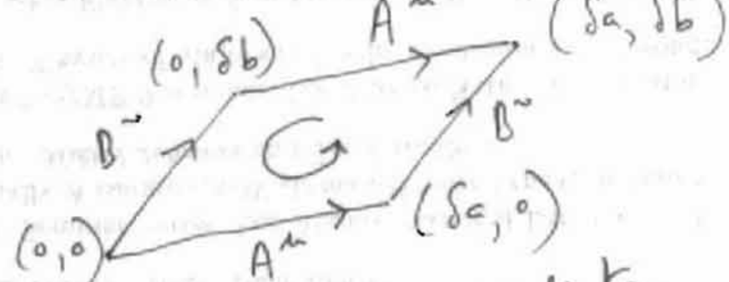
1) 134(1): Fundamental Meaning of the Commutator.

The commutator of covariant derivatives acts on any tensor in Riemann geometry. This method is equivalent to parallel transport of a vector $V^\sigma$ around a closed loop defined by two vectors $A^\nu$ and $B^\mu$ (Carroll p. 74 of 1997 notes.)

Fig. (1)




Parallel transport is independent of coordinates, so there is a tensor or tensors that define the way in which the vector changes when it comes back to its starting point. It must be a linear transformation of a vector and thus involve one upper and one lower index. It depends on the vectors A and B which define the loop, so there must be two additional indices to contract with $A^\nu$ and $B^\mu$. The tensor must be antisymmetric in these two indices because interchanging the vectors corresponds to traversing the loop in the opposite direction. The transformation vanishes if A and B are the same. Therefore:

$$\delta V^\rho = (\delta a)(\delta b) A^\nu B^\mu f^\rho{}_{\sigma\mu\nu} V^\sigma + \dots \quad -(1)$$

In the standard model,

$$f^\rho{}_{\sigma\mu\nu} := R^\rho{}_{\sigma\mu\nu}, \quad -(2)$$

and the torsion is ommitted from eq (1). Therefore it

2)

is assumed arbitrarily that only $R^{\rho}{}_{\sigma\mu\nu}$ can be anti-symmetric. The two additional loop indices $\mu$ and $\nu$ may however contract with any function that has indices $\mu$ and $\nu$, and which has the necessary $\rho$ and $\sigma$ indices of a linear transformation. The loop transformation vanishes if $\mu = \nu$, so all terms vanish if $\mu = \nu$. The above loop transformation is equivalent to:

$$[D_\mu, D_\nu] V^\rho = D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) \quad -(3)$$

It is seen that this operation is identically antisymmetric:

$$\boxed{[D_\mu, D_\nu] V^\rho := -[D_\nu, D_\mu] V^\rho} \quad -(4)$$

Proof

Eq. (4) is:

$$D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) := -\left(D_\nu (D_\mu V^\rho) - D_\mu (D_\nu V^\rho)\right)$$

i.e. $$D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) := D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) \quad -(5)$$

Q.E.D.

The only possible solutions of eq. (5) are:

$$\boxed{D_\mu (D_\nu V^\rho) = - D_\nu (D_\mu V^\rho)} \quad -(6)$$

and

$$\boxed{D_\nu (D_\mu V^\rho) = - D_\mu (D_\nu V^\rho)} \quad -(7)$$

3) Because of antisymmetry in $\mu$ and $\nu$. From fundamental definitions:

$$D_\mu(D_\nu V^\rho) = \partial_\mu(\partial_\nu V^\rho) + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma}\partial_\mu V^\sigma$$
$$- \Gamma^\lambda_{\mu\nu}\partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\sigma} V^\sigma$$
$$+ \Gamma^\rho_{\mu\sigma}\partial_\nu V^\sigma + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} V^\lambda \quad -(8)$$

When we consider $D_\nu(D_\mu V^\rho)$, every term on the right hand side of eq. (8) must change sign. The commutator of two covariant derivatives measures the difference between parallel transporting a tensor, or any term on the right hand side of eq. (8), first one way and then the other, versus the opposite ordering. In this process $V^\rho$ remains constant, and the connections remain the same, the only thing that changes is the sign of each term when:

$$\mu \to \nu, \quad \nu \to \mu. \quad -(9)$$

In the limit of Minkowski spacetime:

$$D_\mu(D_\nu V^\rho) \to \partial_\mu(\partial_\nu V^\rho), \quad -(10)$$

in which case:

$$\partial_\mu(\partial_\nu V^\rho) = -\partial_\nu(\partial_\mu V^\rho) \quad -(11)$$
$$\partial_\nu(\partial_\mu V^\rho) = -\partial_\mu(\partial_\nu V^\rho) \quad -(12)$$

4) However, in Minkowski spacetime:

$$\partial_\mu \left( \partial_\nu V^\rho \right) = \partial_\nu \left( \partial_\mu V^\rho \right) \quad -(13)$$

$$\partial_\nu \left( \partial_\mu V^\rho \right) = \partial_\mu \left( \partial_\nu V^\rho \right). \quad -(14)$$

Eqs (11) to (14) mean that:

$$\boxed{\partial_\mu \left( \partial_\nu V^\rho \right) = \partial_\nu \left( \partial_\mu V^\rho \right) = 0} \quad -(15)$$

For example:

$$\underline{r} = X \underline{i} + Y \underline{j} \quad -(16)$$

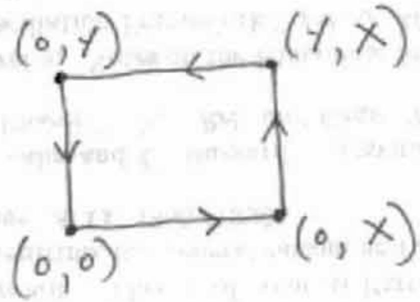



Fig (2)

In this case:

$$\underline{V} = \underline{r} \quad -(17)$$

and

$$\frac{\partial \underline{V}}{\partial X} = \underline{i}, \quad \frac{\partial \underline{V}}{\partial Y} = \underline{j} \quad -(18)$$

$$\frac{\partial}{\partial Y} \left( \frac{\partial \underline{V}}{\partial X} \right) = \frac{\partial}{\partial X} \left( \frac{\partial \underline{V}}{\partial Y} \right) = \underline{0} \quad -(19)$$

Eq. (19) is an example of eq. (15) in two dimensions. In Fig (2), if the loop is traversed clockwise or anticlockwise, the same initial

5) point $(0,0)$ is reached.

Eqs. (11) and (12) are examples of the fact that all the terms on the right hand side of eq. (8) are antisymmetric in $\mu$ and $\nu$. By definition, they all vanish when $\mu$ and $\nu$ are the same.

Working out the algebra:

$$[D_\mu, D_\nu] V^\rho = \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right) V^\sigma$$

$$- \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) D_\lambda V^\rho \quad -(20)$$

Therefore:

$$\boxed{\begin{aligned} \partial_\mu \Gamma^\rho_{\nu\sigma} &= -\partial_\nu \Gamma^\rho_{\mu\sigma} && -(21) \\ \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} &= -\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} && -(22) \\ \Gamma^\lambda_{\mu\nu} &= -\Gamma^\lambda_{\nu\mu} && -(23) \end{aligned}}$$

Reduction to Absurdity Proof of Eqs. (21) - (23)

Take for example eq. (23). Assume that

$$\Gamma^\lambda_{\mu\nu} \neq -\Gamma^\lambda_{\nu\mu}. \quad -(24)$$

In this case $\Gamma^\lambda_{\mu\nu}$ must have a symmetric

6) part:
$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}, \quad -(25)$$

because any asymmetric object with lower indices $\mu$ and $\nu$ is a sum of symmetric and antisymmetric parts. This object may be regarded as a matrix. The connection is not a tensor, because it does not transform as a tensor, but its lower two indices still form a matrix for each $\lambda$.

However, if
$$\mu = \nu \quad -(26)$$
then:
$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} = 0 \quad -(27)$$

so eq. (24) is not true, Q.E.D. Therefore:
$$\boxed{\Gamma^{\lambda}_{\mu\nu} = -\Gamma^{\lambda}_{\nu\mu}} \quad -(28)$$

Eqs. (21) and (22) are proven in the same way.

<u>Errors of the Standard Model</u>

1) <u>It</u> assumes <u>incorrectly</u> <u>that only the sum</u> of <u>the right hand side</u> <u>of the first ~~X~~ four terms</u> <u>of eq. (20)</u> is <u>antisymmetric</u>:
$$R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu} \quad -(29)$$

7) This result is incorrect by omission. Every term of this sum is antisymmetric.

2) The standard model assumes:

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} \quad - (30)$$

in which case, as just proven:

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu} = 0 \quad - (31)$$

and:

$$[D_\mu, D_\nu] V^\rho = 0. \quad - (32)$$

1) 134(2): Antisymmetries of the Torsion and Curvature Tensors

Torsion Tensor

$$T^{\lambda}{}_{\mu\nu} = -T^{\lambda}{}_{\nu\mu} \quad -(1)$$

$$\Gamma^{\lambda}{}_{\mu\nu} = -\Gamma^{\lambda}{}_{\nu\mu} \quad -(2)$$

with:

$$T^{\lambda}{}_{\mu\nu} \neq 0. \quad -(3)$$

The torsion tensor is identically non-zero, and the connection is identically antisymmetric.

Curvature Tensor

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma} \quad -(4)$$

$$R^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\nu\mu} \quad -(5)$$

$$\partial_\mu \Gamma^{\rho}{}_{\nu\sigma} = -\partial_\nu \Gamma^{\rho}{}_{\mu\sigma} \quad -(6)$$

$$\Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} = -\Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}. \quad -(7)$$

In vector format:

$$\underline{\nabla} \times \underline{R}^{\rho}{}_{\sigma 1} = \underline{0} \quad -(8)$$

$$\frac{\partial \underline{R}^{\rho}{}_{\sigma 2}}{\partial t} = \underline{0} \quad -(9)$$

2) where:

$$\underline{R}^{\rho}{}_{\sigma 1} = R^{\rho}{}_{\sigma 01}\,\underline{i} + R^{\rho}{}_{\sigma 02}\,\underline{j} + R^{\rho}{}_{\sigma 03}\,\underline{k} \quad -(10)$$

$$\underline{R}^{\rho}{}_{\sigma 2} = R^{\rho}{}_{\sigma 23}\,\underline{i} + R^{\rho}{}_{\sigma 31}\,\underline{j} + R^{\rho}{}_{\sigma 12}\,\underline{k} \quad -(11)$$

Therefore:

$$\boxed{\underline{\nabla} \times \underline{R}^{\rho}{}_{\sigma 1} + \frac{\partial \underline{R}^{\rho}{}_{\sigma 2}}{\partial t} = \underline{0}} \quad -(12)$$

Therefore $\underline{R}^{\rho}{}_{\sigma 1}$ is irrotational and $\underline{R}^{\rho}{}_{\sigma 2}$ is time independent.

If:

$$R^{\rho}{}_{\sigma\mu\nu} := A^{\rho}{}_{\sigma\mu\nu} + B^{\rho}{}_{\sigma\mu\nu} \quad -(13)$$

where

$$A^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} \quad -(14)$$

$$B^{\rho}{}_{\sigma\mu\nu} = \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \quad -(15)$$

then for each $\rho$ and $\sigma$:

3)

$$A_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu \quad -(16)$$

$$B_{\mu\nu} = \Gamma_{\mu\lambda} \Gamma^\lambda_\nu - \Gamma_{\nu\lambda} \Gamma^\lambda_\mu \quad -(17)$$

Orbital Antisymmetries

In this case:

$$A_{0i} = \partial_0 \Gamma_i - \partial_i \Gamma_0 \quad -(18)$$

$$B_{0i} = \Gamma_{0\lambda} \Gamma^\lambda_i - \Gamma_{i\lambda} \Gamma^\lambda_0 \quad -(19)$$

Define:

$$\Gamma_\mu = \left( \Gamma_0, -\underline{\Gamma} \right) \quad -(20)$$

$$\Gamma_{\mu\lambda} = \left( \Gamma_{0\lambda}, -\underline{\Gamma}_\lambda \right) \quad -(21)$$

$$\Gamma^\lambda_\mu = \left( \Gamma^\lambda_0, -\underline{\Gamma}^\lambda \right) \quad -(22)$$

$$\underline{A_1} = A_{01} \underline{i} + A_{02} \underline{j} + A_{03} \underline{k} \quad -(23)$$

$$\underline{A_2} = A_{23} \underline{i} + A_{31} \underline{j} + A_{12} \underline{k} \quad -(24)$$

Here:

$$\underline{A_1} = -\underline{\nabla} \Gamma_0 - \frac{\partial \underline{\Gamma}}{\partial t} \quad -(25)$$

$$\underline{A_2} = \underline{\nabla} \times \underline{\Gamma} \quad -(26)$$

The antisymmetry law means that:

4)

$$\underline{\nabla}\,\Gamma_0 = \frac{1}{c}\frac{\partial \underline{\Gamma}}{\partial t} \quad -(27)$$

Therefore:

$$\underline{\nabla} \times \underline{A}_1 = \underline{0} \quad -(28)$$

$$\frac{\partial \underline{A}_2}{\partial t} = \underline{0} \quad -(29)$$

Similarly:

$$\underline{\nabla} \times \underline{B}_1 = \underline{0} \quad -(30)$$

$$\frac{\partial \underline{B}_2}{\partial t} = \underline{0} \quad -(31)$$

where:

$$\underline{B}_1 = B_{01}\,\underline{i} + B_{02}\,\underline{j} + B_{03}\,\underline{k} \quad -(32)$$

$$\underline{B}_2 = B_{23}\,\underline{i} + B_{31}\,\underline{j} + B_{12}\,\underline{k} \quad -(33)$$

and:

$$\underline{B}_1 = -\Gamma_0 \times \underline{\Gamma}^{\lambda} + \underline{\Gamma}\times\Gamma_0^{\lambda} \quad -(34)$$

$$\underline{B}_2 = \underline{\Gamma}_\lambda \times \underline{\Gamma}^{\lambda} \quad -(35)$$

Restoring the $\rho$ and $\sigma$ indices to obtain eqs. (8) and (9)

134(3): Cartan Geometry with Constraints.

