137(1): SU(2) Development of the Tetrad Postulate

The tetrad postulate is the most fundamental statement of differential geometry, and is well accepted in mathematics. It is:

$$D_\mu q^a_\nu = 0 \qquad -(1)$$

where:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda \qquad -(2)$$

is defined as the covariant derivative of the tetrad.

From eq. (1):

$$D^\mu (D_\mu q^a_\nu) = \partial^\mu (D_\mu q^a_\nu) = 0 \qquad -(3)$$

Thus:

$$\Box q^a_\nu = \partial^\mu (\Gamma^\lambda_{\mu\nu} q^a_\lambda - \omega^a_{\mu b} q^b_\nu) \qquad -(4)$$

where

$$\Box := \partial^\mu \partial_\mu \qquad -(5)$$

is the d'Alembertian operator. Now define:

$$R q^a_\nu := \partial^\mu (\Gamma^\lambda_{\mu\nu} q^a_\lambda - \omega^a_{\mu b} q^b_\nu) = \partial^\mu (\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}). \qquad -(5)$$

Multiply both sides of eq. (5) by q^ν_a and use

$$q^a_\nu q^\nu_a = 1 \qquad -(6)$$

to obtain:

$$\boxed{\Box q^a_\nu := R q^a_\nu} \qquad -(7)$$

2) This is a fundamental identity of geometry known as the tetrad postulate. It is a wave equation with eigenfunction q^a_μ, the Cartan tetrad. Note carefully that $\Box$ is used in any spacetime. It is defined by:

$$\Box := \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \quad -(8)$$

As shown in paper 136 it may be factorized in the SU(2) representation space:

$$(\sigma^0)^2\,\Box = \left(\frac{\sigma^0}{c}\frac{\partial}{\partial t} + \underline{\sigma}\cdot\underline{\nabla}\right)\left(\frac{\sigma^0}{c}\frac{\partial}{\partial t} - \underline{\sigma}\cdot\underline{\nabla}\right). \quad -(9)$$

which is equivalent to:

$$(\sigma^0)^2\, p^\mu p_\mu = \left(\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p}\right)\left(\sigma^0 p_0 - \underline{\sigma}\cdot\underline{p}\right) \quad -(10)$$

From eq. (9) in eq. (7):

$$(\sigma^0)^2\,\Box\, q^a_\mu = \left(\frac{\sigma^0}{c}\frac{\partial}{\partial t} + \underline{\sigma}\cdot\underline{\nabla}\right)\left(\frac{\sigma^0}{c}\frac{\partial}{\partial t} - \underline{\sigma}\cdot\underline{\nabla}\right) q^a_\mu$$
$$= (\sigma^0)^2 R\, q^a_\mu \quad -(11)$$

This is equivalent to:

$$(\sigma^0)^2\, p^\mu p_\mu\, q^a_\mu = \left(\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p}\right)\left(\sigma^0 p_0 - \underline{\sigma}\cdot\underline{p}\right) q^a_\mu$$
$$= (\sigma^0)^2 \hbar^2 R\, q^a_\mu \quad -(12)$$

3) Eq. (11) is an operator equation while eq. (12) is an algebraic equation. They are equivalent from:

$$p^\mu = i\hbar \partial^\mu \quad -(13)$$

the fundamental operator identity of quantum mechanics.

Eq. (12) means:

$$\boxed{\left(\sigma^0 p_0 + \underline{\sigma} \cdot \underline{p}\right)\left(\sigma^0 p_0 - \underline{\sigma} \cdot \underline{p}\right) = \left(\sigma^0\right)^2 R\hbar^2} \quad -(14)$$

This is an equation of general relativity. It reduces to special relativity when:

$$|R| \rightarrow (mc/\hbar)^2. \quad -(15)$$

In this case:

$$p_0 = mc = i\hbar R^{1/2} \quad -(16)$$

$$E_0 = mc^2 = ic\hbar R^{1/2} \quad -(17)$$

Therefore

$$R = -\left(\frac{mc}{\hbar}\right)^2 \quad -(18)$$

and

$$\left(\Box + \left(\frac{mc}{\hbar}\right)^2\right) q^a_\mu = 0 \quad -(19)$$

from eq. (7). In this limit of special relativity the scalar curvature is determined by mass m within a factor:

4) $$R := q^{\tilde{a}}_{a}\, \partial^{\mu}\left(\Gamma^{a}_{\mu\nu} - \omega^{a}_{\mu\nu}\right) \rightarrow -\left(\frac{mc}{\hbar}\right)^{2} \quad -(20)$$

Now define:

$$\phi^{R} = \left[q^{R}_{1} \quad q^{R}_{2} \right], \quad \phi^{L} = \left[q^{L}_{1} \quad q^{L}_{2} \right] \quad -(21)$$

as in earlier papers. Thus:

$$\phi^{RT} = \begin{bmatrix} q^{R}_{1} \\ q^{R}_{2} \end{bmatrix} \quad -(22)$$

and

$$\phi^{RT}\phi^{L} = \begin{bmatrix} q^{R}_{1} \\ q^{R}_{2} \end{bmatrix} \left[q^{L}_{1} \quad q^{L}_{2} \right] \quad -(23)$$

$$= \begin{bmatrix} q^{R}_{1} q^{L}_{1} & q^{R}_{1} q^{L}_{2} \\ q^{R}_{2} q^{L}_{1} & q^{R}_{2} q^{L}_{2} \end{bmatrix} \quad -(24)$$

$$:= q^{a}_{\tilde{}} \quad -(25)$$

Therefore eq. (12) becomes:

$$(\sigma^{0} p_{0} + \underline{\sigma}\cdot\underline{p})(\sigma^{0} p_{0} - \underline{\sigma}\cdot\underline{p}) \begin{bmatrix} q^{R}_{1} q^{L}_{1} & q^{R}_{1} q^{L}_{2} \\ q^{R}_{2} q^{L}_{1} & q^{R}_{2} q^{L}_{2} \end{bmatrix}$$

$$= (\sigma^{0})^{2} \hbar^{2} R \begin{bmatrix} q^{R}_{1} q^{L}_{1} & q^{R}_{1} q^{L}_{2} \\ q^{R}_{2} q^{L}_{1} & q^{R}_{2} q^{L}_{2} \end{bmatrix} \quad -(26)$$

Possible solutions of eq. (26) are:

$$(\sigma^{0} p_{0} + \underline{\sigma}\cdot\underline{p})(\sigma^{0} p_{0} - \underline{\sigma}\cdot\underline{p})\, q^{R}_{1} q^{L}_{1} = (\sigma^{0})^{2} \hbar^{2} R\, q^{R}_{1} q^{L}_{1} \quad -(27)$$

5) and so on. Write eq. (27) as:

$$(\sigma^0 p_0 - \underline{\sigma}\cdot\underline{p})\, q_1^R\, (\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p})\, q_2^L$$

$$= \sigma^0 \hbar |R|^{1/2} q_1^L\, \sigma^0 \hbar |R|^{1/2} q_1^R \qquad -(28)$$

It is well known from special relativity, and demonstrated in papers 128 onwards to 136 that:

$$(\sigma^0 p_0 - \underline{\sigma}\cdot\underline{p})\, q_1^R = mc\,\sigma^0 q_1^L \qquad -(29)$$

$$(\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p})\, q_1^L = mc\,\sigma^0 q_1^R \qquad -(30)$$

Therefore we choose solutions of eq. (28) to be:

$$\boxed{\begin{aligned}(\sigma^0 p_0 - \underline{\sigma}\cdot\underline{p})\, q_1^R &= \hbar |R|^{1/2} \sigma^0 q_1^L \\ (\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p})\, q_1^L &= \hbar |R|^{1/2} \sigma^0 q_1^R\end{aligned}} \qquad -(31)$$

These are the required SU(2) development of the ECE Lemma. Similarly

$$\boxed{\begin{aligned}(\sigma^0 p_0 - \underline{\sigma}\cdot\underline{p})\, q_2^R &= \hbar |R|^{1/2} \sigma^0 q_2^L \\ (\sigma^0 p_0 + \underline{\sigma}\cdot\underline{p})\, q_2^L &= \hbar |R|^{1/2} \sigma^0 q_2^R\end{aligned}} \qquad -(32)$$

Finally use:

$$p^\mu = i\hbar\, \partial^\mu \qquad -(33)$$

i.e.

$$(p_0, \underline{p}) = i\hbar \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla} \right) \qquad -(34)$$

to obtain:

$$
\left.
\begin{aligned}
i\left(\frac{\sigma^\circ}{c}\frac{\partial}{\partial t}+\underline{\sigma}\cdot\underline{\nabla}\right)q_1^R &= |R|^{1/2}\sigma^\circ q_1^L \\
i\left(\frac{\sigma^\circ}{c}\frac{\partial}{\partial t}-\underline{\sigma}\cdot\underline{\nabla}\right)q_1^L &= |R|^{1/2}\sigma^\circ q_1^R \\
i\left(\frac{\sigma^\circ}{c}\frac{\partial}{\partial t}+\underline{\sigma}\cdot\underline{\nabla}\right)q_2^R &= |R|^{1/2}\sigma^\circ q_2^L \\
i\left(\frac{\sigma^\circ}{c}\frac{\partial}{\partial t}-\underline{\sigma}\cdot\underline{\nabla}\right)q_2^L &= |R|^{1/2}\sigma^\circ q_2^R
\end{aligned}
\right] \quad -(35)
$$

where

$$R = \tilde{q}_a\,\partial^\mu\left(\Gamma^a_{\mu\nu}-\omega^a_{\mu\nu}\right) \quad -(36).$$

1/

137(2): Einstein's Energy Equation in General Relativity.

As in previous notes the ECE Lemma of geometry is derived from the identity:

$$D^\mu (D_\mu q^a_\nu) := 0 \quad -(1)$$

where the covariant derivative implies the use of a non-Minkowski spacetime. The geometrical Lemma (1) can be re-expressed as:

$$\square q^a_\nu := R q^a_\nu \quad -(2)$$

where

$$R = q^\nu_a \partial^\mu (\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}) \quad -(3)$$

The d'Alembertian operator in eq. (2) is valid in any spacetime because it is derived from eq. (1). The Lemma (2) is an eigen equation with eigenoperator $\square$ in any spacetime. The eigen-functions are R, whose expectation values are related to mass density through the hypothesis:

$$R = -kT \quad -(4)$$

as in earlier work. In eq. (2) is defined the d'Alembertian $\square$ as:

$$\square = \partial^\mu \partial_\mu \quad -(5)$$

The covariant derivative is defined as:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \quad -(6)$$

So the partial derivative ∂_μ filters out that part of D_μ which can be expressed in flat or Minkowski spacetime.

Similarly, $\Box$ in eq. (2) filters out that part of eq. (1) which can be expressed in flat or Minkowski spacetime. The latter is the spacetime of special relativity.

The well known operator equivalence is

$$p^{\mu} = i\hbar \partial^{\mu} \quad -(7)$$

$$p_{\mu} = i\hbar \partial_{\mu} . \quad -(8)$$

So $$\Box = \partial^{\mu}\partial_{\mu} = -\frac{1}{\hbar^2} p^{\mu} p_{\mu} . \quad -(9)$$

Therefore eq. (2) becomes:

$$\boxed{p^{\mu} p_{\mu} = -\hbar^2 R} \quad -(10)$$

or $$p^{\mu} p_{\mu} + \hbar^2 R = 0 \quad -(11)$$

Units Check

$$p^{\mu} p_{\mu} = (\mathrm{kgm\, m\, s^{-1}})^2$$

$$\hbar^2 R = (\mathrm{Js})^2 \mathrm{m}^{-2} = (\mathrm{kgm\, m^2 s^{-1}})^2 \mathrm{m}^{-2} \quad \checkmark\checkmark$$

In the limit of special relativity:

$$R \rightarrow -\kappa^2 = -\left(\frac{mc}{\hbar}\right)^2 \quad -(12)$$

so eqn. (11) becomes the Einstein energy equation:

$$p^{\mu} p_{\mu} = m^2 c^2 \quad -(13)$$

Eq. (13) is usually expressed as:

$$E^2 = c^2 p^2 + m^2 c^4 \quad — (14)$$

using:

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 \quad — (15)$$

Eq. (14) is one of special relativity, so there is no acceleration of the particle (e.g. an elementary particle) and no so no force on the particle.

Eq. (11) is one of general relativity, and so R indicates that there is external force on the particle. The particle interacts with a field of force. Eq. (2) is the quantized version of eq. (11). The quantized version of eq. (14) is:

$$(\square + \kappa^2) q^a_\mu = 0 \quad — (16)$$

and is a second order wave equation. Eqs. (14) and (16) are free particle / field equations. The field is the unified field.

Eqs. (2) and (11) are interacting particle / field equations. The field is again the unified field. The latter manifests itself in various elementary particles with mass.

1)

137(3): The Classical Equation of General Relativity and the Rest Volume of a Particle.

The classical equation of general relativity is:

$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 = \hbar^2 k T \quad -(1)$$

where T has the units of density (kilograms per cubic metre), and is defined by

$$\boxed{T = \frac{m}{V} = \frac{E^2 - c^2 p^2}{\hbar^2 c^2 k}} \quad -(2)$$

in any frame of reference for a particle with momentum p. In the rest frame:

$$p = 0 \quad -(3)$$

and

$$E = E_0 = mc^2 \quad -(4)$$

so:

$$T_0 = \frac{m}{V_0} = \frac{E_0^2}{\hbar^2 c^2 k} = \frac{m^2 c^2}{\hbar^2 k} \quad -(5)$$

The rest volume of any elementary particle is:

$$\boxed{V_0 = \frac{\hbar^2 k}{E_0}} \quad -(6)$$

which was first derived in GCUFT1 as eq. (4.91), page 76.

In the case of finite momentum, the ~~rest~~ volume is:

2)

$$\boxed{V = \frac{E^2 - c^2 p^2}{\hbar^2 c^2 k m}} \quad -(7)$$

The mass m of the particle is considered to be a fundamental property of the particle, so the denominator in eq. (7) is a constant for every elementary particle. Denote this by:

$$\beta = \hbar^2 c^2 k m \quad -(8)$$

So:

$$\boxed{V = \frac{1}{\beta}\left(E^2 - c^2 p^2\right)} \quad -(9).$$

<u>Note carefully</u> that this is an equation of <u>general relativity</u>.

In the old physics, the photon has no mass:

$$m = ?\ 0 \quad -(10)$$

and has energy:

$$E = \hbar \omega = c \hbar \kappa = cp. \quad -(11)$$

In this case the volume of the photon is <u>indeterminate</u>:

$$V = ?\ 0/0 \quad -(12)$$

in general. The massless photon has no rest frame. These are obsolete and unsatisfactory concepts.

3) The idea of a massless photon conflicts with the theory of light bending in the relativistic Kepler problem, where a photon of mass m is attracted by an object of mass M in general relativity.

