122(1): Equations of Cosmology and Nanostructure Technology.

Start with the Bianchi identity:

$$D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}^{a\ \mu\nu}_{\ \mu} \qquad -(1)$$

in the usual notation, and the Cartan Evans dual identity:

$$D_\mu T^{a\mu\nu} := R^{a\ \mu\nu}_{\ \mu} \qquad -(2)$$

Eq. (1) gives the homogeneous field equations of cosmology and nanostructure science. Eq. (2) gives the in-homogeneous equations. Expand the covariant derivative:

$$\partial_\mu T^{a\mu\nu} = R^{a\ \mu\nu}_{\ \mu} - \omega^a_{\ \mu b} T^{b\mu\nu} := j^{a\nu} \qquad -(3)$$

$$\partial_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^{a\ \mu\nu}_{\ \mu} - \omega^a_{\ \mu b} \tilde{T}^{b\mu\nu} := \tilde{j}^{a\nu} \qquad -(4)$$

Now use:

$$T^{\kappa\mu\nu} = q^\kappa_{\ a} T^{a\mu\nu} \qquad -(5)$$

$$j^{\kappa\nu} = q^\kappa_{\ a} j^{a\nu} \qquad -(6)$$

$$\omega^\kappa_{\ \mu b} = q^\kappa_{\ a} \omega^a_{\ \mu b} \qquad -(7)$$

to obtain:

$$\partial_\mu T^{\kappa\mu\nu} = j^{\kappa\nu} = R^{\kappa\ \mu\nu}_{\ \mu} - \omega^\kappa_{\ \mu b} T^{b\mu\nu} \qquad -(8)$$

$$\partial_\mu \tilde{T}^{\kappa\mu\nu} = \tilde{j}^{\kappa\nu} = \tilde{R}^{\kappa\ \mu\nu}_{\ \mu} - \omega^\kappa_{\ \mu b} \tilde{T}^{b\mu\nu} \qquad -(9)$$

2)

Now use:

$$\omega^{\kappa}{}_{\mu\lambda} = q^{b}{}_{\lambda}\,\omega^{\kappa}{}_{\mu b} \quad -(10)$$

$$T^{\lambda\mu\nu} = q^{\lambda}{}_{b}\,T^{b\mu\nu} \quad -(11)$$

with

$$q^{b}{}_{\lambda}\,q^{\lambda}{}_{b} = 1 \quad -(12)$$

by convention.

Thus:

$$\partial_{\mu} T^{\kappa\mu\nu} = j^{\kappa\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu} - \omega^{\kappa}{}_{\mu\lambda} T^{\lambda\mu\nu} \quad -(13)$$

$$\partial_{\mu} \tilde{T}^{\kappa\mu\nu} = \tilde{j}^{\kappa\nu} = \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} - \omega^{\kappa}{}_{\mu\lambda} \tilde{T}^{\lambda\mu\nu} \quad -(14)$$

In vector format these are:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{T} &= j = R - \omega T \\ \underline{\nabla} \cdot \underline{\tilde{T}} &= \tilde{j} = \tilde{R} - \omega \tilde{T} \end{aligned}} \quad \begin{aligned} -(15) \\ -(16) \end{aligned}$$

The subjects of cosmology and nanotechnology are developed from this geometrical structure. Eq. (15) gives the generally covariant Newton and Coulomb laws when:

$$\kappa = 0. \quad -(17)$$

3) Generally Covariant Newton Law.

This is defined by:

$$\underline{g} = c^2 \underset{\sim}{T} = c^2 \left(T^{010} \underline{i} + T^{020} \underline{j} + T^{030} \underline{k} \right) \quad -(18)$$

$$\rho_m = \frac{c^2}{4\pi G} \left(R^{0}{}_{1}{}^{10} + R^{0}{}_{2}{}^{20} + R^{0}{}_{3}{}^{30} - \omega^{0}{}_{1\lambda} T^{\lambda 10} - \omega^{0}{}_{2\lambda} T^{\lambda 20} - \omega^{0}{}_{3\lambda} T^{\lambda 30} \right) \quad -(19)$$

where in general:

$$\lambda = 0, 1, 2, 3. \quad -(20)$$

In shorthand, eq. (19) may be written as:

$$\rho_m = \frac{c^2}{4\pi G} \left(R - \omega T \right) \quad -(21)$$

in kilograms per cubic metre. Thus:

$$\boxed{\underline{\nabla} \cdot \underline{g} = 4\pi G \rho_m} \quad -(22)$$

This is a differential equation which must be solved with initial and boundary conditions. It has been shown already that it describes galactic dynamics without the use of dark matter. The initial condition is an initial event in evolution

4) Generally Covariant Coulomb Law

This is defined by:

$$\underline{E} = c A^{(0)} \underline{T} \quad -(23)$$

$$\rho_e = \frac{\epsilon_0}{c A^{(0)}} (R - \omega T) \quad -(24)$$

so:

$$\boxed{\underline{\nabla} \cdot \underline{E} = \rho_e / \epsilon_0} \quad -(25)$$

It is seen that both laws originate in the Cartan Evans dual identity of geometry.