First Cartan Structure Equation

This is:

$$T^a = D \wedge q^a = d \wedge q^a + \omega^a{}_b \wedge q^b \quad -(1)$$

in standard form notation. In tensor notation:

$$T^a{}_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a{}_{\mu b} q^b_\nu - \omega^a{}_{\nu b} q^b_\mu \quad -(2)$$

$$= \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a{}_{\mu\nu} - \omega^a{}_{\nu\mu}$$

To reduce this to vector notation, the torsion is analysed in terms of orbital and spin components. The orbital component is:

$$T^a{}_{0i} = \partial_0 q^a_i - \partial_i q^a_0 + \omega^a{}_{0b} q^b_i - \omega^a{}_{ib} q^b_0 \quad -(3)$$

$$i = 1, 2, 3$$

and the spin component is:

$$T^a{}_{ij} = \partial_i q^a_j - \partial_j q^a_i + \omega^a{}_{ib} q^b_j - \omega^a{}_{jb} q^b_i \quad -(4)$$

$$i, j = 1, 2, 3$$

Now define:

$$\underline{T}^a(\text{orbital}) = T^a{}_{01}\,\underline{i} + T^a{}_{02}\,\underline{j} + T^a{}_{03}\,\underline{k} \quad -(5)$$

$$\underline{T}^a(\text{spin}) = T^a{}_{23}\,\underline{i} + T^a{}_{31}\,\underline{j} + T^a{}_{12}\,\underline{k} \quad -(6)$$

2) The tetrad and spin connection are four-vectors as follows:

$$q_\mu^a = \left( q_0^a, -\underline{q}^a \right) \quad -(7)$$

$$\omega_{\mu\ b}^a = \left( \omega_{0b}^a, -\underline{\omega}^a{}_b \right). \quad -(8)$$

The four-derivative is:

$$\partial_\mu = \left( \frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla} \right). \quad -(9)$$

Note the sign change between the vector parts of (7) and (8) and (9). So:

$$\underline{T}^a(\text{orbital}) = -\frac{1}{c}\frac{\partial \underline{q}^a}{\partial t} - \underline{\nabla} q_0^a - \omega_{0b}^a \underline{q}^b + \underline{\omega}^a{}_b q_0^b \quad -(10)$$

$$\underline{T}^a(\text{spin}) = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b \quad -(11)$$

Details

Eq. (10) is worked out from:

$$T_{01}^a = \partial_0 q_1^a - \partial_1 q_0^a + \omega_{0b}^a q_1^b - \omega_{1b}^a q_0^b$$

$$T_{02}^a = \partial_0 q_2^a - \partial_2 q_0^a + \omega_{0b}^a q_2^b - \omega_{2b}^a q_0^b$$

$$T_{03}^a = \partial_0 q_3^a - \partial_3 q_0^a + \omega_{0b}^a q_3^b - \omega_{3b}^a q_0^b$$

$$-(12)$$

3) From eq. (7):

$$q^a_\mu = \left(q^a_0, q^a_1, q^a_2, q^a_3\right) \quad -(13)$$
$$= \left(q^a_0, -q^a_X, -q^a_Y, -q^a_Z\right).$$

From eq. (8):

$$\omega^a_{\mu b} = \left(\omega^a_{0b}, \omega^a_{1b}, \omega^a_{2b}, \omega^a_{3b}\right) \quad -(14)$$
$$= \left(\omega^a_{0b}, -\omega^a_{Xb}, -\omega^a_{Yb}, -\omega^a_{Zb}\right).$$

We have:

$$\underline{q}^a = q^a_X \underline{i} + q^a_Y \underline{j} + q^a_Z \underline{k} \quad -(15)$$
$$\underline{\omega}^a{}_b = \omega^a_{Xb} \underline{i} + \omega^a_{Yb} \underline{j} + \omega^a_{Zb} \underline{k}. \quad -(16)$$

So for example:

$$\partial_0 q^a_1 = -\frac{1}{c}\frac{\partial}{\partial t} q^a_X \quad -(17)$$
$$-\partial_1 q^a_0 = -\frac{\partial}{\partial X} q^a_0 \quad -(18)$$

etc. This gives the vector equation (10).

Eq. (11) is worked out from:

$$T^a_{23} = \partial_2 q^a_3 - \partial_3 q^a_2 + \omega^a_{2b} q^b_3 - \omega^a_{3b} q^b_2$$
$$T^a_{31} = \partial_3 q^a_1 - \partial_1 q^a_3 + \omega^a_{3b} q^b_1 - \omega^a_{1b} q^b_3$$
$$T^a_{12} = \partial_1 q^a_2 - \partial_2 q^a_1 + \omega^a_{1b} q^b_2 - \omega^a_{2b} q^b_1 \quad -(19)$$

4) So for example:

$$T^a_{12} = -\frac{\partial q^a_Y}{\partial X} + \frac{\partial q^a_X}{\partial Y} + \omega^a_{Xb} q^b_Y - \omega^a_{Yb} q^b_X \quad -(20)$$

$$= T^a_3 = -T^a_Z$$

where we have used:

$$T^a_i = \frac{1}{2} \epsilon_{ijk} T^a_{jk} \quad -(21)$$

We have:

$$\underline{\nabla} \times \underline{q}^a = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial X & \partial/\partial Y & \partial/\partial Z \\ q^a_X & q^a_Y & q^a_Z \end{vmatrix} \quad -(22)$$

$$= \left( \frac{\partial q^a_Y}{\partial X} - \frac{\partial q^a_X}{\partial Y} \right) \underline{k} - \left( \frac{\partial q^a_Z}{\partial X} - \frac{\partial q^a_X}{\partial Z} \right) \underline{j} + \left( \frac{\partial q^a_Z}{\partial Y} - \frac{\partial q^a_Y}{\partial Z} \right) \underline{i} \quad -(23)$$

$$\underline{\omega}^a{}_b \times \underline{q}^b = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \omega^a_{Xb} & \omega^a_{Yb} & \omega^a_{Zb} \\ q^b_X & q^b_Y & q^b_Z \end{vmatrix} \quad -(24)$$

$$= \left( \omega^a_{Xb} q^b_Y - \omega^a_{Yb} q^b_X \right) \underline{k} - \left( \omega^a_{Xb} q^b_Z - \omega^a_{Zb} q^b_X \right) \underline{j} + \left( \omega^a_{Xb} q^b_Y - \omega^a_{Yb} q^b_X \right) \underline{i} \quad -(25)$$

5) So we obtain

$$\underline{T}^a(\text{Spin}) = \underline{\nabla} \times \underline{q}^a - \underline{\omega}^a{}_b \times \underline{q}^b \quad - (26)$$

<u>Antisymmetry</u>

In tensor notation:

$$\Gamma^a_{\mu\nu} = -\Gamma^a_{\nu\mu} \quad - (27)$$

$$= \partial_\mu q^a_\nu + \omega^a_{\mu\nu}$$

$$= \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu$$

<u>So the first</u> <u>Cartan structure equation is constrained by:</u>

$$\partial_\mu q^a_\nu + \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu + \omega^a_{\nu b} q^b_\mu = 0 \quad - (28)$$

<u>which</u> is <u>an entirely</u> <u>new</u> result.

Q.[1] 134(4): <u>Second Cartan Structure Equation</u>

! This is:

$$R^a{}_b = D \wedge \omega^a{}_b = d \wedge \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \quad -(1)$$

In tensor notation:

$$R^a{}_{b\mu\nu} = \partial_\mu \omega^a{}_{\nu b} - \partial_\nu \omega^a{}_{\mu b} + \omega^a{}_{\mu c}\omega^c{}_{\nu b} - \omega^a{}_{\nu c}\omega^c{}_{\mu b} \quad -(2)$$

In vector notation:

$$\underline{R}^a{}_b(\text{orbital}) = -\frac{1}{c}\frac{\partial \underline{\omega}^a{}_b}{\partial t} - \underline{\nabla}\omega^a{}_{0b} - \omega^a{}_{0c}\underline{\omega}^c{}_b + \underline{\omega}^a{}_c\omega^c{}_{0b} \quad -(3)$$

$$\underline{R}^a{}_b(\text{spin}) = \underline{\nabla} \times \underline{\omega}^a{}_b - \underline{\omega}^a{}_c \times \underline{\omega}^c{}_b \quad -(4)$$

where:

$$\underline{R}^a{}_b(\text{orbital}) = R^a{}_{b01}\underline{i} + R^a{}_{b02}\underline{j} + R^a{}_{b03}\underline{k} \quad -(5)$$

$$\underline{R}^a{}_b(\text{spin}) = R^a{}_{b23}\underline{i} + R^a{}_{b31}\underline{j} + R^a{}_{b12}\underline{k} \quad -(6)$$

In Riemann geometry:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} \quad -(7)$$

so:

$$\underline{R}^\rho{}_\sigma(\text{orbital}) = -\frac{1}{c}\frac{\partial \underline{\Gamma}^\rho{}_\sigma}{\partial t} - \underline{\nabla}\Gamma^\rho{}_{0\sigma} - \Gamma^\rho{}_{0\lambda}\underline{\Gamma}^\lambda{}_\sigma + \underline{\Gamma}^\rho{}_\lambda\Gamma^\lambda{}_{0\sigma} \quad -(8)$$

2)

$$\underline{R}^{\rho}{}_{\sigma}(\text{spin}) = \underline{\nabla} \times \underline{\Gamma}^{\rho}{}_{\sigma} - \underline{\Gamma}^{\rho}{}_{\lambda} \times \underline{\Gamma}^{\lambda}{}_{\sigma} \quad -(9)$$

where:

$$\underline{R}^{\rho}{}_{\sigma}(\text{orbital}) = R^{\rho}{}_{\sigma 01}\,\underline{i} + R^{\rho}{}_{\sigma 02}\,\underline{j} + R^{\rho}{}_{\sigma 03}\,\underline{k} \quad -(10)$$

$$\underline{R}^{\rho}{}_{\sigma}(\text{spin}) = R^{\rho}{}_{\sigma 23}\,\underline{i} + R^{\rho}{}_{\sigma 31}\,\underline{j} + R^{\rho}{}_{\sigma 12}\,\underline{k} \quad -(11)$$

The Riemann equations are constrained by:

$$\underline{\nabla} \times \underline{R}^{\rho}{}_{\sigma}(\text{orbital}) = \underline{0} \quad -(12)$$

$$\frac{\partial \underline{R}^{\rho}{}_{\sigma}(\text{spin})}{\partial t} = \underline{0} \quad -(13)$$

The spin and gamma connections are related by:

$$\left.\begin{aligned} \Gamma^{a}_{\mu\nu} &= q^{a}_{\lambda}\,\Gamma^{\lambda}_{\mu\nu} \\ &= \partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu\nu} \\ &= \partial_\mu q^{a}_{\nu} + \omega^{a}_{\mu b}\, q^{b}_{\nu} \end{aligned}\right\} \quad -(14)$$

The curvature form and tensor are related by:

3)

$$R^a{}_{b\mu\nu} = q^a_\rho q^\sigma_b R^\rho{}_{\sigma\mu\nu} \quad - (15)$$

Eq. (14) is the tetrad postulate. Similarly:

$$T^a{}_{\mu\nu} = q^a_\lambda T^\lambda{}_{\mu\nu} \quad - (16)$$

The tetrad postulate is:

$$D_\mu q^a_\nu = 0 \quad - (17)$$

and the tetrad is defined by:

$$V^a = q^a_\mu V^\mu \quad - (18)$$

The commutator shows that:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad - (18)$$

where

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad - (19)$$

So:

$$\Gamma^\lambda_{\mu\nu} = -\Gamma^\lambda_{\mu\nu} \quad - (20)$$

$$\Gamma^a_{\mu\nu} = -\Gamma^a_{\mu\nu} \quad - (21)$$

and from (14):

$$\partial_\mu q^a_\nu + \partial_\nu q^a_\mu + \omega^a{}_{\mu b} q^b_\nu + \omega^a{}_{\nu b} q^b_\mu = 0 \quad - (22)$$

1) Note 134(5) :

The commutator produces :

$$[D_\mu, D_\nu] V^\rho = -T^\lambda_{\mu\nu} D_\lambda V^\rho + R^\rho{}_{\sigma\mu\nu} V^\sigma \quad -(1)$$

where: $$D_\lambda V^\rho = \partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma} V^\sigma \quad -(2)$$

Carroll eq. (3.65), page 75,

Therefore as in 1997 notes :

$$[D_\mu, D_\nu] V^\rho = \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma + \Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma$$

$$- \Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma + \Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma$$

$$+ \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda - (\mu \leftrightarrow \nu) \quad -(3)$$

I have reproduced Carroll's equation exactly from his downloadable notes, available free online.