In special relativistic quantum mechanics the Proca equation was used for a photon with mass:

$$(\Box + \kappa^2) A_\mu = 0 \quad -(13)$$

where

$$\kappa = \frac{mc}{\hbar} \quad -(14)$$

In ECE theory this was generalized to:

$$(\Box + \kappa^2) A^a_\mu = 0 \quad -(15)$$

where a is a polarization index. Eq. (15) is obtained directly from the tetrad postulate:

$$D_\mu q^a_\mu = 0. \quad -(16)$$

Only two simple hypotheses are used in the derivation of eq. (15) from eq. (16):

$$R = -kT \quad -(17)$$

and

$$A^a_\mu = A^{(0)} q^a_\mu. \quad -(18)$$

So the equation of the photon is:

$$(\Box + kT) A^a_\mu = 0. \quad -(19)$$

4) The Proca equation of the nineteen twenties is the limit of eq. (19) when:

$$kT \to \kappa^2 \quad -(20)$$

and for each a. The classical limit of eq. (19) is eq. (1), obtained using:

$$p^\mu = i\hbar \partial^\mu, \quad -(21)$$

and

$$p^\mu p_\mu = -\hbar^2 \Box . \quad -(22)$$

In ECE theory the volume occupied by all elementary particles is given by eq. (7), including the photon and neutrino. In the old theory these used to be regarded as "massless".

In special relativity, $E^2 - c^2 p^2$ is a constant $m^2 c^4$ but in general relativity it is $\hbar^2 c^2 kT$, which varies. The well known equation of special relativity:

$$p^\mu p_\mu = (mc)^2 \quad -(23)$$

is that of a particle where there is no acceleration. The particle's four momentum is:

$$p^\mu = \left(\frac{E}{c}, \underline{p} \right) \quad -(24)$$

$$p_\mu = \left(\frac{E}{c}, -\underline{p} \right). \quad -(25)$$

5) Here $\underline{p}$ is the relativistic momentum:

$$\underline{p} = \gamma m_0 \underline{v} \quad -(26)$$

where:

$$\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \quad -(27)$$

and where u is the speed that one frame moves with respect to another. In the old theory u is constant and c is a universal constant by hypothesis. Usually u is identified with v.

New ideas are challenging the assumption that c is a universal constant.

In eq. (1) outside forces and therefore acceleration of the particle are introduced through $\underline{f} \, \partial kT$. Here:

$$\boxed{R = -kT = q \tilde{V}_a \gamma^\mu \left(\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}\right)} \quad -(28)$$

The volume of an accelerating elementary particle is therefore defined by:

$$T = \frac{m}{V} = \frac{1}{k} q \tilde{V}_a \gamma^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right) \quad -(29)$$

i.e.

$$\boxed{V = \frac{mk}{q \tilde{V}_a \gamma^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right)}} \quad -(30)$$

6) In the rest frame, where $\underline{p}$ is zero:

$$V_0 = \frac{\hbar^2 k}{mc^2} \quad -(31)$$

so:

$$q\tilde{V}_a \gamma^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right) = \frac{1}{k}\left(\frac{mc}{\hbar}\right)^2 \quad -(32)$$

$$= K_0^2$$

Therefore the Compton wavelength is the limit:

$$\boxed{K_0^2 = k\, q\tilde{V}_a \gamma^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right)} \quad -(33)$$

The Compton wavelength K_0 is therefore a rest frame limit of the general relativistic equation:

$$K^2 = q\tilde{V}_a \gamma^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right). \quad -(34)$$

The transition from general relativity to special relativity is the limit:

$$\boxed{q\tilde{V}_a \gamma^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu}\right) \rightarrow \left(\frac{mc}{\hbar}\right)^2} \quad -(35)$$

In this limit the volume of the particle is:

$$V_0 \rightarrow \frac{\hbar^2 k}{E_0}. \quad -(36)$$

If it is assumed that the metric in this limit approaches the Minkowski

7) metric η_{ab} in the equation:

$$g_{\mu\nu} = q^a_\mu \, q^b_\nu \, \eta_{ab} \quad -(37)$$

i.e.

$$q^\nu_a \rightarrow \delta^\nu_a \quad -(38)$$

where

$$\left.\begin{aligned} \delta^\nu_a &= 1, \quad \nu = a \\ &= 0, \quad \nu \neq a \end{aligned}\right\} \quad -(39)$$

Therefore:

$$\boxed{\partial^\mu \left(\omega^a_{\mu\nu} - \Gamma^a_{\mu\nu} \right) \rightarrow \delta^a_\nu \kappa_0^2} \quad -(40)$$

Since:

$$a = \nu \quad -(41)$$

$$\partial^\mu \left(\omega^\nu_{\mu\nu} - \Gamma^\nu_{\mu\nu} \right) \rightarrow \kappa_0^2 \quad -(42)$$

Finally use the tetrad postulate:

$$\begin{aligned} D_\mu q^a_\nu &= \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda \\ &= \partial_\mu q^a_\nu + \omega^a_{\mu\nu} - \Gamma^a_{\mu\nu} \\ &= 0 \end{aligned} \quad -(43)$$

so eq. (40) becomes:

$$\Box q^a_\nu \rightarrow -\delta^a_\nu \kappa_0^2 \quad -(44)$$

or

$$\boxed{\left(\Box + \kappa_0^2 \right) \delta^a_\nu \rightarrow 0} \quad -(45)$$

If

$$q^a_\nu = \delta^a_\nu \quad -(46)$$

8) eq. (45) is an identity because δ^a_μ is a constant.

The existence of any elementary particle in ECE theory depends on the fact that the particle is defined by:

$$\boxed{\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} = \partial_\mu q^a_\nu} \quad -(47)$$

and that

$$R \rightarrow -\kappa_0^2 = \left(\frac{mc}{\hbar}\right)^2 \quad -(48)$$

In this limit the particle is a free particle.

$$(\Box + \kappa_0^2) q^a_\nu = 0 \quad -(49)$$

Note carefully that in a spacetime devoid of mass, there is no scalar curvature R: $\kappa_0 = 0$

$$R = 0, \; m = 0, \quad -(50)$$

and

$$\boxed{\Box q^a_\nu = 0} \quad -(51)$$

One possible solution of eqn. (51) is the Minkowski spacetime:

$$q^a_\nu = \delta^a_\nu, \quad -(52)$$

but if the vacuum is defined by eq. (51) it may have metrics different from the Minkowski metric.

1) 137(3): SIMPLE PROOF OF THE ECE LEMMA

The tetrad postulate is:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda \quad -(1)$$
$$= 0$$

which can be written as:

$$\boxed{\partial_\mu q^a_\nu = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}} \quad -(2)$$

From eq. (1):

$$D^\mu(0) = \partial^\mu(0) = 0 \quad -(3)$$

So in eq. (3):

$$\partial^\mu \partial_\mu q^a_\nu = \partial^\mu(\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}) \quad -(4)$$
$$= \square q^a_\nu$$

Finally write eq. (4) as:

$$\boxed{\square q^a_\nu = R q^a_\nu} \quad -(5)$$

where

$$R = q^\nu_a \partial^\mu(\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}). \quad -(6)$$

By definition:

$$q^a_\nu q^\nu_a = 1. \quad -(7)$$

Eq. (5) is pure geometry. In order to transform it into physics the following hypothesis is used:

2)

$$R = -kT \quad - (8)$$

So:

$$\boxed{(\Box + kT)\, q^{a}_{\sim} = 0.} \quad - (9)$$

Eq. (5) is a second order differential equation that contains the same information as the first order differential equation (2).

In $SU(2)$ representation space eq. (5) factorizes into:

$$i\sigma^{\mu} \partial_{\mu} q^{R}_{1} = |R|^{1/2} \sigma^{\circ} q^{L}_{1} \quad - (10)$$

$$i\sigma^{\mu} \partial_{\mu} q^{L}_{1} = |R|^{1/2} \sigma^{\circ} q^{R}_{1} \quad - (11)$$

From eq. (2):

$$\partial_{\mu} q^{R}_{1} = \Gamma^{R}_{\mu 1} - \omega^{R}_{\mu 1} \quad - (12)$$

$$\partial_{\mu} q^{L}_{1} = \Gamma^{L}_{\mu 1} - \omega^{L}_{\mu 1} \quad - (13)$$

So

$$i\sigma^{\mu} \partial_{\mu} q^{R}_{1} = i\sigma^{\mu} \left(\Gamma^{R}_{\mu 1} - \omega^{R}_{\mu 1}\right) \quad - (14)$$

$$i\sigma^{\mu} \partial_{\mu} q^{L}_{1} = i\sigma^{\mu} \left(\Gamma^{L}_{\mu 1} - \omega^{L}_{\mu 1}\right) \quad - (15)$$

Therefore:

$$\boxed{\begin{aligned} \sigma^{\circ} |R|^{1/2} &= i\, q^{1}_{L} \sigma^{\mu} \left(\Gamma^{R}_{\mu 1} - \omega^{R}_{\mu 1}\right) \\ &= i\, q^{1}_{R} \sigma^{\mu} \left(\Gamma^{L}_{\mu 1} - \omega^{L}_{\mu 1}\right) \end{aligned}} \quad - (16)$$

This gives another expression for the origin of fermion mass. In the limit:

3)

$$|R|^{1/2} \rightarrow \frac{mc}{\hbar} = \kappa \quad -(17)$$

Here:

$$\boxed{\begin{aligned}\sigma^0 \kappa &= i\, q_L^1 \sigma^\mu \left(\Gamma^R_{\mu 1} - \omega^R_{\mu 1}\right) \\ &= i\, q_R\, \sigma^\mu \left(\Gamma^L_{\mu 1} - \omega^L_{\mu 1}\right)\end{aligned}} \quad -(18)$$

where

$$\sigma^\mu = \left(\sigma^0, \sigma^1, \sigma^2, \sigma^3\right) \quad -(19)$$

In eq. (2) the vacuum may be defined by:

$$\Gamma^a_{\mu\nu} = \omega^a_{\mu\nu} \quad -(20)$$

i.e.

$$\partial_\mu q^a_\nu = 0 \quad -(21)$$

and

$$\Box q^a_\nu = 0 \quad -(22)$$

The vacuum metric is:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(23)$$

with:

$$\partial_\mu q^a_\nu = \partial_\nu q^b_\mu = 0 \quad -(24)$$

and

$$\Box q^a_\nu = \Box q^b_\mu = 0 \quad -(25)$$

1) 137(5): First Order Equations in ECE Quantum Mechanics

Consider the tetrad postulate:

$$\partial_\mu q^a_\nu = \Gamma^\lambda_{\mu\nu} q^a_\lambda - \omega^a_{\mu b} q^b_\nu \quad -(1)$$

Relabel indices of summation as follows:

$$\lambda \to \nu, \quad b \to a. \quad -(2)$$

then

$$\partial_\mu q^a_\nu = \Gamma^\nu_{\mu\nu} q^a_\nu - \omega^a_{\mu a} q^a_\nu$$

$$\boxed{\partial_\mu q^a_\nu = \left(\Gamma^\nu_{\mu\nu} - \omega^a_{\mu a}\right) q^a_\nu} \quad -(3)$$

By the rules of covariant geometry eq. (1) or eq. (3) is

$$\partial_\mu q^a_\nu = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}. \quad -(4)$$

$$:= \Delta^a_{\mu\nu}$$

Eqs. (3) and (4) are first order differential equations which may be used in various ways for ECE quantum mechanics. Eq. (4) for example may be written as:

$$\partial_\mu q^a_\nu = \Delta^a_{\mu b} q^b_\nu \quad -(5)$$

or as

$$\partial_\mu q^a_\nu = \Delta^\lambda_{\mu\nu} q^a_\lambda \quad -(6)$$

Eq. (5) gives the fermionic equations in the SU(2) basis, or quark equations in SU(3).

2) If

$$\sim = \lambda \qquad (7)$$

in eq. (6) then:

$$\partial_\mu q^a_\lambda = \Delta^\lambda_{\mu\lambda} q^a_\lambda \qquad (8)$$

$$= \Delta^a_{\mu\lambda}$$

Written out in full, eq. (8) is:

$$\partial_\mu q^a_\sim = \Delta^0_{\mu\lambda} q^a_0 + \Delta^1_{\mu\lambda} q^a_1 + \Delta^2_{\mu\lambda} q^a_2 + \Delta^3_{\mu\lambda} q^a_3 \qquad (9)$$

where

$$\sim = \lambda \qquad (10)$$

This means:

$$\boxed{\begin{aligned} \partial_\mu q^a_0 &= \Delta^0_{\mu 0} q^a_0 \\ \partial_\mu q^a_1 &= \Delta^1_{\mu 1} q^a_1 \\ \partial_\mu q^a_2 &= \Delta^2_{\mu 2} q^a_2 \\ \partial_\mu q^a_3 &= \Delta^3_{\mu 3} q^a_3 \end{aligned}} \qquad \begin{aligned} (11) \\ (12) \\ (13) \\ (14) \end{aligned}$$

Here:

$$\Delta^0_{\mu 0} = \Gamma^0_{\mu 0} - \omega_{\mu}{}^{0}{}_{0} \qquad (15)$$

and so on. The tetrads are defined by:

$$V^a = q^a_\mu V^\mu \qquad (16)$$

i.e.

$$V^a = q^a_0 V^0 + \dots + q^a_3 V^3 \qquad (17)$$

The spin connection is defined by:

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b \qquad (18)$$

and the gamma connection by:

$$D_\mu V^\lambda = \partial_\mu V^\lambda + \Gamma^\lambda_{\mu\nu} V^\nu \quad -(19)$$

The frame labelled by a is different from the frame labelled by μ. For the sake of clarity we put brackets around (a), so eq. (18) is:

$$\begin{bmatrix} V^{(0)} \\ V^{(1)} \\ V^{(2)} \\ V^{(3)} \end{bmatrix} = \begin{bmatrix} q^{(0)}_0 & q^{(0)}_1 & q^{(0)}_2 & q^{(0)}_3 \\ q^{(1)}_0 & q^{(1)}_1 & q^{(1)}_2 & q^{(1)}_3 \\ q^{(2)}_0 & q^{(2)}_1 & q^{(2)}_2 & q^{(2)}_3 \\ q^{(3)}_0 & q^{(3)}_1 & q^{(3)}_2 & q^{(3)}_3 \end{bmatrix} \begin{bmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{bmatrix} \quad -(20)$$

In special relativity the tetrad matrix can be a Lorentz boost matrix or rotation matrix. The frames a and μ can also be two reps. of the same spacetime. In electrodynamics it is convenient to use:

$$a = (0), (1), (2), (3) \quad -(21)$$

and

$$\mu = 0, 1, 2, 3 \quad -(22)$$

So eqns. (11) to (14) become, as in the ECE engineering model:

$$\partial_\mu A^{(0)}_0 = \Delta^0_{\mu 0} A^{(0)}_0 \quad -(23)$$

and so on. For example:

$$\partial_\mu A^{(1)}_1 = \Delta^1_{\mu 1} A^{(1)}_1 \quad -(24)$$

or

$$\partial_\mu A^{(1)}_X = \Delta^1_{\mu 1} A^{(1)}_X \quad -(25)$$

4) In the case of a plane wave:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} \left(\underline{i} - i \underline{j} \right) e^{-i(\omega t - \kappa Z)} \quad -(26)$$

so $$A_X^{(1)} = \frac{A^{(0)}}{\sqrt{2}} e^{-i(\omega t - \kappa Z)} \quad -(27)$$

and $$\partial_0 A_X^{(1)} = \frac{1}{c} \frac{\partial}{\partial t} A_X^{(1)} = -i\kappa A_X^{(1)} \quad -(28)$$

so $$\boxed{\Delta^1_{01} = -i\kappa} \quad -(29)$$

Similarly: $$\boxed{\Delta^1_{31} = i\kappa} \quad -(30)$$

Now introduce quantum equivalence:

$$p_\mu = i\hbar \partial_\mu \quad -(31)$$

so $$-i\hbar p_\mu = \hbar \Delta_\mu{}^1 \quad -(32)$$

Thus: $$-i p_0 = \hbar \Delta^1_{01} = -i\kappa\hbar \quad -(33)$$

$$-i \hbar p_3 = \hbar \Delta^1_{31} = i\kappa\hbar \quad -(34)$$

and $$p_0 = \frac{E}{c} = \hbar\kappa \quad -(35)$$

$$p_3 = -\hbar\kappa \quad -(36)$$

Eq. (35) is Planck's Law

$$E = \hbar\omega \quad -(37)$$

5) So Planck's law (37) and de Broglie's law (36) have been deduced from the tetrad postulate.

