i\phi}, \\ q^X_{(2)} &= \frac{1}{\sqrt{2}} e^{i\phi}, \quad q^Y_{(2)} = -\frac{i}{\sqrt{2}} e^{i\phi} \end{aligned}\right\} \quad -(15)$$

An example of eq. (8) is:

$$q^a_\mu \partial_\nu q^\mu_b = q^{(1)}_X \partial_\nu q^X_{(2)} + q^{(1)}_Y \partial_\nu q^Y_{(2)} \quad -(16)$$

For $\nu = 0$

$$q^{(1)}_X \partial_0 q^X_{(2)} + q^{(1)}_Y \partial_0 q^Y_{(2)}$$

$$= \frac{1}{\sqrt{2}} e^{i\phi} \partial_0 \left(\frac{1}{\sqrt{2}} e^{i\phi}\right) - i e^{i\phi} \partial_0 \left(-\frac{i}{\sqrt{2}} e^{i\phi}\right)$$

$$= \frac{1}{2} e^{i\phi} \left(\partial_0 e^{i\phi} - \partial_0 e^{i\phi}\right)$$

$$= 0 \quad -(17)$$

Q.E.D.

For $\nu = 3$

$$q^{(1)}_X \partial_3 q^X_{(2)} + q^{(1)}_Y \partial_3 q^Y_{(2)}$$

$$= \frac{1}{2} e^{i\phi} \left(\partial_3 e^{i\phi} - \partial_3 e^{i\phi}\right)$$

$$= 0$$

Q.E.D. $-(18)$

4) The second possible example of eq (8) is:

$$q^a_\mu \partial_\nu q^\mu_b = q^{(2)}_X \partial_\nu q^X_{(1)} + q^{(2)}_Y \partial_\nu q^Y_{(1)} \quad -(19)$$

For $\nu = 0$

$$q^{(2)}_X \partial_0 q^X_{(1)} + q^{(2)}_Y \partial_0 q^Y_{(1)}$$

$$= \frac{1}{\sqrt{2}} e^{-i\phi} \partial_0 \left(\frac{1}{\sqrt{2}} e^{-i\phi} \right) + \frac{i}{\sqrt{2}} \partial_0 \left(\frac{i}{\sqrt{2}} e^{-i\phi} \right) \frac{1}{\sqrt{2}} e^{-i\phi}$$

$$= \frac{1}{2} e^{-i\phi} \left(\partial_0 e^{-i\phi} - \partial_0 e^{-i\phi} \right)$$

$$= 0$$

Q.E.D. $-(20)$

For $\nu = 3$

$$q^{(2)}_X \partial_3 q^X_{(1)} + q^{(2)}_Y \partial_3 q^Y_{(1)}$$

$$= \frac{1}{2} e^{-i\phi} \left(\partial_3 e^{-i\phi} - \partial_3 e^{-i\phi} \right)$$

$$= 0$$

Q.E.D. $-(21)$

So the constraint (8) has been tested for the circularly polarized tetrad and shown to be correct.

1)

140(14): Viscosity Effects in Fluid Flow

1) For an inviscid fluid, as in previous notes:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) = -\underline{\nabla} p - \rho\underline{\nabla}\phi \quad -(1)$$

The viscous force, $\underline{f}_v$, is added to the right hand side of eq. (1) to produce:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) = -\underline{\nabla} p - \rho\underline{\nabla}\phi + \underline{f}_v \quad -(2)$$

The most general form of second derivatives that can occur in a vector equation is a linear combination of terms $\nabla^2 \underline{v}$ and $\underline{\nabla}(\underline{\nabla}\cdot\underline{v})$. Therefore:

$$\underline{f}_v = \mu\nabla^2\underline{v} + (\mu+\mu')\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) \quad -(3)$$

where μ and μ' are coefficients. From eqs. (2) and (3):

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v}\right) = -\underline{\nabla} p - \rho\underline{\nabla}\phi + \mu\nabla^2\underline{v} + (\mu+\mu')\underline{\nabla}(\underline{\nabla}\cdot\underline{v}) \quad -(4)$$

The vorticity is defined as:

$$\underline{\Omega} = \underline{\nabla}\times\underline{v} \quad -(5)$$

Using the identity:

$$(\underline{v}\cdot\underline{\nabla})\underline{v} = (\underline{\nabla}\times\underline{v})\times\underline{v} + \frac{1}{2}\nabla(\underline{v}\cdot\underline{v}) \quad -(6)$$

and in an incompressible fluid:

$$\underline{\nabla}\cdot\underline{v} = 0 \quad -(7)$$

2)

$$\frac{\partial \underline{\Omega}}{\partial t} + \underline{\nabla} \times (\underline{\Omega} \times \underline{v}) = \frac{\mu}{\rho} \nabla^2 \underline{\Omega} \quad -(8)$$

which is the equation of motion of a viscous fluid.

The inviscid fluid is

$$\frac{\partial \underline{\Omega}}{\partial t} + \underline{\nabla} \times (\underline{\Omega} \times \underline{v}) = \underline{0} \quad -(9)$$

As in previous notes these bear a similarity to the equations of electrodynamics.