In all these terms, the antisymmetry is in the indices $\mu$ and $\nu$.

In the operation that defines the commutator, the indices $\rho$, $\sigma$ and $\lambda$ remain constant. In general the connection is asymmetric :

2)
$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}(S) + \Gamma^{\lambda}_{\mu\nu}(A) \quad -(4)$$

$$= \frac{1}{2}\left(\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\lambda}_{\nu\mu}\right) + \frac{1}{2}\left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right). \quad -(5)$$

However, if $\mu = \nu$, $\quad -(6)$

then $[D_\mu, D_\nu] = 0 \quad -(7)$

and $\Gamma^{\lambda}_{\mu\nu} = 0 \quad -(8)$

Therefore $\Gamma^{\lambda}_{\mu\nu} = -\Gamma^{\lambda}_{\nu\mu} \quad -(9)$.

and $\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right) \quad -(10)$

In considering terms such as:

$$\Gamma^{\lambda}_{\mu\nu}\left(\Gamma^{\rho}_{\lambda\sigma} V^{\sigma}\right) = -\Gamma^{\lambda}_{\nu\mu}\left(\Gamma^{\rho}_{\lambda\sigma} V^{\sigma}\right) \quad -(11)$$

and $\Gamma^{\rho}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} V^{\lambda} = -\Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda} V^{\lambda} \quad -(12)$

the antisymmetry is in $\mu$ and $\nu$, & is always the case. The antisymmetry is defined by

3) the indices $\mu$ and $\nu$ of the commutator that generates each term. In terms such as (11) and (12), the indices $\rho$, $\sigma$ and $\lambda$ do not change under the action of the commutator. Therefore the antisymmetries in eq. (3) are:

$$[D_\mu, D_\nu] V^\rho = -[D_\nu, D_\mu] V^\rho \quad -(13)$$

$$\partial_\mu \partial_\nu V^\rho = -\partial_\nu \partial_\mu V^\rho \quad -(14)$$

$$(\partial_\mu \Gamma^\rho_{\nu\sigma}) V^\sigma = -(\partial_\nu \Gamma^\rho_{\mu\sigma}) V^\sigma \quad -(15)$$

$$\Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma = -\Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma \quad -(16)$$

$$-\Gamma^\lambda_{\mu\nu} \partial_\lambda V^\rho = \Gamma^\lambda_{\nu\mu} \partial_\lambda V^\rho \quad -(17)$$

$$-\Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} V^\sigma = \Gamma^\lambda_{\nu\mu} \Gamma^\rho_{\lambda\sigma} V^\sigma \quad -(18)$$

$$\Gamma^\rho_{\mu\sigma} \partial_\nu V^\sigma = -\Gamma^\rho_{\nu\sigma} \partial_\mu V^\sigma \quad -(19)$$

$$\Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\nu\lambda} V^\lambda = -\Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\mu\lambda} V^\lambda \quad -(20)$$

In eq. (18), for example $\mu$ is the same as $\nu$, the term is zero by definition. So eq. (9) is the only possible interpretation. The $\Gamma^\rho_{\lambda\sigma} V^\sigma$ part of eq. (18) is interpreted as a constant under the action of $[D_\mu, D_\nu]$ on $V^\rho$. The complicated algebra of eq. (3) is reduced to eq. (1)

4) by straightforward gathering of terms, in which case

$$[D_\mu, D_\nu] V^\rho = -\left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}\right) D_\lambda V^\rho + \dots \quad -(21)$$

$$= -\Gamma^\lambda_{\mu\nu}\left(D_\lambda V^\rho\right) + \dots$$

Here is summation over repeated indices $\lambda$, and $\rho$ is constant. So:

$$\Gamma^\lambda_{\mu\nu} = -\Gamma^\lambda_{\nu\mu} \quad -(22)$$

because if

$$\mu = \nu \quad -(23)$$

then

$$[D_\mu, D_\nu] = 0 \quad -(24)$$

and <u>all</u> the terms (13) to (20) vanish.

<u>Conclusion</u>

The fundamental antisymmetric terms in Riemann geometry are (13) to (20). Each term is antisymmetric in $\mu$ and $\nu$, and in each term, the other indices are constant. So for example in a term such as (20), $\Gamma^\sigma_{\nu\lambda}$ is not antisymmetric in $\nu$ and $\lambda$ <u>and within that term</u>:

$$\Gamma^\sigma_{\nu\lambda} \neq -\Gamma^\sigma_{\lambda\nu} \quad -(25)$$

because the indices of the commutator are $\nu$ and $\mu$.

5) In eq. (21), the connection $\Gamma^\lambda_{\mu\nu}$ is one of the fundamental antisymmetries of Riemann geometry. There are eight fundamental antisymmetries, eqs. (13) to (20). In each case the antisymmetry must be in $\mu$ and $\nu$. In eq. (21) for example:

$$[D_\mu, D_\nu]V^\rho = -\Gamma^\lambda_{\mu\nu}\left(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma}V^\sigma\right) + \cdots \quad -(26)$$

and

$$[D_\nu, D_\mu]V^\rho = \Gamma^\lambda_{\nu\mu}\left(\partial_\lambda V^\rho + \Gamma^\rho_{\lambda\sigma}V^\sigma\right) + \cdots \quad -(27)$$

It is seen that

$$\Gamma^\rho_{\lambda\sigma}V^\sigma = \Gamma^\rho_{\lambda\sigma}V^\sigma \quad -(28)$$

and

$$\partial_\lambda V^\rho = \partial_\lambda V^\rho \quad -(29)$$

but

$$\Gamma^\lambda_{\mu\nu} = -\Gamma^\lambda_{\nu\mu} \quad -(30)$$

Therefore $\mu$ and $\nu$ are interchanged by the commutator but $\rho$, $\lambda$ and $\sigma$ are not. The torsion is defined by the $\mu$ and $\nu$ indices, not by the $\rho$, $\lambda$ and $\sigma$ indices.

1. 134(6) : Geometry of the Homogeneous Field Equation

This is based on the Cartan Bianchi identity:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(1)$$

i.e.

$$d \wedge T^a := j^a \quad -(2)$$

where

$$j^a = R^a{}_b \wedge q^b - \omega^a{}_b \wedge T^b \quad -(3)$$

In tensor notation, eq. (2) is:

$$\partial_\mu T^a{}_{\nu\rho} + \partial_\rho T^a{}_{\mu\nu} + \partial_\nu T^a{}_{\rho\mu} = j^a{}_{\mu\nu\rho} + j^a{}_{\rho\mu\nu} + j^a{}_{\nu\rho\mu} \quad -(4)$$

where

$$j^a{}_{\mu\nu\rho} = R^a{}_{\mu\nu\rho} - \omega^a{}_{\mu b} T^b{}_{\nu\rho} \quad -(5)$$

etc.

Let

$$\mu = 1, \nu = 2, \rho = 3 \quad -(6)$$

then:

$$\partial_1 T^a{}_{23} + \partial_3 T^a{}_{12} + \partial_2 T^a{}_{31} \quad -(7)$$
$$= j^a{}_{123} + j^a{}_{312} + j^a{}_{231}$$

Let:

$$\underline{T}^a(\text{Spin}) = T^a{}_{23}\, \underline{i} + T^a{}_{31}\, \underline{j} + T^a{}_{12}\, \underline{k} \quad -(8)$$

$$= T^a_1 \underline{i} + T^a_2 \underline{j} + T^a_3 \underline{k}$$

$$= - T^a_x \underline{i} - T^a_y \underline{j} - T^a_z \underline{k} \quad -(9)$$

where $T^a_1 = - T^a_x$ etc. $\quad -(10)$

Therefore eq. (7) is:

$$\boxed{\underline{\nabla} \cdot \underline{T}^a (\text{spin}) = j^a_0} \quad -(11)$$

where

$$j^a_0 = - \left( j^a_{123} + j^a_{312} + j^a_{213} \right). \quad -(12)$$

The ECE hypothesis is:

$$\underline{\nabla} \cdot \underline{B}^a = A^{(0)} \underline{T}^a (\text{spin}) \quad -(13).$$

so

$$\boxed{\underline{\nabla} \cdot \underline{B}^a = A^{(0)} j^a_0} \quad -(14)$$

In the experimental absence of a magnetic charge density:

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(15)$$

in which case:

$$R^a{}_b \wedge q^b = \omega^a{}_b \wedge T^b \quad -(16)$$

and

$$d \wedge T^a = 0 \quad -(17)$$

3) If there is no magnetic charge current density, the homogeneous ECE field equation is:

$$d \wedge F^a = 0 \quad -(17)$$

because eq. (2) is:

$$d \wedge T^a = 0 \quad -(18)$$

In vector notation, eq. (17) is:

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(19)$$

$$\underline{\nabla} \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = \underline{0} \quad -(20)$$

where:

$$a = (1), (2), (3). \quad -(21)$$

Now use in eq. (4):

$$\mu = 0, \nu = 1, \rho = 2 \quad -(22)$$

$$\mu = 0, \nu = 3, \rho = 1 \quad -(23)$$

$$\mu = 0, \nu = 2, \rho = 3 \quad -(24)$$

and define:

$$\underline{T}^a(\text{orbital}) = T^a_{01}\,\underline{i} + T^a_{02}\,\underline{j} + T^a_{03}\,\underline{k}$$

$$= T^a_X(\text{orb})\,\underline{i} + T^a_Y(\text{orb})\,\underline{j} + T^a_Z(\text{orb})\,\underline{k}$$

$$-(25)$$

4) Therefore:

$$\partial_0 T^a_{\ 12} + \partial_2 T^a_{\ 01} + \partial_1 T^a_{\ 20} = j^a_{\ 012} + j^a_{\ 201} + j^a_{\ 120}$$

$$\partial_0 T^a_{\ 31} + \partial_1 T^a_{\ 03} + \partial_3 T^a_{\ 10} = j^a_{\ 031} + j^a_{\ 103} + j^a_{\ 310}$$

$$\partial_0 T^a_{\ 23} + \partial_3 T^a_{\ 02} + \partial_2 T^a_{\ 30} = j^a_{\ 023} + j^a_{\ 302} + j^a_{\ 230} \quad -(26)$$

So:

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial T^a_{\ z}}{\partial t}(\text{spin}) + \frac{\partial T^a_{\ y}}{\partial x}(\text{orb}) - \frac{\partial T^a_{\ x}}{\partial y}(\text{orb}) &= j^a_{\ z} \\ \frac{1}{c}\frac{\partial T^a_{\ y}}{\partial t}(\text{spin}) + \frac{\partial T^a_{\ x}}{\partial z}(\text{orb}) - \frac{\partial T^a_{\ z}}{\partial x}(\text{orb}) &= j^a_{\ y} \\ \frac{1}{c}\frac{\partial T^a_{\ x}}{\partial t}(\text{spin}) + \frac{\partial T^a_{\ z}}{\partial y}(\text{orb}) - \frac{\partial T^a_{\ y}}{\partial z}(\text{orb}) &= j^a_{\ z} \end{aligned}\right\} \quad -(27)$$

i.e. $$\boxed{\frac{1}{c}\frac{\partial \underline{T}^a}{\partial t}(\text{spin}) + \underline{\nabla} \times \underline{T}^a(\text{orb}) = \underline{j}^a} \quad -(28)$$

where:

$$\underline{j}^a = j^a_{\ x}\,\underline{i} + j^a_{\ y}\,\underline{j} + j^a_{\ z}\,\underline{k} \quad -(29)$$

$$j^a_{\ x} = -\left(j^a_{\ 012} + j^a_{\ 201} + j^a_{\ 120}\right) \quad -(30)$$

etc.