In special relativity the relativistic momentum is:

$$p = \gamma m v \quad -(38)$$

and the relativistic kinetic energy is:

$$E = T = m c^2 (\gamma - 1) \quad -(39)$$

where:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad -(40)$$

We have:

$$p^2 = (\gamma m v)^2 \quad -(41)$$

$$E^2 = m^2 c^4 (\gamma - 1)^2 \quad -(42)$$

and

$$E^2 - c^2 p^2 = m^2 c^4 \quad -(43)$$

From eq. (31)

$$i \hbar \frac{\partial \psi}{\partial t} = E \psi \quad -(44)$$

$$-i \hbar \nabla \psi = p \psi \quad -(45)$$

$$-\hbar^2 \nabla^2 \psi = p^2 \psi \quad -(46)$$

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = E^2 \psi \quad -(47)$$

$$\left(\Box + \frac{E^2 - p^2 c^2}{\hbar^2}\right) \psi = 0 \quad -(48)$$

6) Eqs. (44) and (45) can be deduced from the tetrad postulate in this note.

In the quantum non-relativistic limit eq. (44) is the time dependent Schroedinger equation, and eq. (46) is the usual form of the Schroedinger equation. In the classical non-relativistic limit

$$p \rightarrow mv, \quad T \rightarrow \frac{1}{2} mv^2 = \frac{p^2}{2m} \quad -(49)$$

so:

$$i\hbar \frac{\partial \phi}{\partial t} = \frac{p^2}{2m} \phi \quad -(50)$$

$$-\frac{\hbar^2 \nabla^2 \phi}{2m} = E\phi \quad -(51)$$

Eq. (51) is:

$$\hat{H}\phi = E\phi \quad -(52)$$

where the Hamiltonian operator is $\hat{H}$. When there is potential energy V:

$$(\hat{H} + V)\phi = E\phi \quad -(53)$$

Note that in eq. (45) position $\underline{r}$ and momentum p enter simultaneously and are both specified. Eq. (53) gives atomic spectra, which have now been observed directly.

137(6): The Equivalence Law and Heisenberg Equation

The Heisenberg equation is, in one dimension:

$$[x, p]\phi = i\hbar\phi \quad -(1)$$

where:

$$[x, p] = xp - px. \quad -(2)$$

The relevant operator equivalence law is:

$$p = -i\hbar\frac{\partial}{\partial x} \quad -(3)$$

So:

$$xp\phi = -ix\hbar\frac{\partial\phi}{\partial x}. \quad -(4)$$

Also:

$$px\phi = p(x\phi) = -i\hbar\frac{\partial}{\partial x}(x\phi) \quad -(5)$$

where

$$\frac{\partial}{\partial x}(x\phi) = \phi\left(\frac{\partial x}{\partial x}\right) + x\frac{\partial\phi}{\partial x} \quad -(6)$$

$$= \phi + x\frac{\partial\phi}{\partial x} \quad -(7)$$

So eq. (1) is:

$$(xp - px)\phi = -ix\hbar\frac{\partial\phi}{\partial x} + ix\hbar\frac{\partial\phi}{\partial x} + i\hbar\phi \quad -(8)$$

$$= i\hbar\phi$$

Q.E.D. So the Heisenberg equation is the equivalence law, and nothing more.

The equivalence law may be applied to classical equations of dynamics. In the non-relativistic limit for example:

2)

$$\underline{p} = m\underline{v} \quad -(9)$$

$$E = T = \frac{1}{2}mv^2 = \frac{p^2}{2m} \quad -(10)$$

The equivalence law is:

$$p^\mu = i\hbar\, \partial^\mu \quad -(11)$$

where

$$p^\mu = \left(\frac{E}{c}, \underline{p}\right) \quad -(12)$$

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right). \quad -(13)$$

So:

$$E = i\hbar\frac{\partial}{\partial t}, \quad \underline{p} = -i\hbar\underline{\nabla}. \quad -(14)$$

Therefore:

$$-i\hbar\,\underline{\nabla}\phi = m\underline{v}\,\phi \quad -(15)$$

$$i\hbar\frac{\partial\phi}{\partial t} = \frac{p^2}{2m}\phi = E\phi. \quad -(16)$$

Also:

$$p^2 = m^2v^2 \quad -(17)$$

so.

$$-\hbar^2\nabla^2\phi = m^2v^2\phi \quad -(18)$$

i.e.

$$-\frac{\hbar^2\nabla^2}{2m}\phi = \frac{1}{2}mv^2\phi = E\phi \quad -(19)$$

or

$$\hat{H}\phi = E\phi \quad -(20)$$

which is the usual Schroedinger equation.

1. 137(7): Criticism of the Uncertainty Principle

The essence of the whole argument of the Copenhagen School is that if:

$$\hat{A}\psi = a\psi, \quad \hat{B}\psi = b\psi \quad -(1)$$

then

$$[\hat{A}, \hat{B}] = 0 \quad -(2)$$

In an illogical leap of thought, this is asserted to mean that $\hat{A}$ and $\hat{B}$ cannot be specified at the same time. For example

$$(\hat{x}\hat{p} - \hat{p}\hat{x})\psi = i\hbar\psi \quad -(3)$$

so

$$[\hat{x}, \hat{p}] \neq 0 \quad -(4)$$

and in the Copenhagen view, "$\hat{x}$ and $\hat{p}$ cannot be specified simultaneously." Eq. (3) is equivalent to

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \quad -(5)$$

The precise point at which the fallacy of Copenhagen starts is as follows:

$$\hat{x}\psi = x\psi \quad -(6)$$

$$\hat{p}\psi = p\psi \quad -(7)$$

which produce:

$$[\hat{x}, \hat{p}]\psi \stackrel{?}{=} 0 \quad -(8)$$

Therefore it is incorrectly assumed that ψ is a wave function soln of $\hat{x}$ and $\hat{p}$.

2. The correct equations are:

$$\hat{x}\,\psi_1 = x\,\psi_1 \quad -(9)$$

$$\hat{p}\,\psi_2 = p\,\psi_2 \quad -(10)$$

These are compatible with eq. (3). The observed x and p are:

$$x = \langle \hat{x} \rangle = \int \psi_1^*\,\hat{x}\,\psi_1\,d\tau \quad -(11)$$

$$p = \langle \hat{p} \rangle = \int \psi_2^*\,\hat{p}\,\psi_2\,d\tau \quad -(12)$$

and the idea of complementarity does not arise. It is replaced by the straightforward use of two different wave functions, ψ_1 and ψ_2, for $\hat{x}$ and $\hat{p}$. There is no reason why:

$$\psi_1 = \psi_2 . \quad -(13)$$

The derivation of the Heisenberg Uncertainty Principle given by Atkins on pp 93 ff of "Molecular Quantum Mechanics" (OUP, 2nd ed 1983) is based on (13). He starts with:

$$\langle \hat{A} \rangle = \int \psi^*\,\hat{A}\,\psi\,d\tau \quad -(14)$$

$$\langle \hat{B} \rangle = \int \psi^*\,\hat{B}\,\psi\,d\tau \quad -(15)$$

so he assumes (13). He then assumes:

$$[\hat{A}, \hat{B}] = i\,\hat{C} \quad -(16)$$

3) He also assumes the integral:

$$I = \int \left| \left(\alpha \Delta \hat{A} - i \Delta \hat{B} \right) \psi \right|^2 d\tau \quad -(17)$$

where:

$$\Delta \hat{A} = \hat{A} - \langle \hat{A} \rangle, \quad \Delta \hat{B} = \hat{B} - \langle \hat{B} \rangle. \quad -(18)$$

It is further assumed that α is real, and finally assumed that this integral should be minimized. These various assumptions are then asserted to lead to:

$$\delta \hat{A} \delta \hat{B} \geq \frac{1}{2} \left| \langle \hat{C} \rangle \right| \quad -(19)$$

where

$$\hat{C} = \frac{1}{i} [\hat{A}, \hat{B}]. \quad -(20)$$

This is an arbitrary mathematical exercise. Prof Croca and his group have shown that the result (19) has no meaning. The HUP has by now been refuted in several experiments, each of which are independent and carried out in different laboratories.

Conclusion

The correct interpretation is eqns (9) and (10), the Schroedinger equations of position and momentum. The basic error made by the Copenhagen School is:

$$\psi_1 = ? \psi_2 \quad -(21)$$

1)

137(8): The Correct Interpretation of Translational Motion in Quantum Mechanics.

Consider the translational motion in the x axis of a free particle of mass m and momentum p. The Schroedinger equation is:

$$\hat{H}\phi = -\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi \quad -(1)$$

where

$$E = T = \frac{p^2}{2m} = \frac{1}{2}mv^2 \quad -(2)$$

is the kinetic energy.

A possible solution of eq. (1) is:

$$\phi = A e^{i\kappa x} \quad -(3)$$

In the position representation:

$$\hat{p}\phi = -i\hbar\frac{\partial\phi}{\partial x} = \hbar\kappa\phi = p\phi \quad -(4)$$

So

$$p = \hbar\kappa \quad (5)$$

This is de Broglie's law.

Using eq. (5) in eq. (3):

$$\boxed{\phi = A\exp\left(\frac{ixp}{\hbar}\right)} \quad -(5)$$

In the momentum representation

$$\hat{x}\phi = i\hbar\frac{\partial\phi}{\partial p} = x\phi \quad -(6)$$

2) Therefore there are three equations:

$$\left.\begin{aligned} \hat{H}\psi &= E\psi \\ \hat{p}\psi &= p\psi \\ \hat{x}\psi &= x\psi \end{aligned}\right\} \quad (7)$$

If it is assumed that the same wavefunction (5) appears in all three equations then:

$$\left.\begin{aligned} [\hat{H}, \hat{p}]\psi &= 0 \\ [\hat{x}, \hat{p}]\psi &= 0 \\ [\hat{H}, \hat{x}]\psi &= 0 \end{aligned}\right\} \quad (8)$$

The result

$$[\hat{x}, \hat{p}]\psi = 0 \quad (9)$$

contradicts the basic equation

$$[\hat{x}, \hat{p}]\psi = i\hbar \quad (10)$$

which is the same as

$$p^{\mu} = i\hbar\partial^{\mu} \quad (11)$$

In the pair of equations:

$$\hat{x}\psi = x\psi \quad (12)$$

$$\hat{p}\psi = p\psi \quad (13)$$

the same wavefunction cannot be used. This is the correct interpretation of eq. (10), the Heisenberg equation. It is seen that:

3) $$\hat{x}\psi_1 = i\hbar\frac{\partial\psi_1}{\partial p} = x\psi_1 \quad -(14)$$

$$\hat{p}\psi_2 = -i\hbar\frac{\partial\psi_2}{\partial p} = p\psi_2 \quad -(15)$$

where:

$$\psi_1 = B\exp\left(-i\frac{xp}{\hbar}\right) \quad -(16)$$

$$\psi_2 = A\exp\left(i\frac{xp}{\hbar}\right) \quad -(17)$$

and:

$$\boxed{\begin{aligned}[\hat{x},\hat{p}]\psi_1 &= i\hbar\psi_1 \quad -(18)\\ [\hat{x},\hat{p}]\psi_2 &= i\hbar\psi_2 \quad -(19)\end{aligned}}$$

There is no indeterminacy, x and p are specified according to these equations.

Also, the same wave function is used in the pair of equations:

$$\hat{H}\psi = E\psi \quad -(20)$$

$$\hat{p}\psi = p\psi \quad -(21)$$

so:

$$\boxed{[\hat{H},\hat{p}]\psi = 0} \quad -(22)$$

This simple result is never mentioned in the Copenhagen dogma. It shows that ~~there~~ in their language, kinetic energy and momentum are simultaneously observable.

1) 137(9): Geometrical Origin of the Operator Equivalence of Quantum Mechanics

From note 135(6), if a curve is parameterized by $x^\mu(\lambda)$, the tangent vector is:

$$V^\mu = dx^\mu / d\lambda \quad -(1)$$

where $d / d\lambda$ is the directional derivative operator. The partial derivative operator ∂_μ acts as a basis set for $d / d\lambda$:

$$\frac{d}{d\lambda} = \left(\frac{dx^\mu}{d\lambda}\right)\partial_\mu \quad -(2)$$

The ∂_μ is the coordinate basis for the tangent spacetime at pt. p of a base manifold. This procedure is a generalization of setting up basis vectors to point along coordinate axes in 3-D Euclidean space.

Now define:

$$\kappa^a = V^a_\mu \partial^\mu \quad -(3)$$

in which both κ^a and ∂^μ are basis sets in the tangent spacetime. Here κ^a has the units of wavenumber (cm^{-1}). The four-momentum by de Broglie's law is:

$$p^a = \hbar \kappa^a \quad -(4)$$

Therefore:

2) $\hat{p}^a = \hbar \hat{\kappa}^a = \hbar q^a_\mu \partial^\mu$ — (5)

is the momentum operator.

On the basis of experimental data described by quantum mechanics

$$\hat{p}^a = i\hbar \partial^a, \quad \text{— (6)}$$

so: $$\boxed{\partial^a = -i\, q^a_\mu \partial^\mu} \quad \text{— (7)}$$

Quantum mechanics therefore originates in the type of geometry. For example:

$$\partial^1 = -i\left(q^1_0 \partial^0 + q^1_1 \partial^1 + q^1_2 \partial^2 + q^1_3 \partial^3\right) \quad \text{— (8)}$$

If the tetrad can be reduced to a diagonal square matrix:

$$\partial^1 = -i\, q^1_1 \partial^1 \quad \text{— (9)}$$

i.e. $$q^1_1 = i \quad \text{— (10)}$$

Similarly: $$\boxed{q^0_0 = q^1_1 = q^2_2 = q^3_3 = i} \quad \text{— (11)}$$

and this is the tetrad that gives eq. (6).