The properties of ρ_m (mass density) and ρ_e (charge density in coulombs per cubic metre) depend on the balance of $R - \omega T$. Different models of the spin connection give different models of cosmology and ~~rot~~ nanotechnology. This gives almost unlimited scope of development using numerical methods and computer power.

In addition there is spin connection resonance.

5) Spin Connection Resonance

This phenomenon occurs throughout nature (e.g. in descaling technologies) and originates in the Cartan structure equation:

$$T^a_{\mu\nu} = (d \wedge q^a)_{\mu\nu} + \omega^a_{\mu b} \wedge q^b_\nu \quad -(26)$$

in differential form notation. Translating into tensor notation:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(27)$$

This equation may be developed in the base manifold as:

$$T^{\kappa\mu\nu} = \partial^\mu q^{\kappa\nu} - \partial^\nu q^{\kappa\mu} + \omega^{\kappa\mu}{}_\lambda q^{\lambda\nu} - \omega^{\kappa\nu}{}_\lambda q^{\lambda\mu} \quad -(28)$$

where

$$q^{\mu\nu} = g^{\mu\nu} \quad -(29)$$

Here $g^{\mu\nu}$ is the inverse metric. Therefore the torsion tensor is defined by the inverse metric tensor:

$$\boxed{T^{\kappa\mu\nu} = \partial^\mu g^{\kappa\nu} - \partial^\nu g^{\kappa\mu} + \omega^{\kappa\mu}{}_\lambda g^{\lambda\nu} - \omega^{\kappa\nu}{}_\lambda g^{\lambda\mu}} \quad -(30)$$

6)

By definition:

$$\omega^{\kappa\mu}{}_{\lambda} = g^{\mu\sigma}\,\omega^{\kappa}{}_{\sigma\lambda} \quad -(31)$$

For application with the generally covariant Newton and Coulomb laws we need:

$$T^{010} = \partial g^{00} - \partial^{0} g^{01} + \omega^{01}{}_{\lambda}\, g^{\lambda 0} - \omega^{00}{}_{\lambda}\, g^{\lambda 1} \quad -(32)$$

where:

$$\partial^{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\underline{\nabla}\right) \quad -(33)$$

So in vector notation:

$$\underline{T} = -\underline{\nabla} g^{00} - \frac{1}{c}\frac{\partial \underline{g}(\alpha)}{\partial t} + \left(\omega^{01}{}_{\lambda}\, g^{\lambda 0} - \omega^{00}{}_{\lambda}\, g^{\lambda 1}\right)\underline{i} + \left(\omega^{02}{}_{\lambda}\, g^{\lambda 0} - \omega^{00}{}_{\lambda}\, g^{\lambda 2}\right)\underline{j} + \left(\omega^{03}{}_{\lambda}\, g^{\lambda 0} - \omega^{00}{}_{\lambda}\, g^{\lambda 3}\right)\underline{k} \quad -(34)$$

with:

$$\underline{g}(\alpha) = g^{01}\,\underline{i} + g^{02}\,\underline{j} + g^{03}\,\underline{k} \quad -(35)$$

The remaining sum over repeated λ indices

7) i.e. eq. (34) may be translated into vector notation using:

$$\omega^{01}{}_{\lambda} = \left(\omega^{01}{}_{0}, -\underline{\omega}^{01} \right) \quad -(36)$$

$$\omega^{02}{}_{\lambda} = \left(\omega^{02}{}_{0}, -\underline{\omega}^{02} \right) \quad -(37)$$

$$\omega^{03}{}_{\lambda} = \left(\omega^{03}{}_{0}, -\underline{\omega}^{03} \right) \quad -(38)$$

$$g^{\lambda 1} = \left(g^{01}, \underline{g}^{1} \right) \quad -(39)$$

$$g^{\lambda 2} = \left(g^{02}, \underline{g}^{2} \right) \quad -(40)$$

$$g^{\lambda 3} = \left(g^{03}, \underline{g}^{3} \right) \quad -(41)$$

Diagonal Metric

If the metric is diagonal, then eq. (34) simplifies to:

$$\underline{T} = -\underline{\nabla} g^{00} + \left(\omega^{01}{}_{0} g^{00} - \omega^{00}{}_{1} g^{11} \right) \underline{i} + \left(\omega^{02}{}_{0} g^{00} - \omega^{00}{}_{2} g^{22} \right) \underline{j} + \left(\omega^{03}{}_{0} g^{00} - \omega^{00}{}_{3} g^{33} \right) \underline{k} \quad -(43)$$

i.e.

$$\underset{\sim}{T} = -\underline{\nabla} g^{00} + \underline{\omega} g^{00} - \left(\omega^{00}{}_{1} g^{11} \underline{i} + \omega^{00}{}_{2} g^{22} \underline{j} + \omega^{00}{}_{3} g^{33} \underline{k} \right) \quad -(44)$$