The ECE hypothesis is:

5)

$$\boxed{\underline{E}^{a} = cA^{(0)}\,\underline{T}^{a}\,(\text{orbital})} \qquad -(31)$$

and so:

$$\boxed{\underline{\nabla} \times \underline{E}^{a} + \frac{\partial \underline{B}^{a}}{\partial t} = cA^{(0)}\,\underline{j}^{a}} \qquad -(32)$$

If there is no magnetic current density:

$$\boxed{\begin{gathered}\underline{\nabla} \times \underline{E}^{a} + \frac{\partial \underline{B}^{a}}{\partial t} = \underline{0} \qquad -(33)\\ \underline{\nabla} \cdot \underline{B}^{a} = 0 \qquad -(34)\end{gathered}}$$

The Torsion Tensor:

For each $a$:

$$T_{\mu\nu} = \begin{bmatrix} 0 & T_{01} & T_{02} & T_{03} \\ T_{10} & 0 & T_{12} & T_{13} \\ T_{20} & T_{21} & 0 & T_{23} \\ T_{30} & T_{31} & T_{32} & 0 \end{bmatrix} = -T_{\nu\mu} \qquad -(35)$$

$$= \begin{bmatrix} 0 & T_x(\text{orb}) & T_y(\text{orb}) & T_z(\text{orb}) \\ -T_x(\text{orb}) & 0 & -T_z(\text{sp}) & T_y(\text{sp}) \\ -T_y(\text{orb}) & T_z(\text{sp}) & 0 & -T_x(\text{sp}) \\ -T_z(\text{orb}) & -T_y(\text{sp}) & T_x(\text{sp}) & 0 \end{bmatrix}$$

1. Notes 134(7): Summary of the Structure of the Field Equations of ECE Electrodynamics.

The homogeneous field equations are:

$$\partial_\mu F^a_{\nu\rho} + \partial_\rho F^a_{\mu\nu} + \partial_\nu F^a_{\rho\mu} = 0 \quad -(1)$$

(assuming no magnetic charge / current density), and the inhomogeneous equations without polarization and magnetization are:

$$\partial_\mu F^{a\mu\nu} = J^{a\nu} / \epsilon_0 \quad -(2)$$

Here:

$$F^{a\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F^a_{\rho\sigma} \quad -(3)$$

where:

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(4)$$

Therefore:

$$F^{a01} = g^{0\rho} g^{1\sigma} F^a_{\rho\sigma} \quad -(5)$$

and so on. Since $g^{\mu\nu} = g_{\mu\nu}$ is diagonal:

$$\left.\begin{aligned} F^{a01} &= g^{00} g^{11} F^a_{01} = -F^a_{01} \\ F^{a02} &= g^{00} g^{22} F^a_{02} = -F^a_{02} \\ F^{a03} &= g^{00} g^{33} F^a_{03} = -F^a_{03} \\ F^{a12} &= g^{11} g^{22} F^a_{12} = F^a_{12} \\ F^{a13} &= g^{11} g^{33} F^a_{13} = F^a_{13} \\ F^{a23} &= g^{22} g^{33} F^a_{23} = F^a_{23} \end{aligned}\right] \quad -(6)$$

a) By hypothesis:

$$F^{a}_{\mu\nu} = A^{(0)} T^{a}_{\mu\nu} \quad -(7)$$

$$F^{a\mu\nu} = A^{(0)} T^{a\mu\nu} \quad -(8)$$

Therefore for each $a$:

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & E_X & E_Y & E_Z \\ -E_X & 0 & -cB_Z & cB_Y \\ -E_Y & cB_Z & 0 & -cB_X \\ -E_Z & -cB_Y & cB_X & 0 \end{bmatrix} \quad -(9)$$

So eq. (1) is:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{B}^{a} &= 0 \quad -(10) \\ \underline{\nabla} \times \underline{E}^{a} + \frac{\partial \underline{B}^{a}}{\partial t} &= 0 \quad -(11) \end{aligned}}$$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E_X & -E_Y & -E_Z \\ E_X & 0 & -cB_Z & cB_Y \\ E_Y & cB_Z & 0 & -cB_X \\ E_Z & -cB_Y & cB_X & 0 \end{bmatrix} \quad -(12)$$

Note that eqns. (9) and (12) are consistent with

3) eqns. (6). For each $a$:

$$\left.\begin{aligned}
F^{01} &= -F_{01} = -E_X \\
F^{02} &= -F_{02} = -E_Y \\
F^{03} &= -F_{03} = -E_Z \\
F^{12} &= F_{12} = -cB_Z \\
F^{13} &= F_{13} = cB_Y \\
F^{23} &= F_{23} = -cB_X
\end{aligned}\right] \quad -(13)$$

Therefore, as in GCUFT1, Appendix B, for:

$$\nu = 0 \quad -(14)$$

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = J^0 / \epsilon_0 \quad -(15)$$

i.e. $$\boxed{\nabla \cdot \underline{E}^a = \rho^a / \epsilon_0} \quad -(16)$$

For $$\nu = 1, 2, 3 \quad -(17)$$

$$\left.\begin{aligned}
\partial_0 F^{01} + \partial_2 F^{21} + \partial_3 F^{31} &= J^1 / \epsilon_0 \\
\partial_0 F^{02} + \partial_1 F^{12} + \partial_3 F^{32} &= J^2 / \epsilon_0 \\
\partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} &= J^3 / \epsilon_0
\end{aligned}\right] \quad -(18)$$

The four current density for each $a$ is:

$$J^\mu = (\rho, \underline{J}/c) \quad -(19)$$

4.

So:

$$-\partial_0 E^1 + c\left(\partial_2 B^3 - \partial_3 B^2\right) = J^1/\epsilon_0$$
$$-\partial_0 E^2 - c\left(\partial_1 B^3 - \partial_3 B^1\right) = J^2/\epsilon_0 \quad -(20)$$
$$-\partial_0 E^3 + c\left(\partial_1 B^2 - \partial_2 B^1\right) = J^3/\epsilon_0$$

e.g.

$$-\frac{1}{c}\frac{\partial E_Z}{\partial t} + c\left(\frac{\partial B_Y}{\partial X} - \frac{\partial B_X}{\partial Y}\right) = \frac{J_Z}{\epsilon_0} \quad -(21)$$

where:

$$\epsilon_0 \mu_0 = \frac{1}{c^2}. \quad -(22)$$

So:

$$\boxed{\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2}\frac{\partial \underline{E}^a}{\partial t} = \mu_0 \underline{J}^a} \quad -(23)$$

Here:

$$a = (1), (2), (3) \quad -(24)$$

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) \quad -(25)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) \quad -(26)$$

$$\underline{e}^{(3)} = \underline{k} \quad -(27)$$

In the presence of polarization $\underline{P}^a$ and magnetization $\underline{M}^a$

5.)

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{D}^{a} &= \rho^{a} \quad -(28) \\ \underline{\nabla} \times \underline{H}^{a} - \frac{\partial \underline{D}^{a}}{\partial t} &= \underline{J}^{a} \quad -(29) \\ \underline{D}^{a} &= \epsilon_0 \underline{E}^{a} + \underline{P}^{a} \quad -(30) \\ \underline{B}^{a} &= \mu_0 \left( \underline{H}^{a} + \underline{M}^{a} \right). \quad -(31) \end{aligned}}$$

where

$\underline{E}^{a}$ = electric field strength (volts per metre)

$\underline{D}^{a}$ = electric displacement (coulombs per $m^2$)

$\rho^{a}$ = charge density (coulombs per $m^3$)

$\underline{H}^{a}$ = magnetic field strength (amps per metre)

$\underline{B}^{a}$ = magnetic flux density (tesla or weber per $m^2$)

$\underline{J}^{a}$ = current density (amps per sq. metre)

$\epsilon_0$ = vacuum permittivity
$= 8.854188 \times 10^{-12} \; J^{-1} C^2 m^{-1}$

$\mu_0$ = vacuum permeability
$= 4\pi \times 10^{-7} \; J s^2 C^{-2} m^{-1}$

# 1. 134(8): Hodge Dual Structures in the Field Equations

The homogeneous field equations are based on:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(1)$$

and the inhomogeneous field equations on:

$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(2)$$

The Hodge dual in eq. (2) is:

$$[D^\mu, D^\nu]_{HD} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} [D_\alpha, D_\beta] \quad -(3)$$

$$\tilde{R}^\rho{}_\sigma{}^{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} R^\rho{}_{\sigma\alpha\beta} \quad -(4)$$

$$\tilde{T}^{\lambda\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} T^\lambda{}_{\alpha\beta} \quad -(5)$$

So the weighting factor $\|g\|^{1/2}$ cancels out. The $\epsilon^{\mu\nu\alpha\beta}$ then refers to Minkowski spacetime.

The Hodge dual of the equation:

$$[D^\mu, D^\nu] V^\rho = R^\rho{}_\sigma{}^{\mu\nu} V^\sigma - T^{\lambda\mu\nu} D_\lambda V^\rho \quad -(6)$$

is eq. (2). The metrics do not enter into this duality because the $\|g\|^{1/2}$ does not enter into eqs (1), (2) and (6). Eq (1) means that:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(7)$$

i.e.

$$\partial_\mu \tilde{T}^{a\mu\nu} = 0 \quad -(8)$$

or

$$\partial_\mu T^a{}_{\nu\rho} + \partial_\rho T^a{}_{\mu\nu} + \partial_\nu T^a{}_{\rho\mu} = 0 \quad -(9)$$

Eq. (2) means that:

$$D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b \quad -(10)$$

or

$$\partial_\mu \tilde{T}^a{}_{\nu\rho} + \partial_\rho \tilde{T}^a{}_{\mu\nu} + \partial_\nu \tilde{T}^a{}_{\rho\mu} = \left(\tilde{j}_{\mu\nu\rho} + \tilde{j}_{\rho\mu\nu} + \tilde{j}_{\nu\rho\mu}\right) / \epsilon_0 \quad -(11)$$

i.e.

$$\partial_\mu T^{a\mu\nu} = j^{a\nu} / \epsilon_0 \quad -(12)$$

In general:

$$T^{a\mu\nu} = g^{\mu\rho} g^{\nu\sigma} T^a{}_{\mu\nu} \quad -(13)$$

and

$$g^{\mu\nu} g_{\mu\nu} = 4 \quad -(14)$$

In general $g^{\mu\nu}$ is not the Minkowski metric.

1.

134(9): Use of the Minkowski Metric in the ECE Field Equations

The homogeneous field equation is based on the Cartan Bianchi identity:

$$D_\mu T^a_{\nu\rho} + D_\rho T^a_{\mu\nu} + D_\nu T^a_{\rho\mu} := R^a_{\mu\nu\rho} + R^a_{\rho\mu\nu} + R^a_{\nu\rho\mu} \quad -(1)$$

which is

$$D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}^{a\ \mu\nu}_{\ \mu} \quad -(2)$$

The inhomogeneous field equation is based on the Cartan Evans identity:

$$D_\mu \tilde{T}^a_{\nu\rho} + D_\rho \tilde{T}^a_{\mu\nu} + D_\nu \tilde{T}^a_{\rho\mu} := \tilde{R}^a_{\mu\nu\rho} + \tilde{R}^a_{\rho\mu\nu} + \tilde{R}^a_{\nu\rho\mu} \quad -(3)$$

which is

$$D_\mu T^{a\mu\nu} := R^{a\ \mu\nu}_{\ \mu} \quad -(4)$$

The Hodge dual in the general four dimensional spacetime of a rank two tensor is another rank two tensor:

$$\tilde{T}^{a\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} T^a_{\alpha\beta} \quad -(5)$$

where $|g|$ is the determinant of the metric. Here, $\epsilon^{\mu\nu\alpha\beta}$ is the totally antisymmetric tensor in Minkowski spacetime. In eqs (1) to (4), $\|g\|^{1/2}$ cancels out, so the Minkowski spacetime Hodge duals can be used in eqs. (2) and (4). The modulus of the metric determinant

2) does not enter into eqs. (2) and (4).

The ECE field equations are:

$$D_\mu \tilde{F}^{a\mu\nu} := A^{(0)} \tilde{R}^{a}{}_{\mu}{}^{\mu\nu} \quad -(6)$$

$$D_\mu F^{a\mu\nu} := A^{(0)} R^{a}{}_{\mu}{}^{\mu\nu} \quad -(7)$$

i.e.

$$\partial_\mu \tilde{F}^{a\mu\nu} = 0 \quad -(8)$$

$$\partial_\mu F^{a\mu\nu} = J^{a\nu} / \epsilon_0 \quad -(9)$$

if it is assumed that there is no magnetic monopole. For each $a$ $\quad -(10)$

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -E_X & -E_Y & -E_Z \\ E_X & 0 & -cB_Z & cB_Y \\ E_Y & cB_Z & 0 & -cB_X \\ E_Z & -cB_Y & cB_X & 0 \end{bmatrix}$$

and

$$J^\nu = (\rho, \underline{J}/c) \quad -(11)$$

So eq. (9) for each $a$ is:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{E} &= \rho / \epsilon_0 \quad -(12) \\ \underline{\nabla} \times \underline{B} - \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} &= \mu_0 \underline{J} \quad -(13) \end{aligned}}$$

These are the Coulomb and Ampère Maxwell laws for each $a$.

3) Note carefully that these laws are now part of general relativity, and are written in a spacetime with torsion and curvature. The electric and magnetic fields are components of spacetime torsion. They are defined in tensor notation in eq. (10), with upper $\mu$ and $\nu$ indices.