) From the tetrad postulate:

$$\partial_\mu q^a_\nu = \Delta^a_{\mu\nu} \quad -(12)$$

where

$$\Delta^a_{\mu\nu} = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \quad -(13)$$

This means that:

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu - \Delta^a_{\mu\nu} = 0. \quad -(14)$$

We have

$$\partial_\mu = q^a_\mu K_a \quad -(15)$$

so

$$\partial_\mu q^a_\nu = q^b_\mu K_b q^a_\nu = q^a_\nu K_\mu .$$

$$= q^a_\mu K_a q^a_\nu \quad -(16)$$

So

$$\boxed{\partial_\mu q^a_\nu = K_\mu q^a_\nu = \Delta^a_{\mu\nu}} \quad -(17)$$

and

$$K_\mu = q^\nu_a \Delta^a_{\mu\nu} \quad -(18)$$

$$\Delta^a_{\mu\nu} = q^a_\nu K_\mu , \quad -(19)$$

i.e.

$$\Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} = q^a_\nu K_\mu \quad -(20)$$

and

$$\boxed{R = q^\nu_a \partial^\mu \Delta^a_{\mu\nu} = \partial^\mu K_\mu} \quad -(21)$$

$$\Box q^a_\nu = R q^a_\nu \quad -(22)$$

1) Notes 137(10) : The fundamental equations of Cartan Geometry

Condensed Notation

$$T = D \wedge q = d \wedge q + \omega \wedge q \quad -(1)$$

$$R = D \wedge \omega = d \wedge \omega + \omega \wedge \omega \quad -(2)$$

$$D \wedge T = d \wedge T + \omega \wedge T := R \wedge q \quad -(3)$$

$$D \wedge \tilde{T} = d \wedge \tilde{T} + \omega \wedge \tilde{T} := \tilde{R} \wedge q \quad -(4)$$

These equations are simple but very abstract. Eqs. (1) and (2) are the first and second structure equations, eqs. (3) and (4) are the identities.

Standard Notation of Differential Geometry

$$T^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad -(5)$$

$$R^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad -(6)$$

$$d \wedge T^a = R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \quad -(7)$$

$$:= j^a$$

$$d \wedge \tilde{T}^a = \tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b \quad -(8)$$

$$:= \tilde{j}^a$$

Tensorial Notation

The equations of relevance are:

2)

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(9)$$

and:

$$D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}^{a\ \mu\nu}_{\ \mu} \quad -(10)$$

$$D_\mu T^{a\mu\nu} := R^{a\ \mu\nu}_{\ \mu} \quad -(11)$$

<u>Vector Notation</u>

This is given in full detail in the ECE engineering model, and is now used routinely.

<u>Computer Code</u>

This has been extensively developed.

The above equations are standard <u>textbook</u> material, as is the <u>tetrad postulate</u>:

$$D_\mu q^a_\nu = 0, \quad -(12)$$

which may be rewritten as:

$$\Box q^a_\mu = R q^a_\mu, \quad -(13)$$

the ECE Lemma.

It is also well known that eq. (7) is equivalent to the fundamental equation of

3) Riemann geometry:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(14)$$

The Hodge dual is defined as:

$$[D_\mu, D_\nu] := \frac{1}{2} \|g\|^{1/2} \epsilon_{\mu\nu}{}^{\alpha\beta} [D_\alpha, D_\beta]_{HD} \quad -(15)$$

where $\|g\|^{1/2}$ is the square root of the modulus of the determinant of the metric. The antisymmetric tensor is defined as follows.

$$\epsilon_{\mu\nu}{}^{\rho\sigma} = g_{\mu\alpha} g_{\nu\beta} \epsilon^{\alpha\beta\rho\sigma} \quad -(16)$$

where:

$$g_{\mu\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(17)$$

and

$$\left.\begin{aligned} \epsilon^{0123} &= -\epsilon^{1230} = \epsilon^{2301} = -\epsilon^{3012} = 1 \\ \epsilon^{1023} &= -\epsilon^{2130} = \epsilon^{3201} = -\epsilon^{0312} = -1 \\ \epsilon^{1032} &= -\epsilon^{2103} = \epsilon^{3210} = -\epsilon^{0321} = 1 \\ \epsilon^{1302} &= -\epsilon^{2013} = \epsilon^{3120} = -\epsilon^{0231} = -1 \end{aligned}\right] \quad -(18)$$

etc.

So:

$$\left.\begin{aligned} \epsilon_{01}{}^{23} &= g_{0\alpha} g_{1\beta} \epsilon^{\alpha\beta 23} \\ &= g_{00} g_{11} \epsilon^{0123} = -1 \end{aligned}\right] \quad -(19)$$

etc.

4) So:

$$\left.\begin{aligned} \epsilon_{01}{}^{23} &= -\epsilon^{0123} = -1 \\ \epsilon_{02}{}^{31} &= -\epsilon^{0231} = -1 \\ \epsilon_{03}{}^{21} &= -\epsilon^{0321} = 1 \end{aligned}\right] \quad -(20)$$

The other indices of $\epsilon_{\mu\nu}{}^{\rho\sigma}$ are the same as $\epsilon^{\mu\nu\rho\sigma}$.

Now denote:

$$[D_\mu, D_\nu] := D_{\mu\nu} \quad -(21)$$

$$[D_\alpha, D_\beta]_{HD} := \tilde{D}_{\alpha\beta}. \quad -(22)$$

It is found that

$$D_{01} = \frac{1}{2}\|g\|^{1/2}\left(\epsilon_{01}{}^{23}\tilde{D}_{23} + \epsilon_{01}{}^{32}\tilde{D}_{32}\right) \quad -(23)$$

$$= -\|g\|^{1/2}\tilde{D}_{23} = \|g\|^{1/2}\tilde{D}_{32}.$$

etc.

Proceeding in this way:

$$\boxed{D_{\mu\nu} = \begin{bmatrix} 0 & D_{01} & D_{02} & D_{03} \\ D_{10} & 0 & D_{12} & D_{13} \\ D_{20} & D_{21} & 0 & D_{23} \\ D_{30} & D_{31} & D_{32} & 0 \end{bmatrix} = \|g\|^{1/2}\begin{bmatrix} 0 & \tilde{D}_{32} & \tilde{D}_{13} & \tilde{D}_{12} \\ \tilde{D}_{23} & 0 & \tilde{D}_{03} & \tilde{D}_{20} \\ \tilde{D}_{13} & \tilde{D}_{30} & 0 & \tilde{D}_{01} \\ \tilde{D}_{12} & \tilde{D}_{02} & \tilde{D}_{10} & 0 \end{bmatrix} \quad -(24)}$$

5)

with similar results for $R^\rho{}_{\sigma\mu\nu}$ and $T^\lambda{}_{\mu\nu}$.

So, for example:

$$D_{01} V^\rho = R^\rho{}_{\sigma 0 1} V^\sigma - T^\lambda{}_{01} D_\lambda V^\rho \quad -(25)$$

$$\Rightarrow \tilde{D}_{32} V^\rho = \tilde{R}^\rho{}_{\sigma 3 2} V^\sigma - \tilde{T}^\lambda{}_{32} D_\lambda V^\rho \quad -(26)$$

etc.

Eq. (26) is the same as eq. (25). In general:

$$D_{\mu\nu} V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(27)$$

$$\Rightarrow \tilde{D}_{\mu\nu} V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(28)$$

It is well known in textbook Cartan geometry that eq. (27) implies:

$$D \wedge T^a = R^a{}_b \wedge q^b \quad -(29)$$

so eq. (28) implies:

$$D \wedge \tilde{T}^a = \tilde{R}^a{}_b \wedge q^b \quad -(30)$$

Q.E.D.

The geometry of the field equations of ECE theory is defined by eqs. (29) and (30):

$$\boxed{\begin{aligned} d \wedge T^a &:= j^a \quad -(31) \\ d \wedge \tilde{T}^a &:= \tilde{j}^a \quad -(32) \end{aligned}}$$

6) where:

$$j^a = R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \quad -(33)$$

$$\tilde{j}^a = \tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b . \quad -(34)$$

Eq. (31) is the homogeneous field equation, and eq. (32) is the inhomogeneous field equation.

<u>Electrodynamics</u> (ECE)

$$d \wedge F^a := A^{(0)} j^a \quad -(35)$$

$$d \wedge \tilde{F}^a := A^{(0)} \tilde{j}^a \quad -(36)$$

$$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b \quad -(37).$$

with the basic postulate:

$$A^a = A^{(0)} q^a \quad -(38)$$

$$F^a = A^{(0)} T^a \quad -(39)$$

<u>Maxwell-Heaviside</u> (MH)

These are given in many textbooks as:

$$d \wedge F = 0 \quad -(40)$$

$$d \wedge \tilde{F} = \epsilon_0 \tilde{J} \quad -(41)$$

$$F = d \wedge A \quad -(42)$$

If it is assumed that:

$$j^a = R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b = 0 \quad -(43)$$

then the ECE equations reduce to:

7)

ECE	MH
$d \wedge F^a = 0$	$d \wedge F = 0$
$d \wedge \tilde{F}^a = \epsilon_0 \tilde{J}^a$	$d \wedge \tilde{F} = \epsilon_0 J$
$F^a = d \wedge A^a + \omega^a{}_b \wedge A^b$	$F = d \wedge A$

in standard differential form notation. Here:

$$\epsilon_0 \tilde{J}^a := A^{(0)} \tilde{j}^a \quad -(44)$$

The main difference is that ECE is general relativity, MH is special relativity

Vector Notation

ECE:

$$\nabla \cdot \underline{B}^a = 0 \quad -(45)$$

$$\nabla \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = \underline{0} \quad -(46)$$

$$\nabla \cdot \underline{E}^a = \rho^a / \epsilon_0 \quad -(47)$$

$$\nabla \times \underline{B}^a - \frac{1}{c^2} \frac{\partial \underline{E}^a}{\partial t} = \mu_0 \underline{J}^a \quad -(48)$$

MH: The same but without the polarization index a.

Philosophically ECE and MH are very different, mathematically very similar.

8) The Field Potential Equations — (49)

ECE:

$$\underline{E}^a = -\underline{\nabla}\phi^a - \frac{\partial \underline{A}^a}{\partial t} - c\omega^a_{0b}\underline{A}^b + cA^b_0\,\underline{\omega}^a_b \quad -(50)$$

$$\underline{B}^a = \underline{\nabla}\times\underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b$$

MH

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \quad -(51)$$

$$\underline{B} = \underline{\nabla}\times\underline{A} \quad -(52)$$

The spin connection enters into the ECE equations and makes a profound difference.

We have:

$$\omega^a_{\mu b} = \left(\omega^a_{0b},\ -\underline{\omega}^a{}_b\right) \quad -(53)$$

Eckardt, Lindstrom, Lichtenberg and myself have extensively developed these equations.

Gravitation

Same structure, with ansatz:

$$\underline{\Phi}^a = \Phi(0)\, q^a \quad -(54)$$

used by Eckardt and myself to give an entirely new cosmology.

1)

137(11): Some Standard Hodge Duals : Cross Checking

In Ryder, "Quantum Field Theory" there are available:

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{bmatrix}, \quad F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} \quad -(1)$$

Therefore:

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & \tilde{F}^{01} & \tilde{F}^{02} & \tilde{F}^{03} \\ \tilde{F}^{10} & 0 & \tilde{F}^{12} & \tilde{F}^{13} \\ \tilde{F}^{20} & \tilde{F}^{21} & 0 & \tilde{F}^{23} \\ \tilde{F}^{30} & \tilde{F}^{31} & \tilde{F}^{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & F^{23} & F^{31} & F^{12} \\ F^{32} & 0 & F^{30} & F^{02} \\ F^{13} & F^{03} & 0 & F^{10} \\ F^{21} & F^{20} & F^{01} & 0 \end{bmatrix} \quad -(2)$$

These results are obtained with:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu}{}_{\alpha\beta}F^{\alpha\beta} \quad -(3)$$

Here:

$$\begin{aligned} \epsilon^{0123} &= -\epsilon^{1230} = \epsilon^{2301} = -\epsilon^{3012} = 1 \\ \epsilon^{1023} &= -\epsilon^{2130} = \epsilon^{3201} = -\epsilon^{0312} = -1 \\ \epsilon^{1032} &= -\epsilon^{2103} = \epsilon^{3210} = -\epsilon^{0321} = 1 \\ \epsilon^{1302} &= -\epsilon^{2013} = \epsilon^{3120} = -\epsilon^{0231} = -1 \end{aligned} \quad -(4).$$

Here

$$\epsilon^{\mu\nu}{}_{\alpha\beta} = g_{\alpha\rho}\, g_{\beta\sigma}\, \epsilon^{\mu\nu\rho\sigma} \quad -(5)$$

$$g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad -(6)$$

Thus:

$$\epsilon^{01}{}_{23} = \epsilon^{0123}, \quad \tilde{F}^{01} = F^{23} \quad -(7)$$

$$\epsilon^{02}{}_{31} = \epsilon^{0231}, \quad \tilde{F}^{02} = F^{31} \quad -(8)$$

$$\epsilon^{03}{}_{12} = \epsilon^{0312}, \quad \tilde{F}^{03} = F^{12} \quad -(9)$$

2) $\epsilon^{12}{}_{03} = -\epsilon^{1203}, \quad \tilde{F}^{12} = F^{30}$ — (10)

$\epsilon^{13}{}_{20} = -\epsilon^{1320}, \quad \tilde{F}^{13} = F^{02}$ — (11)

$\epsilon^{23}{}_{01} = -\epsilon^{2301}, \quad \tilde{F}^{23} = F^{10}$. — (12)

Generalizing to any type of spacetime,

define: $$\tilde{D}^{\mu\nu} = [D^{\mu}, D^{\nu}]_{HD} \quad \text{— (13)}$$

$$D^{\mu\nu} = [D^{\mu}, D^{\nu}] \quad \text{— (14)}$$

and $$\tilde{D}^{\mu\nu} = \frac{\|g\|^{1/2}}{2}\epsilon^{\mu\nu}{}_{\rho\sigma} D^{\rho\sigma} \quad \text{— (15)}$$

So:

$$\tilde{D}^{\mu\nu} = \begin{bmatrix} 0 & \tilde{D}^{01} & \tilde{D}^{02} & \tilde{D}^{03} \\ \tilde{D}^{10} & 0 & \tilde{D}^{12} & \tilde{D}^{13} \\ \tilde{D}^{20} & \tilde{D}^{21} & 0 & \tilde{D}^{23} \\ \tilde{D}^{30} & \tilde{D}^{31} & \tilde{D}^{32} & 0 \end{bmatrix} = \|g\|^{1/2} \begin{bmatrix} 0 & D^{23} & D^{31} & D^{12} \\ D^{32} & 0 & D^{30} & D^{02} \\ D^{13} & D^{03} & 0 & D^{10} \\ D^{21} & D^{20} & D^{01} & 0 \end{bmatrix} \quad \text{— (16)}$$

The key point is that $\epsilon^{\mu\nu}{}_{\rho\sigma}$ in eq. (15) is the Minkowski unit tensor. In the general spacetime it is weighted by $\|g\|^{1/2}$. Apart from this weighting factor, the information contained in $\tilde{D}^{\mu\nu}$ is the same as that contained in $D^{\rho\sigma}$, but the individual entries of the matrices are rearranged. This is an illustration of the fact that the Hodge dual of a two-form in four dimensions is another two-form, containing the same overall information.

In the type of Maxwell Heaviside theory that is used by Ryder, the vacuum field equations

3) are the homogeneous:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad -(17)$$

and the inhomogeneous:

$$\partial_\mu F^{\mu\nu} = 0. \quad -(18)$$

In this type of theory:

$$[D_\mu, D_\nu]\phi = D_{\mu\nu}\phi = -ig\, F_{\mu\nu}\phi \quad -(19)$$

where g is a factor.

Thus:

$$\tilde{D}_{\mu\nu}\phi = -ig\,\tilde{F}_{\mu\nu}\phi \quad -(20)$$

is a simple example of the fact that the Hodge dual field tensor is generated by the Hodge dual commutator.

Eqs. (17) and (18) are Hodge dual invariant, because their structure is unchanged by replacing $F^{\mu\nu}$ by $\tilde{F}^{\mu\nu}$. In form notation eq. (17) is:

$$d \wedge F = 0 \quad -(19)$$

and eq. (18) is

$$d \wedge \tilde{F} = 0. \quad -(20)$$

This is because eq. (17) is the same as:

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0 \quad -(21)$$

which is eq. (19). Eq. (18) is the same as:

$$\partial_\mu \tilde{F}_{\nu\rho} + \partial_\rho \tilde{F}_{\mu\nu} + \partial_\nu \tilde{F}_{\rho\mu} = 0. \quad -(22)$$

which is eq. (20).

4) In form notation, eq. (19) is the Cartan exterior derivative of the two-form F. This translates to the tensor notation (21).

In vector notation, eq. (17) is:

$$\left.\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= \underline{0} \end{aligned}\right\} \quad -(23)$$

using eq. (1). Eq. (18) is:

$$\left.\begin{aligned} \underline{\nabla} \cdot \underline{E} &= 0 \\ \underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} &= \underline{0} \end{aligned}\right\} \quad -(24)$$

Eq. (24) is Hodge dual invariant with eq. (23). All these results are generated by the commutator and the Hodge dual of the commutator. As can be seen in eq. (16) these two metrics contain the same overall information, but coded in a different way. Eq. (24) can be obtained from eq. (23) with:

$$\underline{E} \rightarrow ic\underline{B} \quad -(25)$$

1)

137(12): Lowering Indices in Two-Forms, Standard Hodge Duals.

Minkowski Spacetime

Indices are lowered with the simple Minkowski metric:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(1)$$

so:
$$F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} \quad -(2)$$

For example:
$$F_{01} = g_{0\alpha} g_{1\beta} F^{\alpha\beta} \quad -(3)$$
$$= g_{00} g_{11} F^{01}$$
$$= -F^{01}$$

Therefore:
$$\left.\begin{aligned} F_{01} &= -F^{01}, & F_{12} &= F^{12}, \\ F_{02} &= -F^{02}, & F_{13} &= F^{13}, \\ F_{03} &= -F^{03}, & F_{23} &= F^{23}. \end{aligned}\right] \quad -(4)$$

The electromagnetic field tensor is: $-(5)$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E_X & -E_Y & -E_Z \\ E_X & 0 & -cB_Z & cB_Y \\ E_Y & cB_Z & 0 & -cB_X \\ E_Z & -cB_Y & cB_X & 0 \end{bmatrix}$$

Thus:
$$F_{01} = -F^{01} = E_X = -E_1$$
$$F_{02} = -F^{02} = E_Y = -E_2$$
$$F_{03} = -F^{03} = E_Z = -E_3$$

2)

$$F_{12} = F^{12} = -cB_Z = cB_3$$

$$F_{13} = F^{13} = cB_Y = -cB_2$$

$$F_{23} = F^{23} = -cB_X = cB_1 \qquad -(6)$$

and:

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{bmatrix} \qquad -(7)$$

The Hodge dual of this tensor in Minkowski spacetime is:

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu}{}^{\alpha\beta} F_{\alpha\beta} \qquad -(8)$$

Adopt the convention:

$$\epsilon^{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}, \qquad -(9)$$

Then:

$$\epsilon_{0123} = -\epsilon_{1230} = \epsilon_{2301} = -\epsilon_{3012} = 1$$

$$\epsilon_{1023} = -\epsilon_{2130} = \epsilon_{3201} = -\epsilon_{0312} = -1$$

$$\epsilon_{1032} = -\epsilon_{2103} = \epsilon_{3210} = -\epsilon_{0321} = 1$$

$$\epsilon_{1302} = -\epsilon_{2013} = \epsilon_{3120} = -\epsilon_{0231} = -1 \qquad -(10)$$

and

$$\epsilon_{\mu\nu}{}^{\alpha\beta} = g^{\alpha\rho} g^{\beta\sigma} \epsilon_{\mu\nu\rho\sigma} \qquad -(11)$$

It is found that:

3) It is found that:

$$
\begin{aligned}
\epsilon_{01}{}^{23} &= \epsilon_{0123}, & \tilde{F}_{01} &= F_{23}, \\
\epsilon_{02}{}^{31} &= \epsilon_{0231}, & \tilde{F}_{02} &= F_{31}, \\
\epsilon_{03}{}^{12} &= \epsilon_{0312}, & \tilde{F}_{03} &= F_{12}, \\
\epsilon_{12}{}^{03} &= -\epsilon_{1203}, & \tilde{F}_{12} &= F_{30}, \\
\epsilon_{13}{}^{20} &= \epsilon_{1320}, & \tilde{F}_{13} &= F_{02}, \\
\epsilon_{23}{}^{01} &= -\epsilon_{2301}, & \tilde{F}_{23} &= F_{10}.
\end{aligned} \quad -(12)
$$

So:

$$
\tilde{F}_{\mu\nu} = \begin{bmatrix} 0 & cB_1 & cB_2 & cB_3 \\ -cB_1 & 0 & E_3 & -E_2 \\ -cB_2 & -E_3 & 0 & E_1 \\ -cB_3 & E_2 & -E_1 & 0 \end{bmatrix} \quad -(13)
$$

Summary for Minkowski Spacetime

$$
F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix}, \quad F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{bmatrix},
$$

$$
\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{bmatrix}, \quad \tilde{F}_{\mu\nu} = \begin{bmatrix} 0 & cB_1 & cB_2 & cB_3 \\ -cB_1 & 0 & E_3 & -E_2 \\ -cB_2 & -E_3 & 0 & E_1 \\ -cB_3 & E_2 & -E_1 & 0 \end{bmatrix}.
$$

$-(14)$

4)

$$\tilde{F}^{01} = F^{23} \quad , \quad \tilde{F}_{01} = F_{23}$$
$$\tilde{F}^{02} = F^{31} \qquad \tilde{F}_{02} = F_{31}$$
$$\tilde{F}^{03} = F^{12} \qquad \tilde{F}_{03} = F_{12}$$
$$\tilde{F}^{12} = F^{30} \qquad \tilde{F}_{12} = F_{30}$$
$$\tilde{F}^{13} = F^{02} \qquad \tilde{F}_{13} = F_{02}$$
$$\tilde{F}^{23} = F^{10} \qquad \tilde{F}_{23} = F_{10} \qquad -(15)$$

General Four Dimensional Spacetime

Indices are raised and lowered by the Minkowski metric, but this is no longer the Minkowski metric, (1) in general. The Hodge dual transformations are:

$$\tilde{D}_{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\mu\nu}{}^{\alpha\beta} D_{\alpha\beta} \qquad -(16)$$

$$\tilde{D}^{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu}{}_{\alpha\beta} D^{\alpha\beta} \qquad -(17)$$

where $\|g\|^{1/2}$ is the square root of the determinant of the metric. In eqs. (16) and (17) it is important to note that $\epsilon_{\mu\nu}{}^{\alpha\beta}$ and $\epsilon^{\mu\nu}{}_{\alpha\beta}$ are still defined in Minkowski spacetime.

The most important object is the commutator operator:

$$D_{\mu\nu} := [D_\mu, D_\nu] \qquad -(18)$$

Its Hodge dual operator is:

5)

$$\tilde{D}_{\mu\nu} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu}{}^{\alpha\beta}D_{\alpha\beta} \quad -(19)$$

Thus:

$$\tilde{D}_{01} = \|g\|^{1/2} D_{23}$$
$$\tilde{D}_{02} = \|g\|^{1/2} D_{31}$$
$$\tilde{D}_{03} = \|g\|^{1/2} D_{12}$$
$$\tilde{D}_{12} = \|g\|^{1/2} D_{30}$$
$$\tilde{D}_{13} = \|g\|^{1/2} D_{02}$$
$$\tilde{D}_{23} = \|g\|^{1/2} D_{10}. \quad -(20)$$

The commutator generates the basic curvature and torsion tensors in any spacetime:

$$D_{\mu\nu} V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(21)$$

where:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \quad -(22)$$

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad -(23)$$

These two tensors (22) and (23) are related by the Cartan identity. In shorthand:

$$D \wedge T := R \wedge q \quad -(24)$$

6) The identity is written out in tensor format in paper 102, eqns. (9.12) to (9.20). It is a rigorously correct identity which states that the cyclic sum of three curvature tensors (lhs of eq. (22)) is identically equal to the same cyclic sum of the definitions of the same three tensors (rhs of eq. (22)).

If we take any pair of indices μ and ν for which $D_{\mu\nu}$ is non-zero, then

$$\mu \neq \nu \quad -(25)$$

For example:

$$D_{23} V^\rho = R^\rho{}_{\sigma 23} V^\sigma - T^\lambda{}_{23} D_\lambda V^\rho \quad -(26)$$

The Hodge duals of $R^\rho{}_{\sigma\mu\nu}$ and $T^\lambda{}_{\mu\nu}$ are defined in the same way as the Hodge dual of the commutator, so:

$$\tilde{R}^\rho{}_{\sigma 01} = \|g\|^{1/2} R^\rho{}_{\sigma 23} \quad -(27)$$

etc.

and

$$\tilde{T}^\lambda{}_{01} = \|g\|^{1/2} T^\lambda{}_{23} \quad -(28)$$

etc. Thus eq. (26) is the same as:

$$\tilde{D}_{01} V^\rho = \tilde{R}^\rho{}_{\sigma 01} V^\sigma - \tilde{T}^\lambda{}_{01} D_\lambda V^\rho \quad -(29)$$

and the $\|g\|^{1/2}$ factor cancels out.

Proceeding in this way it is found that:

7)

$$\left(\tilde{D}_{01} V^\rho = \tilde{R}^\rho{}_{\sigma 01} V^\sigma - \tilde{T}^\lambda_{01} D_\lambda V^\rho\right) \leftrightarrow \left(D_{23} V^\rho = R^\rho{}_{\sigma 23} V^\sigma - T^\lambda_{23} D_\lambda V^\rho\right)$$

$$\left(\tilde{D}_{02} V^\rho = \tilde{R}^\rho{}_{\sigma 02} V^\sigma - \tilde{T}^\lambda_{02} D_\lambda V^\rho\right) \leftrightarrow \left(D_{31} V^\rho = R^\rho{}_{\sigma 31} V^\sigma - T^\lambda_{31} D_\lambda V^\rho\right)$$

$$\left(\tilde{D}_{03} V^\rho = \tilde{R}^\rho{}_{\sigma 03} V^\sigma - \tilde{T}^\lambda_{03} D_\lambda V^\rho\right) \leftrightarrow \left(D_{12} V^\rho = R^\rho{}_{\sigma 12} V^\sigma - T^\lambda_{12} D_\lambda V^\rho\right)$$

$$\left(\tilde{D}_{12} V^\rho = \tilde{R}^\rho{}_{\sigma 12} V^\sigma - \tilde{T}^\lambda_{12} D_\lambda V^\rho\right) \leftrightarrow \left(D_{30} V^\rho = R^\rho{}_{\sigma 30} V^\sigma - T^\lambda_{30} D_\lambda V^\rho\right)$$

$$\left(\tilde{D}_{13} V^\rho = \tilde{R}^\rho{}_{\sigma 13} V^\sigma - \tilde{T}^\lambda_{13} D_\lambda V^\rho\right) \leftrightarrow \left(D_{02} V^\rho = R^\rho{}_{\sigma 02} V^\sigma - T^\lambda_{02} D_\lambda V^\rho\right)$$

$$\left(\tilde{D}_{23} V^\rho = \tilde{R}^\rho{}_{\sigma 23} V^\sigma - \tilde{R}^\lambda_{23} D_\lambda V^\rho\right) \leftrightarrow \left(D_{10} V^\rho = R^\rho{}_{\sigma 10} V^\sigma - T^\lambda_{10} D_\lambda V^\rho\right) \quad -(30)$$

i.e:

$$\left(\tilde{D}_{\mu\nu} V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda_{\mu\nu} D_\lambda V^\rho\right) \leftrightarrow \quad -(31)$$

$$\left(D_{\mu\nu} V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} D_\lambda V^\rho\right),$$

$$\mu \neq \nu.$$

The symbol $\leftrightarrow$ means that for the pairs of indices in eq. (30), each equation linked by $\leftrightarrow$ is the same equation. This property is summarized in eq. (31): Hodge dual invariance.

It follows ~~that~~:

$$\left(D \wedge T^a_{\mu\nu} := R^a{}_{b\mu\nu} \wedge q^b_\sigma\right) \quad -(32)$$

$$\leftrightarrow \left(D \wedge \tilde{T}^a_{\mu\nu} := \tilde{R}^a{}_{b\mu\nu} \wedge q^b_\sigma\right)$$

8) For example:

$$D \wedge T^a{}_{(0)} := R^a{}_{b(0)} \wedge q^b{}_\sigma \quad -(33)$$

is the same equation as:

$$D \wedge \tilde{T}^a{}_{23} := \tilde{R}^a{}_{b23} \wedge q^b{}_\sigma \quad -(34)$$

and so on.

Thus if

$$\boxed{\begin{aligned} D \wedge T &:= R \wedge q \quad -(35) \\ D \wedge \tilde{T} &:= \tilde{R} \wedge q \quad -(36) \end{aligned}}$$

then

In tensor format eq. (35) is

$$D_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^a{}_\mu{}^{\mu\nu} \quad -(37)$$

and eq. (36) is:

$$D_\mu T^{a\mu\nu} := R^a{}_\mu{}^{\mu\nu} \quad -(38)$$

An example of eq. (38) is:

$$\boxed{D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}} \quad -(39)$$

which has been evaluated by computer algebra to show that the metrics of the Einstein field equation are incorrect due to neglect of torsion:

$$T^{\kappa\mu\nu} \stackrel{?}{=} 0 \quad -(40)$$

137(B): The Fundamental Importance of the Hodge Dual in Four Dimensions.

The fundamental importance of the Hodge dual in four dimensions is based on the fact that the Hodge dual of a two-form in four dimensions is another two-form. Therefore the Hodge dual of the commutator is another commutator. If the commutator is defined by:

$$D_{\mu\nu} = [D_\mu, D_\nu] \qquad -(1)$$

then its Hodge dual is defined by:

$$
\begin{aligned}
\tilde{D}_{01} &= \|g\|^{1/2} D_{23} \\
\tilde{D}_{02} &= \|g\|^{1/2} D_{31} \\
\tilde{D}_{03} &= \|g\|^{1/2} D_{12} \\
\tilde{D}_{12} &= \|g\|^{1/2} D_{30} \\
\tilde{D}_{13} &= \|g\|^{1/2} D_{02} \\
\tilde{D}_{23} &= \|g\|^{1/2} D_{10}
\end{aligned} \qquad -(2)
$$

It can be seen that apart from sign change in $\tilde{D}_{01}$, $\tilde{D}_{02}$ and $\tilde{D}_{03}$, and the weighting factor $\|g\|^{1/2}$, the left hand column has been "turned upside down". The elements have been rearranged, but the two-form remains a two-form. It is a Hodge invariant.

It is well known that the equation:

$$D_{\mu\nu} V^\sigma = R^\sigma{}_{\rho\mu\nu} V^\rho - T^\lambda{}_{\mu\nu} D_\lambda V^\sigma \qquad -(3)$$

2) is equivalent to the Cartan identity:

$$d \wedge T^a + \omega^a{}_b \wedge T^b := R^a{}_b \wedge q^b \quad -(4)$$

or:

$$\boxed{d \wedge T^a := j = R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b} \quad -(5)$$

This is the geometry of the homogeneous field equation, both of dynamics and electrodynamics.