The Hodge dual field tensor for each $a$ is:

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -cB_X & -cB_Y & -cB_Z \\ cB_X & 0 & E_Z & -E_Y \\ cB_Y & -E_Z & 0 & E_X \\ cB_Z & E_Y & -E_X & 0 \end{bmatrix}. \quad -(14)$$

so eq. (8) for each $a$ is:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 & -(15) \\ \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= \underline{0} & -(16) \end{aligned}}$$

Here $\tilde{F}^{\mu\nu}$ is again defined with upper indices. The relation between $\tilde{F}^{\mu\nu}$ and $F^{\mu\nu}$ is defined by the Minkowski $\epsilon^{\mu\nu\rho\sigma}$, i.e.:

$$\boxed{\begin{aligned} \tilde{F}^{01} &= F^{23}, \ \tilde{F}^{02} = F^{31}, \ \tilde{F}^{03} = F^{12} & -(17) \\ \tilde{F}^{12} &= F^{30}, \ \tilde{F}^{31} = F^{20}, \ \tilde{F}^{23} = F^{10} & -(18) \end{aligned}}$$

It can be seen that this is a rearrangement of a four dimensional antisymmetric tensor

4) to give another 4-D antisymmetric tensor. The indices in eq. (17) are in cyclic permutation:

$$0123,\ 0231,\ 0312 \quad -(19)$$

and also those in eq. (18):

$$1230,\ 3120,\ 2310. \quad -(20)$$

The antisymmetric tensor is:

$$\epsilon^{0123} = \epsilon^{0231} = \epsilon^{0312} = 1 \quad -(21)$$

$$\epsilon^{1230} = \epsilon^{3120} = \epsilon^{2310} = -1. \quad -(22)$$

These are elements in Minkowski spacetime. More generally:

$$\epsilon^{0123} = -\epsilon^{1230} = \epsilon^{2301} = -\epsilon^{3012} = 1$$

$$\epsilon^{1023} = -\epsilon^{2130} = \epsilon^{3201} = -\epsilon^{0312} = -1$$

$$\epsilon^{1032} = -\epsilon^{2103} = \epsilon^{3210} = -\epsilon^{0321} = 1$$

$$\epsilon^{1302} = -\epsilon^{2013} = \epsilon^{3120} = -\epsilon^{0231} = -1$$

$$-(23)$$

etc.

Therefore eqs. (17) and (18) mean that there are two ways of writing an antisymmetric tensor in four dimensions.

In the ECE engineering model therefore these two tensors are:

5)

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_Z \\ E_x & 0 & -cB_Z & cB_y \\ E_y & cB_Z & 0 & -cB_x \\ E_Z & -cB_y & cB_x & 0 \end{bmatrix}, \quad \tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB_x & -cB_y & -cB_Z \\ cB_x & 0 & E_Z & -E_y \\ cB_y & -E_Z & 0 & E_x \\ cB_Z & E_y & -E_x & 0 \end{bmatrix} \quad -(24)$$

Therefore in the engineering model it is worth emphasizing how the basic fields are defined, as in eq. (24), and in eqs. (17) and (18).

Eqs. (2) and (4) are Hodge invariant. This means that a 4-D tensor appearing in the equations can be replaced by its Hodge dual.

Eqs. (1) and (3) are also Hodge invariant, but the indices in these equations are lowered.

134(10): The Fundamental Hodge Duality of the Field Equations.

Consider the fundamental commutator structure of Riemann geometry:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(1)$$

using eqs. (17) and (18) of note 134(9), the Hodge duals of the commutator operator is defined as follows:

$$[D_0, D_1]_{HD} V^\rho = [D_2, D_3] V^\rho \quad -(2)$$
$$[D_0, D_2]_{HD} V^\rho = [D_3, D_1] V^\rho \quad -(3)$$
$$[D_0, D_3]_{HD} V^\rho = [D_1, D_2] V^\rho \quad -(4)$$
$$[D_1, D_2]_{HD} V^\rho = [D_3, D_0] V^\rho \quad -(5)$$
$$[D_3, D_1]_{HD} V^\rho = [D_2, D_0] V^\rho \quad -(6)$$
$$[D_2, D_3]_{HD} V^\rho = [D_1, D_0] V^\rho \quad -(7)$$

Proof

From eq. (1), raise indices using the metric:

$$[D^\mu, D^\nu] = g^{\mu\alpha} g^{\nu\beta} [D_\alpha, D_\beta] \quad -(8)$$
$$R^\rho{}_\sigma{}^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R^\rho{}_{\sigma\alpha\beta} \quad -(9)$$
$$T^{\lambda\mu\nu} = g^{\mu\alpha} g^{\nu\beta} T^\lambda{}_{\alpha\beta} . \quad -(10)$$

Therefore:

$$g^{\mu\alpha} g^{\nu\beta} [D_\alpha, D_\beta] V^\rho = g^{\mu\alpha} g^{\nu\beta} \left( R^\rho{}_{\sigma\alpha\beta} V^\sigma - T^\lambda{}_{\alpha\beta} D_\lambda V^\rho \right) \quad -(11)$$

and

$$[D^\mu, D^\nu] V^\rho = R^\rho{}_\sigma{}^{\mu\nu} V^\sigma - T^{\lambda\mu\nu} D_\lambda V^\rho \quad -(12)$$

Take the Hodge dual of eq. (12) term by term, using the definitions:

$$[D_\mu, D_\nu]_{HD} V^\rho = \frac{1}{2} \|g\|^{1/2} \epsilon_{\mu\nu\alpha\beta} [D^\alpha, D^\beta] V^\rho \quad -(13)$$

$$\tilde{R}^\rho{}_{\sigma\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\mu\nu\alpha\beta} R^\rho{}_\sigma{}^{\alpha\beta} \quad -(14)$$

$$\tilde{T}^\lambda{}_{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\mu\nu\alpha\beta} T^{\lambda\alpha\beta} \quad -(15)$$

Thus:

$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(16)$$

if

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(17)$$

Therefore indices are raised in transforming eq. (1) to eq. (12), and are lowered again in transforming eq. (12) to eq. (16). The relation between eq. (16) and eq. (17) therefore does not involve the metric. Also, $\|g\|^{1/2}$ cancels out. The antisymmetric tensor $\epsilon_{\mu\nu\alpha\beta}$ is defined (see Carroll) as the Minkowski spacetime tensor.

Now consider each set of indices individually.

For example:

) $[D_2, D_3] V^\rho = 1$

Therefore:

$$[D^2, D^3] V^\rho = R^\rho{}_\sigma{}^{23} V^\sigma - T^{\lambda 23} D_\lambda V^\rho \quad -(19)$$

and $[D_0, D_1]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma 01} V^\sigma - \tilde{T}^\lambda{}_{01} D_\lambda V^\rho \quad -(20)$

The transformation from (19) to (20) is:

$$\frac{1}{2} \|g\|^{1/2} \left( \epsilon_{0123} [D^2, D^3] + \epsilon_{0132} [D^3, D^2] \right) V^\rho = R.H.S. \quad -(21)$$

This means that we can write:

$$[D_0, D_1]_{HD} V^\rho = [D^2, D^3] V^\rho \quad -(22)$$

$$\tilde{R}^\rho{}_{\sigma 01} V^\sigma = R^\rho{}_\sigma{}^{01} V^\sigma \quad -(23)$$

$$\tilde{T}^\lambda{}_{01} D_\lambda V^\rho = T^{\lambda 01} D_\lambda V^\rho \quad -(24)$$

Therefore if:

$$[D_2, D_3] V^\rho = R^\rho{}_{\sigma 23} V^\sigma - T^\lambda{}_{23} D_\lambda V^\rho \quad -(25)$$

then $[D_0, D_1]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma 01} V^\sigma - \tilde{T}^\lambda{}_{01} D_\lambda V^\rho \quad -(26)$

This is true for any metric, because the metrics cancel out on both sides of eq. (11). So:

$$[D_0, D_1]_{HD} V^\rho = [D_2, D_3] V^\rho \quad -(27)$$

etc.

Q.E.D.

Eq. (17) means that:

4)

$$D \wedge T^a = R^a{}_b \wedge q^b \quad - (28)$$

and eq. (16) means that

$$D \wedge \tilde{F}^a = \tilde{R}^a{}_b \wedge q^b \quad - (29)$$

From eqs. (2) to (7) the duality between eqs. (28) and (29) is the same as the duality the Maxwell Heaviside equations. So the ECE field equations are:

$$\partial_\mu \tilde{F}^{a\mu\nu} = 0 \quad - (30)$$

$$\partial_\mu F^{a\mu\nu} = J^{a\nu} / \epsilon_0 \quad - (31)$$

For each $a$

$$\boxed{\begin{matrix} \tilde{F}^{01} = F^{23}, & \tilde{F}^{02} = F^{31}, & \tilde{F}^{03} = F^{12} \\ \tilde{F}^{12} = F^{30}, & \tilde{F}^{31} = F^{20}, & \tilde{F}^{23} = F^{10} \end{matrix}} \quad - (32)$$

For each $a$, and for any four dimensional spacetime

$$\left.\begin{aligned} \nabla \cdot \underline{B} &= 0 \\ \nabla \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= \underline{0} \\ \nabla \cdot \underline{E} &= \rho / \epsilon_0 \\ \nabla \times \underline{B} - \frac{\partial \underline{E}}{\partial t} &= \mu_0 \underline{J} \end{aligned}\right\} - (33)$$

in which eqs. (32) apply exactly as in the Maxwell Heaviside equations.

5) Note carefully that this rule applies only if raised indices are used for $\tilde{F}^{a\mu\nu}$ and $F^{a\mu\nu}$. If indices are raised as follows for example:

$$R^{\rho}{}_{\sigma}{}^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R^{\rho}{}_{\sigma\alpha\beta} \quad (34)$$

then the metric appropriate to the spacetime being considered must be used, as in paper 93 and similar papers.

The basic point is that the commutator acting on $V^{\rho}$ fixes the relation between $R^{\rho}{}_{\sigma\mu\nu}$ and $T^{\lambda}{}_{\mu\nu}$. As in previous papers this relation is equivalent to the Cartan Bianchi identity (28). The relation between $R^{\rho}{}_{\sigma\mu\nu}$ and $\tilde{T}^{\lambda}{}_{\mu\nu}$ is the same as the relation between $R^{\rho}{}_{\sigma\mu\nu}$ and $T^{\lambda}{}_{\mu\nu}$. This leads to the Cartan Evans identity (29). The weighting factor $\|g\|^{1/2}$ does not enter into eq. (16), so this means that the Hodge dual in eq. (30) is defined by eq. (32). This leads to a key feature of the ECE field equations, i.e. that the metric is subsumed into their structure. If, however, indices are raised or lowered in a particular individual tensor, ~~the~~ the metric of the spacetime must be used.

1) 134(11) : Fundamental Definitions

'u(1) Electrodynamics

The potential is

$$A^\mu = (A_0, \underline{A}), \quad A_\mu = (A_0, -\underline{A}) \quad -(1)$$

$$A^\mu = g^{\mu\nu} A_\nu \quad -(2)$$

$$g_{\mu\nu} = g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad -(3)$$

$$\partial^\mu = \left( \frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla} \right) \quad -(4)$$

$$\partial_\mu = \left( \frac{1}{c}\frac{\partial}{\partial t}, \underline{\nabla} \right) \quad -(5)$$

The field is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad -(6)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad -(7)$$

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \quad -(8)$$

Therefore

$$A^0 = A_0 = \phi / c \quad -(9)$$

$$A^1 = -A_1 = A_x \quad -(10)$$

$$A^2 = -A_2 = A_y \quad -(11)$$

$$A^3 = -A_3 = A_z \quad -(12)$$

$$\boxed{\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} \; ; \quad \underline{B} = \underline{\nabla} \times \underline{A}}$$

2)
$$F^{01} = -F_{01} = -E^{1} = E_1 = -E_X \quad -(13)$$
$$F^{02} = -F_{02} = -E^{2} = E_2 = -E_Y \quad -(14)$$
$$F^{03} = -F_{03} = -E^{3} = E_3 = -E_Z \quad -(15)$$
$$F^{12} = F_{12} = -cB^{3} = cB_3 = -cB_Z \quad -(16)$$
$$F^{31} = F_{31} = -cB^{2} = cB_2 = -cB_Y \quad -(17)$$
$$F^{23} = F_{23} = -cB^{1} = cB_1 = -cB_X \quad -(18)$$

<u>ECE Electrodynamics</u>

$$A^{a\mu} = \left(A_0^a, \underline{A}^a\right), \; A_\mu^a = \left(A_0^a, -\underline{A}^a\right) \quad -(19)$$

$$A^{a\mu} = g^{\mu\nu} A_\nu^a \quad -(20)$$

In general $g^{\mu\nu} g_{\mu\nu} = 4 \quad -(21)$

but $g_{\mu\nu} \neq g^{\mu\nu} \quad -(22)$

and the metric tensor is that of a spacetime with torsion and curvature in four dimensions. In general the metric is not known. To circumvent this difficulty a self-consistent scheme of fundamental definitions is used as follows.