The reason for this is that eq. (3) implies:

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad -(6)$$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \quad -(7)$$

and using the rule:

$$T^a{}_{\mu\nu} = q^a{}_\lambda T^\lambda{}_{\mu\nu} \quad -(8)$$

$$R^a{}_{b\mu\nu} = q^a{}_\rho q^\sigma{}_b R^\rho{}_{\sigma\mu\nu} \quad -(9)$$

the two tensors (6) and (7) are related by (4). This is standard, textbook Cartan geometry (e.g. Carroll, chapter 3). This is proven in all detail in the GCUFT series, using the tetrad postulate:

$$D_\mu q^a{}_\nu = 0. \quad -(10)$$

It may be proven that the Cartan identity is an exact identity (e.g. paper 103). Eq. (4) may

3) be written as:

$$R^{\rho}{}_{\sigma\mu\nu} + \dots := \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} + \dots \quad -(11)$$

where $\sigma\mu\nu$ are permuted cyclically, for example:

$$R^{\rho}{}_{123} + R^{\rho}{}_{312} + R^{\rho}{}_{231} := \dots \quad -(12)$$

It becomes obvious that eq. (4) consists of a sum of three $R^{\rho}{}_{\sigma\mu\nu}$ tensors on one side, and the same sum of the definition of the same tensors on the other.

So write down, for example:

$$\begin{aligned} R^{\rho}{}_{123} &= \partial_2 \Gamma^{\rho}_{31} - \partial_3 \Gamma^{\rho}_{21} + \Gamma^{\rho}_{2\lambda}\Gamma^{\lambda}_{31} - \Gamma^{\rho}_{3\lambda}\Gamma^{\lambda}_{21} \\ R^{\rho}{}_{312} &= \partial_1 \Gamma^{\rho}_{23} - \partial_2 \Gamma^{\rho}_{13} + \Gamma^{\rho}_{1\lambda}\Gamma^{\lambda}_{23} - \Gamma^{\rho}_{2\lambda}\Gamma^{\lambda}_{13} \\ R^{\rho}{}_{231} &= \partial_3 \Gamma^{\rho}_{12} - \partial_1 \Gamma^{\rho}_{32} + \Gamma^{\rho}_{3\lambda}\Gamma^{\lambda}_{12} - \Gamma^{\rho}_{1\lambda}\Gamma^{\lambda}_{32} \\ \hline \text{sum} &= \text{sum} \end{aligned} \quad -(13)$$

and add each equation to give the Cartan identity.

It becomes clear that the identity is precise and exact.

Its fundamental building block is eq. (7).

The tensor $R^{\rho}{}_{\sigma\mu\nu}$ is a two form so its Hodge dual in four dimensions is constructed in the same way as equation (2). For example:

$$\tilde{R}^{\rho}{}_{\sigma 23} = \|g\|^{1/2} R^{\rho}{}_{\sigma 10} \quad -(14)$$

So it follows that:

4) $\tilde{R}^{\rho}{}_{123} = \|g\|^{1/2} R^{\rho}{}_{110}$

$\tilde{R}_{312} = \|g\|^{1/2} R^{\rho}{}_{330}$

$\tilde{R}_{231} = \|g\|^{1/2} R^{\rho}{}_{220}$ — (15)

Therefore:

$$\tilde{R}^{\rho}{}_{123} + \tilde{R}^{\rho}{}_{312} + \tilde{R}^{\rho}{}_{231} = \|g\|^{1/2}\left(R^{\rho}{}_{110} + R^{\rho}{}_{330} + R^{\rho}{}_{220}\right)$$ — (16)

Taking Hodge duals on both sides of eq. (3) gives:

$$\tilde{D}_{\mu\nu} V^{\sigma} = \tilde{R}^{\sigma}{}_{\rho\mu\nu} V^{\rho} - \tilde{T}^{\lambda}{}_{\mu\nu} D_{\lambda} V^{\sigma},$$ — (17)

which is equivalent to:

$$d \wedge \tilde{T}^{a} + \omega^{a}{}_{b} \wedge \tilde{T}^{b} := \tilde{R}^{a}{}_{b} \wedge q^{b}.$$ — (18)

This means that, for example:

$$\tilde{R}^{a}{}_{123} + \tilde{R}^{a}{}_{312} + \tilde{R}^{a}{}_{231} := \partial_1 \tilde{T}^{a}{}_{23} + \partial_2 \tilde{T}^{a}{}_{31} + \partial_3 \tilde{T}^{a}{}_{12} + \omega^{a}{}_{1b} \tilde{T}^{b}{}_{23} + \omega^{a}{}_{2} \tilde{T}^{a}{}_{31} + \omega^{a}{}_{3} \tilde{T}^{a}{}_{12}$$ — (19)

which is:

$$R^{a}{}_{110} + R^{a}{}_{330} + R^{a}{}_{220} = \partial_1 T^{a}{}_{10} + \partial_3 T^{a}{}_{30} + \partial_2 T^{a}{}_{20} + \omega^{a}{}_{1b} T^{b}{}_{10} + \omega^{a}{}_{3b} T^{b}{}_{30} + \omega^{a}{}_{2b} T^{a}{}_{20}$$ — (20)

Eq. (18) gives the inhomogeneous field equation geometry

$$d \wedge \tilde{T}^a = \tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b$$
$$:= J^a \qquad -(21)$$

The two equations are therefore:

$$\boxed{\begin{aligned} d \wedge T^a &= j^a \qquad -(22) \\ d \wedge \tilde{T}^a &= J^a \qquad -(23) \end{aligned}}$$

The homogeneous current is:

$$j^a = R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b, \qquad -(24)$$

and the inhomogeneous current is:

$$J^a = \tilde{R}^a{}_b \wedge q^b - \omega^a{}_b \wedge \tilde{T}^b. \qquad -(25)$$

These two currents contain different physics. This is the fundamental importance of the Hodge dual.

In tensor notation, eq. (24) is

$$j^a_{\mu\nu\rho} + j^a_{\rho\mu\nu} + j^a_{\nu\rho\mu} \qquad -(26)$$
$$= R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu}$$
$$- \left(\omega^a_{\mu b} T^b_{\nu\rho} + \omega^a_{\rho b} T^b_{\mu\nu} + \omega^a_{\nu b} T^b_{\rho\mu}\right)$$

6) and is a cyclic sum of three forms. The Hodge dual of this cyclic sum is a one-form in four dimensions. This fact simplifies the structure of eqns. (22) and (23) to:

$$\partial_\mu \tilde{T}^{a\mu\nu} = j^{a\nu} \quad -(27)$$

$$\partial_\mu T^{a\mu\nu} = J^{a\nu}. \quad -(28)$$

Fundamentally, eqs. (27) and (28) follow from the fact that $R^\rho{}_{\sigma\mu\nu}$ has a Hodge dual, so we may write eq. (16). The structure of $\tilde{R}^\rho{}_{\sigma 23}$ is defined by eq. (14) through the structure of $R^\rho{}_{\sigma 10}$. Thus:

$$\tilde{R}^\rho{}_{\sigma 123} = \|g\|^{1/2} R^\rho{}_{110}$$

$$= \|g\|^{1/2} \left(\partial_1 \Gamma^\rho_{01} - \partial_0 \Gamma^\rho_{11} + \Gamma^\rho_{1\lambda} \Gamma^\lambda_{01} - \Gamma^\rho_{0\lambda} \Gamma^\lambda_{11} \right)$$

$$= \|g\|^{1/2} \left(\partial_1 \Gamma^\rho_{01} + \Gamma^\rho_{1\lambda} \Gamma^\lambda_{01} \right) \quad -(29)$$

because of the antisymmetry of the connection. Similarly:

$$\tilde{R}^\rho{}_{312} = \|g\|^{1/2} \left(\partial_3 \Gamma^\rho_{03} + \Gamma^\rho_{3\lambda} \Gamma^\lambda_{03} \right) \quad -(30)$$

$$\tilde{R}^\rho{}_{231} = \|g\|^{1/2} \left(\partial_2 \Gamma^\rho_{02} + \Gamma^\rho_{2\lambda} \Gamma^\lambda_{02} \right). \quad -(31)$$

By definition:

7)

$$\tilde{R}^{\rho}{}_{123} = \left(\partial_2 \Gamma^{\rho}_{31} - \partial_3 \Gamma^{\rho}_{21} + \Gamma^{\rho}_{2\lambda}\Gamma^{\lambda}_{31} - \Gamma^{\rho}_{3\lambda}\Gamma^{\lambda}_{21}\right)_{HD} \quad -(32)$$

and so on, therefore

$$\boxed{\begin{aligned}\left(\partial_2 \Gamma^{\rho}_{31} - \partial_3 \Gamma^{\rho}_{21} + \Gamma^{\rho}_{2\lambda}\Gamma^{\lambda}_{31} - \Gamma^{\rho}_{3\lambda}\Gamma^{\lambda}_{21}\right)_{HD} &= \|g\|^{1/2}\left(\partial_1 \Gamma^{\rho}_{01} + \Gamma^{\rho}_{1\lambda}\Gamma^{\lambda}_{01}\right) \\ &= \tilde{R}^{\rho}{}_{123}\end{aligned}} \quad -(33)$$

Similarly:

$$\tilde{T}^{\lambda}_{\mu\nu} = \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right)_{HD} \quad -(34)$$

so for example

$$\tilde{T}^{\lambda}_{01} = \left(\Gamma^{\lambda}_{01} - \Gamma^{\lambda}_{10}\right)_{HD} = \|g\|^{1/2}\left(\Gamma_{23} - \Gamma_{32}\right) \quad -(35)$$

From the antisymmetry of the commutator:

$$\tilde{\Gamma}^{\lambda}_{01} = \|g\|^{1/2}\,\Gamma_{23} \quad -(36)$$

$$\tilde{\Gamma}^{\lambda}_{10} = \|g\|^{1/2}\,\Gamma_{32} \quad -(37)$$

and

$$\left(\Gamma^{\lambda}_{01} - \Gamma^{\lambda}_{10}\right)_{HD} = \tilde{\Gamma}^{\lambda}_{01} - \tilde{\Gamma}^{\lambda}_{10} \quad -(38)$$

In general:

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2}\|g\|^{1/2}\,\epsilon_{\mu\nu}{}^{\alpha\beta}\,\Gamma^{\lambda}_{\alpha\beta} \quad -(39)$$

8) So:

$$\tilde{\Gamma}^{\lambda}_{01} = \|g\|^{1/2}\,\Gamma^{\lambda}_{23}$$
$$\tilde{\Gamma}^{\lambda}_{02} = \|g\|^{1/2}\,\Gamma^{\lambda}_{31}$$
$$\tilde{\Gamma}^{\lambda}_{03} = \|g\|^{1/2}\,\Gamma^{\lambda}_{12}$$
$$\tilde{\Gamma}^{\lambda}_{12} = \|g\|^{1/2}\,\Gamma^{\lambda}_{30} \quad -(40)$$
$$\tilde{\Gamma}^{\lambda}_{13} = \|g\|^{1/2}\,\Gamma^{\lambda}_{02}$$
$$\tilde{\Gamma}^{\lambda}_{23} = \|g\|^{1/2}\,\Gamma^{\lambda}_{10}$$

With these definitions, the cyclic sum of three eqs. (3) gives the identity (19), which states that the cyclic sum of three Hodge duals such as $\tilde{R}^{\rho}{}_{123}$ is identically equal to the same cyclic sum of the definitions of the same Hodge duals.

Finally the most succinct proof of (19) is as follows. Take as example of eq. (3):

$$D_{10}V^{\rho} = R^{\rho}{}_{110}V^{1} - T^{\lambda}{}_{10}D_{\lambda}V^{\rho} \quad -(41)$$

and use the Hodge dual rules of type (2) to obtain:

$$\tilde{D}_{23}V^{\rho} = \tilde{R}^{\rho}{}_{123}V^{1} - \tilde{T}^{\lambda}{}_{23}D_{\lambda}V^{\rho} \quad -(42)$$

It follows from eq. (42) that:

$$D_1\tilde{T}^{a}{}_{23} + D_3\tilde{T}^{a}{}_{12} + D_2\tilde{T}^{a}{}_{31} := \tilde{R}^{a}{}_{123} + \tilde{R}^{a}{}_{312} + \tilde{R}^{a}{}_{231} \quad -(43)$$

Q.E.D.

1) 137(14): A Simple Proof of the Incorrectness of the Einstein Field Equation

Define the commutator by:

$$D_{\mu\nu} := [D_\mu, D_\nu] \quad -(1)$$

$$D_{\mu\nu} = -D_{\nu\mu}$$

Then:

$$D_{\mu\nu} V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(2)$$

$$= -(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}) D_\lambda V^\rho + R^\rho{}_{\sigma\mu\nu} V^\sigma \quad -(3)$$

$$\boxed{D_{\mu\nu} V^\rho = -\Gamma^\lambda{}_{\mu\nu} D_\lambda V^\rho + \cdots}$$

If $\mu = \nu$ then $\Gamma^\lambda{}_{\mu\nu} = 0$, $\quad -(4)$

because $D_{\mu\nu} = 0$. $\quad -(5)$

Therefore:

$$\boxed{\Gamma^\lambda{}_{\mu\nu} = -\Gamma^\lambda{}_{\nu\mu}.} \quad -(6)$$

The Einstein equation uses the incorrect:

$$\Gamma^\lambda{}_{\mu\nu} \stackrel{?}{=} \Gamma^\lambda{}_{\nu\mu} \neq ? \; 0 \quad -(7)$$

The Einstein equation is therefore incorrect, QED.

The connection $\Gamma^\lambda{}_{\mu\nu}$ is antisymmetric in its lower two indices μ and ν. It is therefore possible to define the Hodge dual connection:

$$\tilde{\Gamma}^\lambda{}_{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\mu\nu}{}^{\alpha\beta} \Gamma^\lambda{}_{\alpha\beta} \quad -(8)$$

2) Thus:

$$\tilde{\Gamma}^{\lambda}_{01} = \|g\|^{1/2}\,\Gamma^{\lambda}_{23}$$
$$\tilde{\Gamma}^{\lambda}_{02} = \|g\|^{1/2}\,\Gamma^{\lambda}_{31}$$
$$\tilde{\Gamma}^{\lambda}_{03} = \|g\|^{1/2}\,\Gamma^{\lambda}_{12}$$
$$\tilde{\Gamma}^{\lambda}_{12} = \|g\|^{1/2}\,\Gamma^{\lambda}_{30}$$
$$\tilde{\Gamma}^{\lambda}_{13} = \|g\|^{1/2}\,\Gamma^{\lambda}_{02}$$
$$\tilde{\Gamma}^{\lambda}_{23} = \|g\|^{1/2}\,\Gamma^{\lambda}_{10} \qquad -(9)$$

The Hodge dual torsion is:

$$\boxed{\tilde{T}^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu} - \tilde{\Gamma}^{\lambda}_{\nu\mu}} \qquad -(10)$$

and transforms as a tensor under the general coordinate transformation. The Hodge dual transform applied to eqn. (2) produces:

$$\tilde{D}_{\mu\nu} V^{\rho} = \tilde{R}^{\rho}{}_{\sigma\mu\nu} V^{\sigma} - \tilde{T}^{\lambda}_{\mu\nu} D_{\lambda} V^{\rho} \qquad -(11)$$

where the structure of the $\tilde{T}^{\lambda}_{\mu\nu}$ tensor is given by eq. (10). This result is produced by using the covariant derivative:

$$\boxed{D_{\mu} V^{\rho} = \partial_{\mu} V^{\rho} + \tilde{\Gamma}^{\rho}_{\mu\lambda} V^{\lambda}} \qquad -(12)$$

in the operation:

3)

$$\tilde{D}_{\mu\nu} V^\rho := D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) \quad -(13)$$

It follows that:

$$\boxed{\tilde{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \tilde{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \tilde{\Gamma}^\rho_{\mu\sigma} + \tilde{\Gamma}^\rho_{\mu\lambda} \tilde{\Gamma}^\lambda_{\nu\sigma} - \tilde{\Gamma}^\rho_{\nu\lambda} \tilde{\Gamma}^\lambda_{\mu\sigma}} \quad -(14)$$

by directly working out the algebra in eq. (13) using eq. (12). This algebra gives eq. (10) self consistently. From eqs. (10) and (14) it follows that:

$$\boxed{D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b} \quad -(15)$$

which is

$$D_\mu \tilde{T}^a{}_{\nu\rho} + D_\rho \tilde{T}^a{}_{\mu\nu} + D_\nu \tilde{T}^a{}_{\rho\mu} := \tilde{R}^a{}_{\mu\nu\rho} + \tilde{R}^a{}_{\rho\mu\nu} + \tilde{R}^a{}_{\nu\rho\mu} \quad -(16)$$

i.e.

$$\boxed{D_\mu T^{a\mu\nu} := R^{a}{}_{\mu}{}^{\mu\nu}} \quad -(17)$$

A special case of eq (17) is:

$$\boxed{D_\mu T^{\kappa\mu\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad -(18)$$

4)

Evaluation by Computer Algebra.