3)

1) The partial derivatives in this spacetime are defined from fundamentals to be the same as eqns. (4) and (5).

2) For each $a$:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_X & -E_Y & -E_Z \\ E_X & 0 & -cB_Z & cB_Y \\ E_Y & cB_Z & 0 & -cB_X \\ E_Z & -cB_Y & cB_X & 0 \end{bmatrix} \quad (23)$$

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -cB_X & -cB_Y & -cB_Z \\ cB_X & 0 & E_Z & -E_Y \\ cB_Y & -E_Z & 0 & E_X \\ cB_Z & E_Y & -E_X & 0 \end{bmatrix} \quad (24)$$

Therefore:

$$\left.\begin{aligned} \tilde{F}^{01} &= F^{23}, & \tilde{F}^{02} &= F^{31}, & \tilde{F}^{03} &= F^{12} \\ \tilde{F}^{12} &= F^{30}, & \tilde{F}^{31} &= F^{20}, & \tilde{F}^{23} &= F^{10} \end{aligned}\right\} \quad (25)$$

3) For each $a$:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + A^{(0)}\left(\omega^{\mu\nu} - \omega^{\nu\mu}\right) \quad (26)$$

Therefore contravariant indices are used throughout.

4) The field equations for each $a$ are:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad -(27)$$

$$\partial_\mu F^{\mu\nu} = J^\nu / \epsilon_0 \quad -(28)$$

where $J^\nu = \left( c J^a_0, \underline{J} \right) \quad -(29)$

This method subsumes the metric into the definitions of $\phi$, $\underline{A}$, $\underline{E}$ and $\underline{B}$, so that knowledge of the metric is not required. Note carefully that in the general four dimensional spacetime, the contravariant definitions are different from the covariant definitions. This is because indices are raised and lowered by the metric, which is not the Minkowski metric (3). It is also important to note that the vector notation, although useful, hides the presence of the metric.

The end result is:

$$\underline{E}^a = -c \nabla A^a_0 - \frac{\partial \underline{A}^a}{\partial t} - c\omega^a_{0b} \underline{A}^b + c A^b_0 \underline{\omega}^a_b \quad (30)$$

$$\underline{B}^a = \nabla \times \underline{A}^a - \underline{\omega}^a_b \times \underline{A}^b \quad -(31)$$

5)

$$\underline{\nabla} \cdot \underline{B}^a = 0 \quad -(32)$$

$$\underline{\nabla} \times \underline{E}^a + \frac{\partial \underline{B}^a}{\partial t} = \underline{0} \quad -(33)$$

$$\underline{\nabla} \cdot \underline{E}^a = \rho^a / \epsilon_0 \quad -(34)$$

$$\underline{\nabla} \times \underline{B}^a - \frac{1}{c^2} \frac{\partial \underline{E}^a}{\partial t} = \mu_0 \underline{J}^a \quad -(35)$$

$$J^{a\mu} = \left( c\rho^a_0, \underline{J}^a \right) \quad -(36)$$

Hidden in these definitions is the fact that for self consistency, contravariant definitions are used throughout.

Antisymmetry law

For each $a$

$$\partial^\mu A^\nu + \partial^\nu A^\mu + A^{(0)}\left(\omega^{\mu\nu} + \omega^{\nu\mu}\right) = 0 \quad -(37)$$

or in full:

$$\partial^\mu A^{a\nu} + \partial^\nu A^{a\mu} + A^{(0)}\left(\omega^{a\mu\nu} + \omega^{a\nu\mu}\right) = 0 \quad -(38)$$

Here:

6) $$\omega^{a\mu\nu} = \omega^{a\mu}{}_{b}\, q^{b\nu} \quad -(39)$$

$$\omega^{a\nu\mu} = \omega^{a\nu}{}_{b}\, q^{b\mu} \quad -(40)$$

The contravariant definitions are:

$$\omega^{a\mu}{}_{b} = \left(\omega^{a}{}_{0b},\ \underline{\omega}^{a}{}_{b}\right) \quad -(41)$$

$$q^{b\nu} = \left(q^{b}{}_{0},\ \underline{q}^{b}\right) \quad -(42)$$

$$A^{a\mu} = \left(A^{a}{}_{0},\ \underline{A}^{a}\right) \quad -(43)$$

The Polarization Indices

In four dimensions, the contravariant spacetime vector $x^{\mu}$ is represented by the Cartesian:

$$x^{\mu} = (ct, X, Y, Z) \quad -(44)$$

$$= (x^{0}, x^{1}, x^{2}, x^{3})$$

or by the complex circular:

$$x^{a} = \left(x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}\right) \quad -(45)$$

The spacelike unit vectors of the complex circular basis are:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) \quad -(46)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) \quad -(47)$$

$$\underline{e}^{(3)} = \underline{k} \quad -(48)$$

7) The covariant tetrad is defined by:

$$x^a = q^a_\mu \, x^\mu, \quad -(49)$$

and the covariant potential is:

$$A^a_\mu = A^{(0)} q^a_\mu \quad -(50)$$

The contravariant tetrad is defined by:

$$x^a = q^{a\mu} x_\mu \quad -(51)$$

and the contravariant potential is defined by:

$$A^{a\mu} = A^{(0)} q^{a\mu} \quad -(52)$$

Here: $A^{a\mu} = g^{\mu\nu} A^a_\nu \quad -(53)$

where $g^{\mu\nu}$ is the general unknown. So $A^{a\mu}$ is not the same as $A^a_\nu$.

From eqs. (46) – (48), it is known that some components of $A^a_\mu$ and $A^{a\mu}$ vanish by definition:

$$\boxed{\begin{aligned} A^{(1)}_Z &= A^{(2)}_Z = 0 \quad -(54) \\ A^{(3)}_X &= A^{(3)}_Y = 0 \quad -(55) \end{aligned}}$$

1) 134(12): The Fundamental Hodge Duality

U(1) Electrodynamics

In the usual U(1) approach:

$$[D^\mu, D^\nu]\phi = -ig\left(\partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]\right)\phi \quad -(1)$$

It is incorrectly asserted that:

$$[A^\mu, A^\nu] =? 0 \quad -(2)$$

so in U(1) electrodynamics there is no IFE, contrary to experiment. Accepting this incorrect dogma for the sake of illustration only, we have:

$$[D^\mu, D^\nu]\phi = -ig F^{\mu\nu}\phi \quad -(3)$$

or $$[D_\mu, D_\nu]\phi = -ig F_{\mu\nu}\phi. \quad -(4)$$

Take the Hodge dual of both sides of eqn. (4), using:

$$[D^\mu, D^\nu]_{HD}\phi = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}[D_\alpha, D_\beta], \quad -(5)$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \quad -(6)$$

then:

$$[D^\mu, D^\nu]_{HD}\phi = -ig\tilde{F}^{\mu\nu}\phi \quad -(7)$$

Lower indices:

$$[D_\mu, D_\nu]_{HD}\phi = g_{\mu\alpha}g_{\nu\beta}[D^\alpha, D^\beta]_{HD}\phi \quad -(8)$$

$$\tilde{F}_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}\tilde{F}^{\alpha\beta} \quad -(9)$$

i.e.

$$[D_\mu, D_\nu]_{HD}\,\phi = -ig\,\tilde{F}_{\mu\nu}\,\phi \quad -(10)$$

if

$$[D_\mu, D_\nu]\,\phi = -ig\,F_{\mu\nu}\,\phi \quad -(11)$$

Therefore the existence of the Hodge dual field tensor follows from the existence of the Hodge dual of the commutator. For individual indices:

$$[D^0, D^1]_{HD} = [D^2, D^3] \quad -(12)$$

$$[D^0, D^2]_{HD} = [D^3, D^1] \quad -(13)$$

$$[D^0, D^3]_{HD} = [D^1, D^2] \quad -(14)$$

$$[D^1, D^2]_{HD} = [D^3, D^0] \quad -(15)$$

$$[D^3, D^1]_{HD} = [D^2, D^0] \quad -(16)$$

$$[D^2, D^3]_{HD} = [D^1, D^0] \quad -(17)$$

Therefore eq. (10) is an example of eq. (11). On the U(1) level this is an obvious result, but at the ECE level it has a deeper significance and leads to the Cartan Evans identity.

<u>ECE Electrodynamics</u>

The geometrical structure of ECE is based on:

3)

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(18)$$

The structure of electrodynamics is found by multiplying eq. (18) by $A^{(0)}$, giving:

$$[D_\mu, D_\nu] A^\rho = R^\rho{}_{\sigma\mu\nu} A^\sigma - F^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(19)$$

Taking the Hodge dual term by term of eq. (18) results in:

$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(20)$$

where the Hodge dual commutator is related to the original commutator by eqs. (12) to (17). This means that the tensors $\tilde{R}^\rho{}_{\sigma\mu\nu}$ and $\tilde{T}^\lambda{}_{\mu\nu}$ ~~both~~ are related to each other in the same way as the tensors $R^\rho{}_{\sigma\mu\nu}$ and $T^\lambda{}_{\mu\nu}$. This is seen from the fact that eq. (20) is a rearrangement of eq. (18). Similarly eq. (10) is a rearrangement of eq. (11).

Eq. (18) leads to:

$$D \wedge T^a := R^a{}_b \wedge q^b, \quad -(21)$$

the Cartan Bianchi identity. Eq. (20) leads to:

$$D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b, \quad -(22)$$

the Cartan Evans identity. The two identities in tensor notation are the field equations of ECE theory. The field tensors $F^a_{\mu\nu}$ and $\tilde{F}^a_{\mu\nu}$

4) are related in the way the commutators are related in eqs. (12) to (17). Thus, for each $a$:

$$\left.\begin{array}{l}\tilde{F}^{01} = F^{23}, \ \tilde{F}^{02} = F^{31}, \ \tilde{F}^{03} = F^{12}, \\ \tilde{F}^{12} = F^{30}, \ \tilde{F}^{31} = F^{20}, \ \tilde{F}^{23} = F^{10}.\end{array}\right\} \quad (23)$$

Similarly, for each $a$:

$$\left.\begin{array}{l}\tilde{T}^{01} = T^{23}, \ \tilde{T}^{02} = T^{31}, \ \tilde{T}^{03} = T^{12}, \\ \tilde{T}^{12} = T^{30}, \ \tilde{T}^{31} = T^{20}, \ \tilde{T}^{23} = T^{10}.\end{array}\right\} \quad (24)$$

For each $a$ and $b$:

$$\left.\begin{array}{l}\tilde{R}^{01} = R^{23}, \ \tilde{R}^{02} = R^{31}, \ \tilde{R}^{03} = R^{12}, \\ \tilde{R}^{12} = R^{30}, \ \tilde{R}^{31} = R^{20}, \ \tilde{R}^{23} = R^{10}.\end{array}\right\} \quad (25)$$

Therefore, taking a particular example of eq. (18):

$$[D_0, D_1] V^\rho = R^\rho{}_{\sigma 01} V^\sigma - T^\lambda{}_{01} D_\lambda V^\rho \quad (26)$$

Raising indices term by term:

$$g^{00} g^{11} [D_0, D_1] V^\rho = g^{00} g^{11} \left( R^\rho{}_{\sigma 01} V^\sigma - T^\lambda{}_{01} D_\lambda V^\rho \right) \quad (27)$$

i.e.

$$[D^0, D^1] V^\rho = R^\rho{}_\sigma{}^{01} V^\sigma - T^{\lambda 01} D_\lambda V^\rho \quad (28)$$

From eqs. (24) and (25):

$$[D^2, D^3]_{HD} V^\rho = \tilde{R}^\rho{}_\sigma{}^{23} V^\sigma - \tilde{T}^{\lambda 23} D_\lambda V^\rho \quad (29)$$

Thus eq. (29) is the same as eq. (28).

The relation between eq. (18) and eq. (21) follows from the fact that eq. (18) defines

5)

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma} \quad -(30)$$

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu}. \quad -(31)$$

These two tensors are related by eq. (21), which is:

$$D_\mu T^a{}_{\nu\rho} + D_\rho T^a{}_{\mu\nu} + D_\nu T^a{}_{\rho\mu} := R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} \quad -(32)$$

As shown in previous work, eqs. (30) and (31) prove that eq. (32) is an exact identity. Eq. (32) shows that the cyclic sum of three Riemann tensors is identically equal to the same cyclic sum of the definitions of the three tensors. Therefore the Cartan Bianchi identity is constructed directly from the definition (30). The procedure is as follows.

1) Define $R^\rho{}_{\sigma\mu\nu}$ as in eq. (30).
2) Form the sum $R^\rho{}_{\sigma\mu\nu} + R^\rho{}_{\nu\sigma\mu} + R^\rho{}_{\mu\nu\sigma}$.
3) State that this sum is identically equal to the same sum.
4) On the left hand side use 2), on the right hand side use the sum of definitions.
5) Rearrange the identity as eq. (32) using eq. (31) and the tetrad postulate.

The Cartan Evans Identity

This is eq. (22), which is:

$$D_\mu \tilde{T}^a{}_{\nu\rho} + D_\rho \tilde{T}^a{}_{\mu\nu} + D_\nu \tilde{T}^a{}_{\rho\mu} := \tilde{R}^a{}_{\mu\nu\rho} + \tilde{R}^a{}_{\rho\mu\nu} + \tilde{R}^a{}_{\nu\rho\mu} \quad -(33)$$

The tensor components in eq. (33) are defined by eqs. (24) and (25). Therefore for example:

6)
$$R^\rho{}_{\sigma 10} = \tilde{R}^\rho{}_{\sigma 23} \quad -(34)$$
and steps (1) to (5) may be repeated with $\tilde{R}$ instead of $R$ and $\tilde{T}$ instead of $T$. This procedure gives eq. (33).
The two basic geometrical identities are therefore eqs. (32) and (33). They may be written equivalently as:
$$D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}^{a}{}_{\mu}{}^{\mu\nu} \quad -(35)$$
and
$$D_\mu T^{a\mu\nu} := R^{a}{}_{\mu}{}^{\mu\nu} \quad -(36)$$
respectively. To see this, take an example of eq. (32):
$$D_1 T^a{}_{23} + D_3 T^a{}_{12} + D_2 T^a{}_{31} := R^a{}_{123} + R^a{}_{312} + R^a{}_{231}. \quad -(37)$$
Now use:
$$T^a{}_{23} = \|g\|^{1/2} \tilde{T}^{01} \quad -(38)$$
$$T^a{}_{12} = \|g\|^{1/2} \tilde{T}^{03} \quad -(39)$$
$$T^a{}_{31} = \|g\|^{1/2} \tilde{T}^{02} \quad -(40)$$
$$R^a{}_{123} = \|g\|^{1/2} \tilde{R}^a{}_1{}^{01} \quad -(41)$$
$$R^a{}_{312} = \|g\|^{1/2} \tilde{R}^a{}_3{}^{03} \quad -(42)$$
$$R^a{}_{231} = \|g\|^{1/2} \tilde{R}^a{}_2{}^{02} \quad -(43)$$
so:
$$D_1 \tilde{T}^{a01} + D_2 \tilde{T}^{a02} + D_3 \tilde{T}^{a03} := \tilde{R}^a{}_1{}^{01} + \tilde{R}^a{}_2{}^{02} + \tilde{R}^a{}_3{}^{03} \quad -(44)$$
i.e.
$$D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}^a{}_\mu{}^{\mu\nu} \quad -(45)$$
for the case
$$\nu = 0. \quad -(46)$$

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134(13): The Basic Role of the a Index

The a index of ECE theory is basic to all physics. The existence of wave equations depends on it, as does the existence of fermions, bosons, weak field bosons and quarks. In gravitation and electrodynamics it represents states of polarization. It is defined on the fundamental level as follows. Consider the vector field:

$$x = x^{\mu} e_{\mu} = x^{a} e_{a} \quad -(1)$$

In four dimensional spacetime:

$$x^{\mu} = (x^{0}, x^{1}, x^{2}, x^{3}) \quad -(2)$$

$$x^{a} = (x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}) \quad -(3)$$

Here $x$ may be any vector field in any 4-D spacetime. If we consider the coordinate vector field, and use Cartesian coordinates for the space-like components, then:

$$x^{\mu} = (ct, X, Y, Z) \quad -(4)$$

Define the complex circular space-like representation by:

$$a = (1), (2), (3). \quad -(5)$$

Its unit vectors are:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) \quad -(6)$$

$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}} (\underline{i} + i\underline{j}) \quad -(7)$$

$$\underline{e}^{(3)} = \underline{k} \quad -(8)$$

The time-like part is:

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$$\underline{e}^{(3)} = \underline{k} \quad -(8)$$

The time-like part is:

2)

$$a = (0). \quad -(9)$$

Define
$$x^0 = x^{(0)} = ct. \quad -(10)$$

The complex circular basis is a choice of basis. It is convenient because it is a natural basis for circular polarization and non-linear optics. The conjugate product of the complex circular basis is defined by a vector cross product of potentials defined by:

$$\underline{A}^{(1)} = A^{(0)} \underline{e}^{(1)} e^{i\phi} \quad -(11)$$
$$\underline{A}^{(2)} = A^{(0)} \underline{e}^{(2)} e^{-i\phi}. \quad -(12)$$

Thus:
$$\underline{B}^{(3)*} = -ig \underline{A}^{(1)} \times \underline{A}^{(2)} \quad -(13)$$

where $\underline{B}^{(3)}$ is a longitudinal and fundamental magnetic flux density observed in the inverse Faraday effect. <u>However</u>, <u>the a index is completely general</u>, <u>and is not restricted to the basis (6) to (8).</u>

The basis (6) to (8) has the properties:

$$\left.\begin{aligned} \underline{e}^{(1)} \times \underline{e}^{(2)} &= i\underline{e}^{(3)*} \\ \underline{e}^{(3)} \times \underline{e}^{(1)} &= i\underline{e}^{(2)*} \\ \underline{e}^{(2)} \times \underline{e}^{(3)} &= i\underline{e}^{(1)*} \end{aligned}\right\} -(14)$$

The Cartesian basis has the properties:

$$\left.\begin{aligned} \underline{i} \times \underline{j} &= \underline{k} \\ \underline{k} \times \underline{i} &= \underline{j} \\ \underline{j} \times \underline{k} &= \underline{i} \end{aligned}\right\} \quad -(15)$$

Eqs. (14) and (15) have the same cyclic symmetries and are equivalent space-like representations. Also:

$$\left.\begin{aligned} \underline{e}^{(1)} \cdot \underline{e}^{(2)} &= 1 \\ \underline{e}^{(2)} \cdot \underline{e}^{(1)} &= 1 \\ \underline{e}^{(3)} \cdot \underline{e}^{(3)} &= 1 \end{aligned}\right\} \quad -(16)$$

i.e.:
$$\underline{e}^{(1)} \cdot \underline{e}^{(1)*} = \underline{e}^{(2)} \cdot \underline{e}^{(2)*} = \underline{e}^{(3)} \cdot \underline{e}^{(3)*}$$
$$= 1 \quad -(17)$$

where $*$ denotes complex conjugate. The complex conjugate is defined by:

$$i \longrightarrow -i \quad -(18)$$

In the Cartesian basis:

$$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1 \quad -(19)$$

and
$$\underline{i} \cdot \underline{k} = \underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{k} = 0 \quad -(20)$$

Therefore $\underline{i}$, $\underline{j}$ and $\underline{k}$ are orthogonal.

Similarly:

$$\underline{e}^{(1)} \cdot \underline{e}^{(2)*} = \underline{e}^{(1)} \cdot \underline{e}^{(3)*} = \underline{e}^{(2)} \cdot \underline{e}^{(3)*}$$
$$= 0 \quad -(21)$$

4) In the Cartesian system:

$$x^\mu = (ct, \underline{r}) \quad - (22)$$

where:

$$\underline{r} = X\underline{i} + Y\underline{j} + Z\underline{k} \quad - (23)$$

Therefore:

$$r^2 = \underline{r} \cdot \underline{r} = X^2 + Y^2 + Z^2 \quad - (24)$$

The vector $\underline{r}$ is known as the position vector. It follows that

$$x^\mu x_\mu = c^2t^2 - X^2 - Y^2 - Z^2. \quad - (25)$$

In the complex circular basis the position vector is:

$$\underline{r} = x^{(1)} \underline{e}^{(1)} + x^{(2)} \underline{e}^{(2)} + x^{(3)} \underline{e}^{(3)} \quad - (26)$$

and:

$$r^2 = \underline{r} \cdot \underline{r}^* = x^{(1)} x^{(1)*} + x^{(2)} x^{(2)*} + x^{(3)} x^{(3)*}$$

$$= X^2 + Y^2 + Z^2 \quad - (27)$$

From orthogonality:

$$X^2 = x^{(1)} x^{(1)*} \quad - (28)$$

$$Y^2 = x^{(2)} x^{(2)*} \quad - (29)$$

$$Z^2 = x^{(3)} x^{(3)*} \quad - (30)$$

so:

$$x^{(1)} = \frac{X}{\sqrt{2}} (1 - i) = x^{(2)*} \quad - (31)$$

$$x^{(2)} = \frac{Y}{\sqrt{2}} (1 + i) = x^{(1)*} \quad - (32)$$

$$x^{(3)} = Z \quad - (33).$$

The two representations are related by a *generalization* of the Cartan tetrad to:

$$x^a = q^a_\mu x^\mu \quad -(34)$$

$$e_a = q^\mu_a e_\mu. \quad -(35)$$

The covariant derivatives for $\mu$ and $a$ are:

$$D_\mu x^\nu = \partial_\mu x^\nu + \Gamma^\nu_{\mu\lambda} x^\lambda \quad -(36)$$

$$D_\mu x^a = \partial_\mu x^a + \omega^a_{\mu b} x^b \quad -(37)$$

It follows from eqs. (1), and (34) to (37), that

$$D_\mu q^a_\nu = \partial_\mu q^a_\nu + \omega^a_{\mu b} q^b_\nu - \Gamma^\lambda_{\mu\nu} q^a_\lambda$$

$$= 0 \quad -(38)$$

The tetrad is normalized by the Kronecker delta:

$$q^a_\mu q^\nu_a = \delta^\nu_\mu. \quad -(39)$$

By definition:

$$\omega^a_{\mu\nu} = \omega^a_{\mu b} q^b_\nu \quad -(40)$$

$$\Gamma^a_{\mu\nu} = \Gamma^\lambda_{\mu\nu} q^a_\lambda \quad -(41)$$

so:

$$\partial_\mu q^a_\nu = \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu}. \quad -(42)$$

Thus:

$$\Box q^a_\nu = \partial^\mu \partial_\mu q^a_\nu = \partial^\mu \left( \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \right)$$

$$-(43)$$

Define:

$$R\, q^a_\nu := \partial^\mu \left( \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \right) \quad -(44)$$

So:

$$R := q^\nu_a\, \partial^\mu \left( \Gamma^a_{\mu\nu} - \omega^a_{\mu\nu} \right) \quad -(45)$$

and

$$\boxed{\Box\, q^a_\nu := R\, q^a_\nu} \quad -(46)$$

This geometrical identity is known as the ECE Lemma and defines all the wave equations of physics. By hypothesis:

$$R = -kT \quad -(47)$$

The existence of R and T depend on the fact that a is different from $\nu$. So a is fundamental to quantum mechanics.

If:

$$a = \nu \quad -(48)$$

in eq. (46) then:

$$q^a_\nu = \delta^a_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad -(49)$$

and

$$\partial_\mu q^a_\nu = 0 \quad -(50)$$

so

$$\Gamma^a_{\mu\nu} = \omega^a_{\mu\nu}, \quad -(51)$$

$$R = 0 \quad -(52)$$

7) The existence of quantum mechanics also depends on the existence of:

$$\boxed{\partial_\mu q^a_{\sim} \neq 0} \quad . \quad -(53)$$

There must be a phase factor present in the definition of the vector $x^a$.

For a wave propagating in Z, the phase factor is

$$\phi = \omega t - \kappa Z \quad -(54)$$

where $\omega$ is the angular frequency at instant t, and $\kappa$ is the wavenumber at point Z. Thus:

$$\boxed{X^a = x^a e^{i\phi}} \quad -(54)$$

The frame $X^a$ is rotating and translating with respect to $x^\mu$. This is a concept of relativity because $x^\mu$ is rotating and translating with respect to $X^a$.

In order to fulfill the basic vector field equation (1), the phase must enter into the basis vectors as follows:

$$\epsilon_a = e_a e^{-i\phi} \quad -(55)$$

so:

$$x = x^\mu e_\mu = X^a \epsilon_a \quad -(56)$$

Therefore:

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}}\left(\underline{i} - i\underline{j}\right) e^{-i\phi}$$
$$\underline{e}^{(2)} = \frac{1}{\sqrt{2}}\left(\underline{i} + i\underline{j}\right) e^{i\phi} \quad -(57)$$
$$\underline{e}^{(3)} = \underline{k}$$

The tetrad components from eq. (57) are:

$$q^{(1)}_x = \frac{1}{\sqrt{2}} e^{-i\phi}$$
$$q^{(1)}_y = -\frac{i}{\sqrt{2}} e^{-i\phi}$$
$$q^{(2)}_x = \frac{1}{\sqrt{2}} e^{i\phi} \quad -(58)$$
$$q^{(2)}_y = \frac{i}{\sqrt{2}} e^{i\phi}$$
$$q^{(3)}_z = 0.$$

By definition:

$$q^{(1)}_z = q^{(2)}_z = 0 \quad -(59)$$
$$q^{(3)}_x = q^{(3)}_y = 0 \quad -(60)$$

The spacelike basis components from eq. (57) may be arranged in column vectors. These will define the tetrads (58) as in the next note.