Eq. (18) is amenable to evaluation by computer algebra. This has been carried out in papers 93 ff for many metrics that are solutions of the incorrect Einstein field equation. The latter assumes zero torsion because of the incorrect assumption (7). So the Einstein field equation gives the incorrect result:

$$\boxed{R^{\kappa}{}_{\mu}{}^{\mu\nu} \overset{?}{=} 0} \quad \text{— (19)}$$

The computer algebra gives, in general:

$$\boxed{R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq 0} \quad \text{— (20)}$$

proving beyond doubt that the metrics of the Einstein field equation are incorrect.

Some Details

$$R^{\kappa}{}_{\mu}{}^{\mu\nu} = R^{\kappa}{}_{0}{}^{0\nu} + R^{\kappa}{}_{1}{}^{1\nu} + R^{\kappa}{}_{2}{}^{2\nu} + R^{\kappa}{}_{3}{}^{3\nu} \quad \text{— (21)}$$

For $\nu = 0$

$$R^{\kappa}{}_{\mu}{}^{\mu 0} = R^{\kappa}{}_{1}{}^{10} + R^{\kappa}{}_{2}{}^{20} + R^{\kappa}{}_{3}{}^{30} \quad \text{— (22)}$$

This sum is evaluated from elements of the Riemann tensor as follows. For diagonal metrics:

$$R^{\kappa}{}_{1}{}^{10} = g^{11} g^{00} R^{\kappa}{}_{110} \quad \text{— (23)}$$

5) $$R^{\kappa}{}_{2}{}^{20} = g^{22} g^{00} R^{\kappa}{}_{220} \quad -(24)$$

$$R^{\kappa}{}_{3}{}^{30} = g^{33} g^{00} R^{\kappa}{}_{330} \quad -(25)$$

For any κ the inverse metric elements g^{00}, g^{11}, g^{22} and g^{33} are known from the Einstein field equation, and from these the Riemann tensor elements can be computed. This was first carried out in 2007, by code that was thoroughly checked in several ways. It is far too difficult by hand; the calculations but computer algebra can do quickly.

Off Diagonal Metrics

In this case eqs. (23) to (25) become:

$$R^{\kappa}{}_{1}{}^{10} = g^{1\alpha} g^{0\beta} R^{\kappa}{}_{1\alpha\beta} \quad -(26)$$

$$R^{\kappa}{}_{2}{}^{20} = g^{2\alpha} g^{0\beta} R^{\kappa}{}_{2\alpha\beta} \quad -(27)$$

$$R^{\kappa}{}_{3}{}^{30} = g^{3\alpha} g^{0\beta} R^{\kappa}{}_{3\alpha\beta} \quad -(28)$$

with summation over α and β.

1) 137(15): Detailed Mathematics of the Cartan Bianchi Identity.

In shorthand notation this is:

$$D \wedge T := R \wedge q \quad -(1)$$

which in standard notation is:

$$D \wedge T^a := R^a{}_b \wedge q^b . \quad -(2)$$

In tensor notation:

$$D_\mu T^a{}_{\nu\rho} + D_\rho T^a{}_{\mu\nu} + D_\nu T^a{}_{\rho\mu} := R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} \quad -(3)$$

i.e.:

$$\partial_\mu T^a{}_{\nu\rho} + \omega^a{}_{\mu b} T^b{}_{\nu\rho} + \dots := R^\lambda{}_{\mu\nu\rho}\, q^a{}_\lambda + \dots \quad -(4)$$

By definition:

$$T^a{}_{\mu\nu} = \left(\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\right) q^a{}_\lambda \quad -(5)$$

So eq. (4) is

$$\partial_\mu \left(\left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) q^a{}_\lambda \right) + \omega^a{}_{\mu b} \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) q^b{}_\lambda + \dots := R^\lambda{}_{\mu\nu\rho}\, q^a{}_\lambda + \dots \quad -(6)$$

Use the Leibniz rule to give:

$$\partial_\mu \left(\left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) q^a{}_\lambda \right) = q^a{}_\lambda\, \partial_\mu \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + \left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) \partial_\mu q^a{}_\lambda . \quad -(7)$$

So eq. (6) is:

$$\left(\partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\mu \Gamma^\lambda{}_{\rho\nu}\right) q^a{}_\lambda + \left(\partial_\mu q^a{}_\lambda + \omega^a{}_{\mu b} q^b{}_\lambda\right)\left(\Gamma^\lambda{}_{\nu\rho} - \Gamma^\lambda{}_{\rho\nu}\right) + \dots := R^\lambda{}_{\mu\nu\rho}\, q^a{}_\lambda \quad -(8)$$

2)

Now re-label summation indices to give:

$$\left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu}\right) q^a_\lambda + \left(\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma\right)\left(\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu}\right) + \dots := R^\lambda_{\mu\nu\rho}\, q^a_\lambda + \dots, \quad -(9)$$

and use the tetrad postulate:

$$\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma = \Gamma^\lambda_{\mu\sigma} q^a_\lambda, \quad -(10)$$

to obtain:

$$\left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu}\right) q^a_\lambda + \Gamma^\lambda_{\mu\sigma}\left(\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu}\right) q^a_\lambda + \dots := R^\lambda_{\mu\nu\rho}\, q^a_\lambda + \dots \quad -(11)$$

A solution of eq. (11) is:

$$\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\mu\sigma}\left(\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu}\right) + \dots = R^\lambda_{\mu\nu\rho} + \dots \quad -(12)$$

The curvature tensor in eq. (12) is defined by:

$$R^\lambda_{\mu\nu\rho} := \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} \quad -(13)$$

Writing out eq. (12) in full gives a cyclic sum of terms, and when this cyclic sum is re-arranged the meaning of the Cartan-Bianchi identity becomes obvious. The procedure is as follows:

3)

$$
\begin{aligned}
&: \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\mu\sigma}\left(\Gamma^\sigma_{\nu\rho} - \Gamma^\sigma_{\rho\nu}\right) \\
&+ \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\rho\sigma}\left(\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}\right) \\
&+ \partial_\nu \Gamma^\lambda_{\rho\mu} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\nu\sigma}\left(\Gamma^\sigma_{\rho\mu} - \Gamma^\sigma_{\mu\rho}\right) \qquad -(14) \\
&:= R^\lambda_{\mu\nu\rho} + R^\lambda_{\rho\mu\nu} + R^\lambda_{\nu\rho\mu} .
\end{aligned}
$$

Rearrange terms of the left hand side of eq. (14) to give:

$$
\begin{aligned}
&\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} \\
&+ \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\rho\nu} \\
&+ \partial_\nu \Gamma^\lambda_{\rho\mu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\rho\mu} - \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\nu\mu} \qquad -(15) \\
&:= R^\lambda_{\mu\nu\rho} + R^\lambda_{\rho\mu\nu} + R^\lambda_{\nu\rho\mu}
\end{aligned}
$$

where:

$$
\begin{aligned}
R^\lambda_{\mu\nu\rho} &= \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} , \\
R^\lambda_{\rho\mu\nu} &= \partial_\rho \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\rho\nu} + \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\rho\nu} , \\
R^\lambda_{\nu\rho\mu} &= \partial_\nu \Gamma^\lambda_{\rho\mu} - \partial_\rho \Gamma^\lambda_{\nu\mu} + \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\rho\mu} - \Gamma^\lambda_{\rho\sigma}\Gamma^\sigma_{\nu\mu} . \qquad -(16)
\end{aligned}
$$

It is seen that eq. (2) of differential geometry is simply the cyclic sum of the three

4) definitions in eq. (16). These definitions originate in:

$$[D_\mu, D_\nu] V^\lambda = R^\lambda{}_{\rho\mu\nu} V^\rho - T^\sigma{}_{\mu\nu} D_\sigma V^\lambda \qquad -(17)$$

<u>Errors</u> of the <u>Standard Model</u>

1) $$\Gamma^\lambda_{\mu\nu} =? \Gamma^\lambda_{\nu\mu} \neq ? 0. \qquad -(18)$$

2) $$T^\lambda_{\mu\nu} =? 0 \qquad -(19)$$

3) $$R^\lambda{}_{\mu\nu\rho} + R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\nu\rho\mu} =? 0. \qquad -(20)$$

Here, the connection is defined by:

$$D_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\lambda} V^\lambda \qquad -(21)$$

The error (19) means that:

$$R^\lambda{}_{\mu\nu\rho} =? 0. \qquad -(22)$$

4) $$\mu =? \nu \qquad -(23)$$

in eq. (17).

The correct identity is the exact identity (15), which is based on eq. (17). This exact identity may be elegantly written as the <u>Cartan Bianchi identity</u> (2).

1) 137(16): Detailed Mathematics of the Cartan Evans Identity

In shorthand notation this is:

$$\tilde{D} \wedge \tilde{T} := \tilde{R} \wedge q \quad -(1)$$

which in standard notation is:

$$\tilde{D} \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b \quad -(2)$$

Define the Hodge dual connection by

$$\tilde{D}_\mu V^\nu = \partial_\mu V^\nu + \tilde{\Gamma}^\nu_{\mu\lambda} V^\lambda \quad -(3)$$

Then:

$$[\tilde{D}_\mu, \tilde{D}_\nu] V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(4)$$

where

$$\tilde{T}^\lambda{}_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\nu\mu} \quad -(5)$$

$$\tilde{R}^\lambda{}_{\mu\nu\rho} = \partial_\mu \tilde{\Gamma}^\lambda_{\nu\rho} - \partial_\nu \tilde{\Gamma}^\lambda_{\mu\rho} + \tilde{\Gamma}^\lambda_{\mu\sigma} \tilde{\Gamma}^\sigma_{\nu\rho} - \tilde{\Gamma}^\lambda_{\nu\sigma} \tilde{\Gamma}^\sigma_{\mu\rho} \quad -(6)$$

As in note 137(15), these two tensors are related by eq. (2), which in tensor notation is:

$$\tilde{D}_\mu \tilde{T}^a_{\nu\rho} + \tilde{D}_\rho \tilde{T}^a_{\mu\nu} + \tilde{D}_\nu \tilde{T}^a_{\rho\mu} := \tilde{R}^a_{\mu\nu\rho} + \tilde{R}^a_{\rho\mu\nu} + \tilde{R}^a_{\nu\rho\mu} \quad -(7)$$

i.e.

$$\boxed{\tilde{D}_\mu T^{a\mu\nu} := R^a{}_\mu{}^{\mu\nu}} \quad -(8)$$

A special case of eq. (8) is:

$$\tilde{D}_\mu T^{\kappa\mu\nu} := R^\kappa{}_\mu{}^{\mu\nu} \quad -(9)$$

2) The proof of eq. (9) is exactly the same as the proof of the Cartan Bianchi identity given in note 137 (15). The Hodge dual connection is defined by

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2}\|g\|^{1/2}\,\epsilon_{\mu\nu}{}^{\alpha\beta}\,\Gamma^{\lambda}_{\alpha\beta}. \quad -(10)$$

The Hodge dual tetrad postulate is:

$$\tilde{D}_{\mu} q^{a}_{\nu} = \partial_{\mu} q^{a}_{\nu} + \omega^{a}_{\mu b} q^{b}_{\nu} - \tilde{\Gamma}^{\lambda}_{\mu\nu} q^{a}_{\lambda} = 0 \quad -(11)$$

Error in the Einstein Field Equation

This violates the geometry (9), producing:

$$T^{\kappa\mu\nu} = ?\,0,\quad R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq 0 \quad -(12)$$

1. INTRODUCTION

The subject of correlated or collective spontaneous emission by a system of a large number of atoms was first discussed by Dicke [1], who introduced the concept of [illegible] on the [illegible] field produced by the other atomic dipoles [illegible] with an enhanced spontaneous emission rate. The shortening of the atomic lifetime resulting from the interaction between N atoms would manifest as enhancement of the intensity of radiation up to N².

The earliest investigations into correlated spontaneous emission from multiatom systems were motivated by attempts to detect coherent effects in the interaction of light with resonant atomic systems [2–4]. Another intrinsic feature of correlated spontaneous emission is that the emitted field exhibits strong nonlinear and directional behavior [5]. Moreover, the interest in correlated spontaneous emission lies in its close connection with the quantum and classical as well as with the spontaneous and stimulated aspects of atomic emission [6].

The phenomenon of collective emission is, in general, characteristic of macroscopic systems with a large number of emitting atoms confined a region much smaller than the optical wavelength. However, to understand collective effects from a macroscopic system of atoms, it is necessary to have a microscopic formulation of the interaction between the atoms and the electromagnetic field. Therefore, some previous work has been devoted to study collective effects in the case of few atoms [7–10]. Although a system of two or three atoms is admittedly an elementary model, it offers some advantages over the multiatom problem. Because of its simplicity, one obtains detailed and almost exact dynamical solutions with a variety of initial conditions. Many of the results predicted for the system of two or three atoms are analogous to phenomena that one could expect in multiatom systems. Early treatments of two or three-atom systems assumed a constant interatomic separation during the radiation process. When averaged over all such possible interatomic separations the collective effects average out, which made them difficult to observe experimentally.

In the 1990s, advances in trapping and cooling of small number of ions and neutral atoms greatly renewed the interest in collective effects in the interaction

137(17): Relation to Previous Proofs of the Cartan Evans Identity

In previous proofs the connection was denoted by:

$$D_\mu V^\rho = \partial_\mu V^\rho + \Lambda^\rho_{\mu\lambda} V^\lambda. \quad -(1)$$

In the new proof of paper 137 it is demonstrated for the first time that:

$$\boxed{\Lambda^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu}{}^{\alpha\beta}\Gamma^\lambda_{\alpha\beta}} \quad -(2)$$

This connection is enabled through the realization that the connection is antisymmetric in μ and ν, so has a Hodge dual denoted $\tilde{\Gamma}^\lambda_{\mu\nu}$. In paper 137, the covariant derivative is now written in a fully consistent way as:

$$\tilde{D}_\mu V^\rho = \partial_\mu V^\rho + \tilde{\Gamma}^\rho_{\mu\lambda} V^\lambda. \quad -(3)$$

Eqs (1) and (3) are the same.

Below are given complete details of the proof of the Cartan Evans identity. The result is a self-checking, precise identity.

Proof: Start with the fundamental equation of the Riemannian manifold:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} D_\lambda V^\rho. \quad -(4)$$

Take the Hodge of the two forms on both sides of eq. (4). These Hodge duals are defined by:

2)

$$[D_\mu, D_\nu]_{HD} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu}{}^{\alpha\beta}[D_\alpha, D_\beta] \quad -(5)$$

$$\tilde{R}^\rho{}_{\sigma\mu\nu} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu}{}^{\alpha\beta}\cancel{R_{\mu\nu}{}^{\alpha\beta}}\, R^\rho{}_{\sigma\alpha\beta} \quad -(6)$$

$$\tilde{T}^\lambda{}_{\mu\nu} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\mu\nu}{}^{\alpha\beta}T^\lambda{}_{\alpha\beta} \quad -(7)$$

From eqs. (5) to (7) in eq. (4):

$$\boxed{[D_\alpha, D_\beta]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma\alpha\beta}V^\sigma - \tilde{T}^\lambda{}_{\alpha\beta}D_\lambda V^\rho \quad -(8)}$$

Relabel indices in eq. (8) to give:

$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu}V^\sigma - \tilde{T}^\lambda{}_{\mu\nu}D_\lambda V^\rho \quad -(9)$$

This is eq. (4) of note 137(16) in which the connection is defined as in eq. (2) of this note. Thus:

$$[D_\mu, D_\nu]_{HD} V^\rho = D_\mu(D_\nu V^\rho) - D_\nu(D_\mu V^\rho) \quad -(10)$$

where:

$$D_\mu V^\rho = \partial_\mu V^\rho + \Lambda^\rho{}_{\mu\lambda}V^\lambda \quad -(11)$$

$$D_\nu V^\rho = \partial_\nu V^\rho + \Lambda^\rho{}_{\nu\lambda}V^\lambda \quad -(12)$$

Now work out the algebra in eq. (10) following paper 99, to give:

3)

$$\tilde{T}^{\lambda}_{\mu\nu} = \Lambda^{\lambda}_{\mu\nu} - \Lambda^{\lambda}_{\nu\mu} \quad -(13)$$

$$\tilde{R}^{\lambda}_{\mu\nu\rho} = \partial_\mu \Lambda^{\lambda}_{\nu\rho} - \partial_\nu \Lambda^{\lambda}_{\mu\rho} + \Lambda^{\lambda}_{\mu\sigma}\Lambda^{\sigma}_{\nu\rho} - \Lambda^{\lambda}_{\nu\sigma}\Lambda^{\sigma}_{\mu\rho} \quad -(14)$$

These are the same as eqs. (5) and (6) of note 137(16).

Now prove the Cartan Evans identity as follows. The identity is

$$\boxed{d \wedge \tilde{T}^a + \omega^a{}_b \wedge \tilde{T}^b := \tilde{R}^a{}_b \wedge q^b} \quad -(15)$$

which in tensor notation, valid in the Riemannian manifold, is:

$$D_\mu \tilde{T}^a{}_{\nu\rho} + D_\rho \tilde{T}^a{}_{\mu\nu} + D_\nu \tilde{T}^a{}_{\rho\mu} := \tilde{R}^a{}_{\mu\nu\rho} + \tilde{R}^a{}_{\rho\mu\nu} + \tilde{R}^a{}_{\nu\rho\mu} \quad -(16)$$

Note carefully that eqs. (15) and (16) are the same, eq. (15) is written in standard textbook notation of differential geometry with the manifold indices μ and ν omitted because they are always the same on both sides of any equation of differential geometry in any base manifold, not only the Riemannian manifold. ECE theory is defined in the Riemannian manifold. It uses the Riemannian torsion and curvature.

4) Now proceed to prove eq. (16) is exactly the same way as the proof of eq. (3) of note 137(15), but using Λ instead of Γ. For the sake of completeness we write out the proof here. We have to prove that

$$\partial_\mu \tilde{T}^a_{\nu\rho} + \omega^a_{\mu b}\tilde{T}^b_{\nu\rho} + \ldots := \tilde{R}^\lambda_{\mu\nu\rho}\, q^a_\lambda + \ldots \qquad (17)$$

By definition:

$$\tilde{T}^a_{\mu\nu} = \left(\Lambda^\lambda_{\mu\nu} - \Lambda^\lambda_{\nu\mu}\right) q^a_\lambda \qquad (18)$$

so eq. (17) is:

$$\partial_\mu\left(\left(\Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu}\right) q^a_\lambda\right) + \omega^a_{\mu b}\left(\Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu}\right) q^b_\lambda + \ldots := \tilde{R}^\lambda_{\mu\nu\rho}\, q^a_\lambda + \ldots \qquad (19)$$

Use the Leibniz rule to give:

$$\partial_\mu\left(\left(\Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu}\right) q^a_\lambda\right) = q^a_\lambda\, \partial_\mu\left(\Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu}\right) + \left(\Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu}\right)\partial_\mu q^a_\lambda \qquad (20)$$

So eq. (19) is:

$$\left(\partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\mu \Lambda^\lambda_{\rho\nu}\right) q^a_\lambda + \left(\partial_\mu q^a_\lambda + \omega^a_{\mu b} q^b_\lambda\right)\left(\Lambda^\lambda_{\nu\rho} - \Lambda^\lambda_{\rho\nu}\right) + \ldots := \tilde{R}^\lambda_{\mu\nu\rho}\, q^a_\lambda \qquad (21)$$

+) Relabel summation indices to give:

$$(\partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\mu \Lambda^\lambda_{\rho\nu}) q^a_\lambda + (\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma)(\Lambda^\sigma_{\nu\rho} - \Lambda^\sigma_{\rho\nu}) + \dots := \tilde{R}^\lambda_{\mu\nu\rho}\, q^a_\lambda + \dots \qquad -(22)$$

Now use the tetrad postulate with Λ connection:

$$\partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma = \Lambda^\lambda_{\mu\sigma} q^a_\lambda \qquad (23)$$

This follows from eqs. (11) and (12).

Eq. (23) in eq. (22) gives

$$(\partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\mu \Lambda^\lambda_{\rho\nu}) q^a_\lambda + \Lambda^\lambda_{\mu\sigma}(\Lambda^\sigma_{\nu\rho} - \Lambda^\sigma_{\rho\nu}) q^a_\lambda + \dots := \tilde{R}^\lambda_{\mu\nu\rho}\, q^a_\lambda + \dots \qquad -(24)$$

A solution of eq. (24) is:

$$\begin{aligned}
&\partial_\mu \Lambda^\lambda_{\nu\rho} - \partial_\mu \Lambda^\lambda_{\rho\nu} + \Lambda^\lambda_{\mu\sigma}(\Lambda^\sigma_{\nu\rho} - \Lambda^\sigma_{\rho\nu}) \\
&+ \partial_\rho \Lambda^\lambda_{\mu\nu} - \partial_\rho \Lambda^\lambda_{\nu\mu} + \Lambda^\lambda_{\rho\sigma}(\Lambda^\sigma_{\mu\nu} - \Lambda^\sigma_{\nu\mu}) \\
&+ \partial_\nu \Lambda^\lambda_{\rho\mu} - \partial_\nu \Lambda^\lambda_{\mu\rho} + \Lambda^\lambda_{\nu\sigma}(\Lambda^\sigma_{\rho\mu} - \Lambda^\sigma_{\mu\rho}) \\
&:= \tilde{R}^\lambda_{\mu\nu\rho} + R^\lambda_{\rho\mu\nu} + R^\lambda_{\nu\rho\mu}
\end{aligned} \qquad -(25)$$

Rearrange terms on the left hand side of eq. (25) to give an exact, self-checking, identity:

5)

$$\tilde{R}^{\lambda}_{\mu\nu\rho} + \tilde{R}^{\lambda}_{\rho\mu\nu} + \tilde{R}^{\lambda}_{\nu\rho\mu} :=$$

$$\begin{aligned} &\partial_\mu \Lambda^{\lambda}_{\nu\rho} - \partial_\nu \Lambda^{\lambda}_{\mu\rho} + \Lambda^{\lambda}_{\mu\sigma}\Lambda^{\sigma}_{\nu\rho} - \Lambda^{\lambda}_{\nu\sigma}\Lambda^{\sigma}_{\mu\rho} \\ &+ \partial_\rho \Lambda^{\lambda}_{\mu\nu} - \partial_\mu \Lambda^{\lambda}_{\rho\nu} + \Lambda^{\lambda}_{\rho\sigma}\Lambda^{\sigma}_{\mu\nu} - \Lambda^{\lambda}_{\mu\sigma}\Lambda^{\sigma}_{\rho\nu} \\ &+ \partial_\nu \Lambda^{\lambda}_{\rho\mu} - \partial_\rho \Lambda^{\lambda}_{\nu\mu} + \Lambda^{\lambda}_{\nu\sigma}\Lambda^{\sigma}_{\rho\mu} - \Lambda^{\lambda}_{\rho\sigma}\Lambda^{\sigma}_{\nu\mu} \end{aligned} \quad -(26)$$

where:

$$\begin{aligned} \tilde{R}^{\lambda}_{\mu\nu\rho} &= \partial_\mu \Lambda^{\lambda}_{\nu\rho} - \partial_\nu \Lambda^{\lambda}_{\mu\rho} + \Lambda^{\lambda}_{\mu\sigma}\Lambda^{\sigma}_{\nu\rho} - \Lambda^{\lambda}_{\nu\sigma}\Lambda^{\sigma}_{\mu\rho} \\ \tilde{R}^{\lambda}_{\rho\mu\nu} &= \partial_\rho \Lambda^{\lambda}_{\mu\nu} - \partial_\mu \Lambda^{\lambda}_{\rho\nu} + \Lambda^{\lambda}_{\rho\sigma}\Lambda^{\sigma}_{\mu\nu} - \Lambda^{\lambda}_{\mu\sigma}\Lambda^{\sigma}_{\rho\nu} \\ \tilde{R}^{\lambda}_{\nu\rho\mu} &= \partial_\nu \Lambda^{\lambda}_{\rho\mu} - \partial_\rho \Lambda^{\lambda}_{\nu\mu} + \Lambda^{\lambda}_{\nu\sigma}\Lambda^{\sigma}_{\rho\mu} - \Lambda^{\lambda}_{\rho\sigma}\Lambda^{\sigma}_{\nu\mu} \end{aligned} \quad -(27)$$

quod erat demonstrandum (Q.E.D)

Special Case of Eq. (16)

Use:

$$\tilde{T}^{\kappa}_{\nu\rho} = q^{\kappa}_{a} T^{a}_{\nu\rho} \quad -(28)$$

$$\tilde{R}^{\kappa}_{\mu\nu\rho} = q^{\kappa}_{a} R^{a}_{\mu\nu\rho} \quad -(29)$$

and so on.

6) Therefore:

$$D_\mu \tilde{T}^a{}_{\nu\rho} = D_\mu \left(q^a{}_\kappa \tilde{T}^\kappa{}_{\nu\rho} \right) \quad (30)$$

$$= \left(D_\mu q^a{}_\kappa \right) \cancel{D_\mu} \tilde{T}^\kappa{}_{\nu\rho} + q^a{}_\kappa D_\mu \tilde{T}^\kappa{}_{\nu\rho} \quad (31)$$

using the Leibniz rule. Now use the tetrad postulate:

$$D_\mu q^a{}_\kappa = 0 \quad (32)$$

to find:

$$D_\mu \tilde{T}^a{}_{\nu\rho} = q^a{}_\kappa D_\mu \tilde{T}^\kappa{}_{\nu\rho} \quad (33)$$

Eq. (16) becomes:

$$\boxed{D_\mu \tilde{T}^\kappa{}_{\nu\rho} + D_\rho \tilde{T}^\kappa{}_{\mu\nu} + D_\nu \tilde{T}^\kappa{}_{\rho\mu} = \tilde{R}^\kappa{}_{\mu\nu\rho} + \tilde{R}^\kappa{}_{\rho\mu\nu} + \tilde{R}^\kappa{}_{\nu\rho\mu}} \quad (34)$$

where the covariant derivatives are defined with the convention (2). The Hodge duals in eq. (34) are defined by eqs. (6) and (7). It follows that eq. (34) is the same as:

$$\boxed{D_\mu T^{\kappa\mu\nu} = R^\kappa{}_\mu{}^{\mu\nu}} \quad (35)$$

where the covariant derivative is defined by eq. (2).

7) The easiest way to see this is to take a particular example of eq. (34), for example:

$$D_1 \tilde{T}^{\kappa}{}_{23} + D_3 \tilde{T}^{\kappa}{}_{12} + D_2 \tilde{T}^{\kappa}{}_{31} := \tilde{R}^{\kappa}{}_{123} + \tilde{R}^{\kappa}{}_{312} + \tilde{R}^{\kappa}{}_{231} \quad -(36)$$

From eqs. (6) and (7), take Hodge duals term by term in eq. (36). The $\|g\|^{1/2}$ factor cancels to give:

$$D_1 T^{\kappa 01} + D_3 T^{\kappa 03} + D_2 T^{\kappa 02} := R^{\kappa}{}_{1}{}^{01} + R^{\kappa}{}_{3}{}^{03} + R^{\kappa}{}_{2}{}^{02} \quad -(37)$$

which is an example of eq. (35) Q.E.D.

Testing the Einstein Field Equation

Eq. (35) has been used to show that the Einstein field equation is incorrect because it produces:

$$T^{\kappa\mu 0} \to 0 \quad -(38)$$

$$R^{\kappa}{}_{\mu}{}^{\mu 0} \neq 0 \quad -(39)$$

Therefore twentieth century cosmology is incorrect. It has been replaced by ECE cosmology.

8) The Covariant Derivative in Eq. (34)

This is defined by:

$$D_\sigma \tilde{T}^{\kappa}_{\mu\nu} = \partial_\sigma \tilde{T}^{\kappa}_{\mu\nu} + \Lambda^{\kappa}_{\sigma\lambda} \tilde{T}^{\lambda}_{\mu\nu} - \Lambda^{\lambda}_{\sigma\mu} \tilde{T}^{\kappa}_{\lambda\nu} - \Lambda^{\lambda}_{\sigma\nu} \tilde{T}^{\kappa}_{\mu\lambda} \quad - (40)$$

(see papers 50, 100, 102 and 109 for example).

Eq. (40) uses the Λ connection defined by eq. (2) and by the rule for taking the covariant derivative of a rank three tensor in Riemann geometry. A new theorem of the Riemannian torsion was derived.

The Covariant Derivative in Eq. (15)

This is defined by the wedge derivative:

$$D \wedge \tilde{T}^a := d \wedge \tilde{T}^a + \omega^a{}_b \wedge \tilde{T}^b \quad - (41)$$

where the spin connection is defined in terms of the Λ connection by the tetrad postulate:

$$\boxed{D_\mu q^a_\sigma = \partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma - \Lambda^\lambda_{\mu\sigma} q^a_\lambda = 0} \quad - (42)$$