1) 134(14): The a Index and Tetrads in Electromagnetism.

Consider the plane wave as an example. The plane wave of electromagnetic potential is a two dimensional vector:

$$\underline{A}^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{i\phi} \quad -(1)$$

where

$$\underline{e}^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) \quad -(2)$$

$$\phi = \omega t - \kappa Z. \quad -(3)$$

Therefore:

$$\underline{A}^{(1)} = A^{(1)} \underline{e}^{(1)} \quad -(4)$$

where:

$$A^{(1)} = \frac{A^{(0)}}{\sqrt{2}} (1 - i) e^{i\phi} \quad -(5)$$

From eq. (1):

$$A^{(1)}_X = \frac{A^{(0)}}{\sqrt{2}} e^{i\phi} \quad -(6)$$

$$A^{(1)}_Y = -i\frac{A^{(0)}}{\sqrt{2}} e^{i\phi} \quad -(7)$$

and

$$\boxed{\begin{aligned} \underline{A}^{(1)} &= A^{(1)}_X \underline{i} + A^{(1)}_Y \underline{j} \\ &= A^{(1)} \underline{e}^{(1)} \end{aligned}} \quad -(8)$$

In this example the a index is:

$$a = (1). \quad -(9)$$

The potential four vector is:

2)

$$A^a_\mu = A^{(1)}_\mu = \left(A^{(1)}_0, -\underline{A}^{(1)}\right) \quad -(10)$$

$$= \left(\frac{\phi^{(1)}}{c}, -\underline{A}^{(1)}\right) \quad -(11)$$

Therefore $\phi^{(1)}$ is the scalar potential of an electromagnetic wave with polarization index (1). The latter is defined by the unit vector $\underline{e}^{(1)}$ in eq. (2). So:

$$e^{(1)}_X = \frac{1}{\sqrt{2}}, \; e^{(1)}_Y = -\frac{i}{\sqrt{2}}. \quad -(12)$$

These quantities are multiplied by the phase $e^{-i\phi}$ as discussed in the previous note, so:

$$\epsilon^{(1)}_X = e^{-i\phi}/\sqrt{2}, \; \epsilon^{(1)}_Y = -ie^{-i\phi}/\sqrt{2} \quad -(13)$$

Now denote the static Cartesian unit vectors as:

$$\underline{e}^1 = \underline{i}, \quad \underline{e}^2 = \underline{j}, \quad -(14)$$

So

$$e^1_X = 1, \quad e^2_Y = 1. \quad -(15)$$

Construct the column vectors:

$$\epsilon^{(1)} = \frac{e^{-i\phi}}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}, \quad e^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad -(16)$$

$$= \epsilon^a \qquad\qquad\qquad = e^\mu$$

The tetrad is defined by:

$$\epsilon^a = q^a_\mu e^\mu \quad -(17)$$

3)

i.e.
$$\begin{bmatrix} \epsilon^{(1)}_x \\ \epsilon^{(1)}_y \end{bmatrix} = \begin{bmatrix} q_1^{(1)} & 0 \\ 0 & q_2^{(1)} \end{bmatrix} \begin{bmatrix} e^1_x \\ e^2_y \end{bmatrix} \quad -(18)$$

This can be seen from:
$$\epsilon^{(1)} = q^{(1)}_1 e^1 + q^{(1)}_2 e^2, \quad -(19)$$

so:
$$\epsilon^{(1)}_x = q^{(1)}_1 e^1_x + q^{(1)}_2 e^2_x \quad -(20)$$
$$\epsilon^{(1)}_y = q^{(1)}_1 e^1_y + q^{(1)}_2 e^2_y \quad -(21)$$

Here:
$$e^2_x = e^1_y = 0 \quad -(22)$$

thus giving eq. (18). Therefore:
$$q^{(1)}_1 = \epsilon^{(1)}_x = \frac{1}{\sqrt{2}} e^{-i\phi} \quad -(23)$$
$$q^{(1)}_2 = \epsilon^{(1)}_y = -\frac{i}{\sqrt{2}} e^{-i\phi} \quad -(24)$$

The tetrad vector is:
$$\underline{q}^{(1)} = \frac{1}{\sqrt{2}} (\underline{i} - i\underline{j}) e^{-i\phi} \quad -(25)$$
$$\underline{q}^{(2)} = \frac{1}{\sqrt{2}} (\underline{i} + i\underline{j}) e^{i\phi} \quad -(26)$$

This was first introduced in 2003. We have
$$\underline{q}^{(3)*} = \underline{k} = -i\, \underline{q}^{(1)} \times \underline{q}^{(2)} \quad -(27)$$

ECE Hypothesis
$$\left\{ \begin{array}{l} \underline{A}^{(1)} = A^{(0)} \underline{q}^{(1)} \quad -(28) \\ \underline{A}^{(2)} = A^{(0)} \underline{q}^{(2)}. \quad -(29) \end{array} \right.$$

Polarization Indices of the Electromagnetic Potential

$$A_\mu^{(1)} = \left( \frac{\phi^{(1)}}{c}, -\underline{A}^{(1)} \right) \quad -(30)$$

$$A_\mu^{(2)} = \left( \frac{\phi^{(2)}}{c}, -\underline{A}^{(2)} \right) \quad -(31)$$

$$A_\mu^{(3)} = \left( \frac{\phi^{(3)}}{c}, -\underline{A}^{(3)} \right) \quad -(32)$$

$$A_\mu^{(0)} = \left( \frac{\phi^{(0)}}{c}, -\underline{A}^{(0)} \right) \quad -(33)$$

In these equations, $\phi^a$ is the scalar potential for each $a$. The spacelike polarizations are

$$a = (1), (2), (3), \quad -(34)$$

thus:

$$\boxed{\begin{aligned} \phi^{(1)} &= c A_0^{(1)} \quad -(35) \\ \phi^{(2)} &= c A_0^{(2)} \quad -(36) \\ \phi^{(3)} &= c A_0^{(3)} \quad -(37) \end{aligned}}$$

Therefore $A_0^{(1)}$, $A_0^{(2)}$ and $A_0^{(3)}$ have a clear interpretation. They are the scalar potentials in polarizations $\underline{(1)}$, $\underline{(2)}$ and $\underline{(3)}$ divided by $c$.

In eq. (33),

$$\underline{A}^{(0)} = \underline{0} \quad -(38)$$

because $(0)$ is the timelike index, while $\underline{A}$ is spacelike. So:

$$\boxed{A_\mu^{(0)} = A_0^{(0)} = \phi^{(0)} / c} \quad -(39)$$

## Summary

$$A_\mu^{a} = \begin{bmatrix} A_0^{(0)} & 0 & 0 & 0 \\ A_0^{(1)} & A_1^{(1)} & A_2^{(1)} & A_3^{(1)} \\ A_0^{(2)} & A_1^{(2)} & A_2^{(2)} & A_3^{(2)} \\ A_0^{(3)} & A_1^{(3)} & A_2^{(3)} & A_3^{(3)} \end{bmatrix} \quad - (40)$$

for the general basis $a = (0), (1), (2), (3)$.

$$A_\mu^{a} = \begin{bmatrix} A_0^{(0)} & 0 & 0 & 0 \\ A_0^{(1)} & A_1^{(1)} & A_2^{(1)} & 0 \\ A_0^{(2)} & A_1^{(2)} & A_2^{(2)} & 0 \\ A_0^{(3)} & 0 & 0 & A_3^{(3)} \end{bmatrix} \quad - (41)$$

for the complex circular basis.

1) 134(15): Polarization Indices

In purely mathematical terms the electromagnetic potential in ECE theory is a rank two tensor in four dimensions, so has sixteen components mathematically. In electromagnetic theory however it is defined by:

$$A^a_\mu = \left( \frac{\phi^a}{c}, -\underline{A}^a \right) \quad -(1)$$

where $\phi^a$ is the scalar potential in volts and where $\underline{A}^a$ is the spacelike vector potential. The scalar potential is timelike, and $\phi^a$ is scalar valued. Here, $\underline{A}^a$ is vector-valued. By definition:

$$\phi^a = cA^a_0 \quad -(2)$$

so $A^a_0$ is scalar valued for each $a$. Quantities such as $A^a_i$, $i = 1, 2, 3$, are components of the spacelike vector part of $A^a_\mu$, i.e. of $\underline{A}^a$.

Magnetic Field of ECE Theory

This is defined by:

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad -(3)$$

Only the $i = 1, 2, 3$ indices of $\underline{A}^a$ enter into the definition, and only $a = (1), (2), (3)$. So:

$$A^a_\mu(\text{magnetic}) = \begin{bmatrix} A^{(1)}_X & A^{(1)}_Y & A^{(1)}_Z \\ A^{(2)}_X & A^{(2)}_Y & A^{(2)}_Z \\ A^{(3)}_X & A^{(3)}_Y & A^{(3)}_Z \end{bmatrix} \quad -(4)$$

In the complex circular basis, some components of

2) the matrix (4) are zero by definition:

$$A^{(1)}_Z = A^{(2)}_Z = A^{(3)}_X = A^{(3)}_Y = 0 \quad -(5)$$

so:

$$A^a_\mu(\text{magnetic}) = \begin{bmatrix} A^{(1)}_X & A^{(1)}_Y & 0 \\ A^{(2)}_X & A^{(2)}_Y & 0 \\ 0 & 0 & A^{(3)}_Z \end{bmatrix} \quad -(6)$$

Therefore only these five components enter into eq. (3) in general.

<u>Electric</u> <u>Field</u> in <u>ECE</u> <u>Theory</u>

This is defined by:

$$\underline{E}^a = -\underline{\nabla}\phi^a - \frac{\partial \underline{A}^a}{\partial t} - c\,\omega^a{}_{0b}\,\underline{A}^b + c\,\underline{\omega}^a{}_b A^b_0 \quad -(7)$$

or in tensor notation:

$$E^a_{0i} = c\left(\partial_0 A^a_i - \partial_i A^a_0 + \omega^a_{0b} A^b_i - \omega^a_{ib} A^b_0\right) \quad -(8)$$

By definition, an electric field is a <u>space</u> <u>like</u> vector quantity, in which:

$$a = (1), (2), (3) \quad -(9)$$

Therefore

$$E^{(0)}_{0i} = -E^{(0)}_i = 0, \quad -(10)$$

$$i = 1, 2, 3$$

3) However, components such as $A^a_0$ enter into the definition of the electric field. In contrast, the magnetic field is:

$$\underline{B}^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + \omega^a_{ib} A^b_j - \omega^a_{jb} A^b_i \quad -(11)$$

where

$$i, j = 1, 2, 3 \quad -(12)$$

In eqs. (8) and (12), the b index runs from (0) to (3), so in purely mathematical terms tensor components such as $A^{(0)}_i$ may enter into the definition of $\underline{E}^a$ and $\underline{B}^a$.

To investigate the meaning of $A^{(0)}_i$ in physics, it is seen that they are components of a putative spacelike vector $\underline{A}^{(0)}$. As for the electric field and magnetic field, such a vector must be zero, because it must have spacelike polarization (1), (2) and (3) only. Therefore:

$$\underline{E}^{(0)} = \underline{B}^{(0)} = \underline{A}^{(0)} = \underline{0} \quad -(13)$$

and

$$E^{(0)}_i = B^{(0)}_i = A^{(0)}_i = 0. \quad -(14)$$

Therefore

$$\boxed{A^{(0)}_\mu = \left( \frac{\phi^{(0)}}{c}, \underline{0} \right)} \quad -(15)$$

where

$$\phi^{(0)} = c A^{(0)}_0. \quad -(16)$$

4)

Engineering Model

$$\underline{B}^a = \underline{\nabla} \times \underline{A}^a - \underline{\omega}^a{}_b \times \underline{A}^b \quad -(17)$$

$$a = (1), (2), (3)$$
$$b = (1), (2), (3)$$

$$\underline{E}^a = -\underline{\nabla}\phi^a - \frac{\partial \underline{A}^a}{\partial t} - c\omega^a{}_{0b}\underline{A}^b + c\underline{\omega}^a{}_b A_0^b \quad -(18)$$

$$a = (1), (2), (3)$$
$$b = (1), (2), (3), \text{ but } A_0^{(0)} \neq 0.$$

Physical Meaning of $A_0^a$:

In the definition of the electric field, these tensor components appear. As defined in eq. (2) they are time-like and therefore scalar valued. They are not components of a space-like quantity.

Therefore

$$\boxed{A_\mu^a = \left(\frac{\phi^a}{c}, \underline{A}^a\right) \quad -(19) \qquad a = (1), (2), (3)}$$

$$\boxed{A_\mu^{(0)} = \left(\frac{\phi^{(0)}}{c}, \underline{0}\right) \quad -(20)}$$

Therefore $\phi^{(0)}$ is the scalar potential of a scalar wave; $\phi^{(i)}$ is the scalar potential of a wave with space polarizations $(i) = (1), (2), (3)$