8) where:

$$\underline{\omega} = \omega^{0}{}_{0}{}^{1}\,\underline{i} + \omega^{0}{}_{0}{}^{2}\,\underline{j} + \omega^{0}{}_{0}{}^{3}\,\underline{k} \quad -(45)$$

In order to simplify eq. (44) further, it may be assumed that:

$$\omega^{00}{}_{1} = \omega^{00}{}_{2} = \omega^{00}{}_{3} = 0 \quad -(46)$$

so:

$$\underline{T} = -\underline{\nabla}\, g^{00} + \underline{\omega}\, g^{00} \quad -(47)$$

i.e.

$$\underline{g} = -c^2\underline{\nabla}\, g^{00} + c^2\underline{\omega}\, g^{00}.$$

$$\boxed{\underline{g} = -\underline{\nabla}\,\Phi + \underline{\omega}\,\Phi} \quad -(48)$$

where the gravitational potential is:

$$\boxed{\Phi = c^2 g^{00}} \quad -(49)$$

Similarly:

$$\boxed{\underline{E} = -\underline{\nabla}\,\phi + \underline{\omega}\,\phi} \quad -(50)$$

where the electric scalar potential is:

$$\boxed{\phi = cA^{(0)} g^{00}} \quad -(51)$$

It is seen that both Φ and ϕ are determined by g^{00}. In the Newton and Coulomb laws therefore, g^{00} must have a $1/r$ dependence.

Cosmological Equation

This is:

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{g} &= 4\pi G \rho_m \\ \underline{g} &= -\underline{\nabla} \Phi + \underline{\omega} \Phi \end{aligned}} \quad -(52)$$

Nanotechnology

This is governed by the use of the Coulomb law in the Schroedinger equation as follows

$$\boxed{\begin{aligned} \underline{\nabla} \cdot \underline{E} &= \rho_e / \epsilon_0 \\ \underline{E} &= -\underline{\nabla} \phi + \underline{\omega} \phi \end{aligned}} \quad -(53)$$

Both eqns. (52) and (53) can give Bernoulli - Euler resonance, i.e. spin connection resonance.

THIS GIVES RISE TO NEW COSMOLOGIES AND NANOTECHNOLOGIES

122(2): Relation between Gamma and Spin Connection

Start with the definition of the torsion tensor in Cartan geometry:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(1)$$

and use the definitions:

$$q^\kappa_a q^a_\nu = q^\kappa_\nu = \delta^\kappa_\nu \quad -(2)$$

$$q^\kappa_a q^a_\mu = q^\kappa_\mu = \delta^\kappa_\mu \quad -(3)$$

where:

$$\delta^\kappa_\mu = 1 \text{ if } \kappa = \mu \quad -(4)$$

$$\delta^\kappa_\mu = 0 \text{ if } \kappa \neq \mu. \quad -(5)$$

Multiply eq (1) by q^κ_a and use:

$$T^\kappa_{\mu\nu} = q^\kappa_a T^a_{\mu\nu} = q^\kappa_a (D \wedge q^a)_{\mu\nu} \quad -(6)$$

In the case $\quad a = \kappa \quad -(7)$

eq. (1) becomes:

$$T^\kappa_{\mu\nu} = \partial_\mu q^\kappa_\nu - \partial_\nu q^\kappa_\mu + \omega^\kappa_{\mu b} q^b_\nu - \omega^\kappa_{\nu b} q^b_\mu \quad -(8)$$

$$= \omega^\kappa_{\mu b} q^b_\nu - \omega^\kappa_{\nu b} q^b_\mu$$

$$= \omega^\kappa_{\mu\nu} - \omega^\kappa_{\nu\mu} \quad -(9)$$

$$= \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu}$$

2)

Therefore eq. (7) means:

$$\boxed{\begin{gathered}\omega^{\kappa}_{\ \mu\nu} = \Gamma^{\kappa}_{\ \mu\nu} \\ \text{if } a = \kappa\end{gathered}} \qquad -(10)$$

The form of eq. (8) needed in ECE for its engineering model is:

$$T^{\kappa\mu\nu} = \partial^{\mu} q^{\kappa\nu} - \partial^{\nu} q^{\kappa\mu} + \omega^{\kappa\mu}_{\ \ \ b}\, q^{b\nu} - \omega^{\kappa\nu}_{\ \ \ b}\, q^{b\mu} \qquad -(11)$$

where:

$$q^{\kappa\nu} = g^{\kappa\nu} \qquad -(12)$$

$$q^{\kappa\mu} = g^{\kappa\mu} \qquad -(13)$$

where $g^{\kappa\nu}$ is the inverse metric. From eq. (10):

$$\omega^{\kappa\mu}_{\ \ \ b}\, q^{b\nu} = \omega^{\kappa\mu\nu} = \Gamma^{\kappa\mu\nu} \qquad -(14)$$

so:

$$\boxed{\begin{aligned} T^{\kappa\mu\nu} &= \partial^{\mu} g^{\kappa\nu} - \partial^{\nu} g^{\kappa\mu} + \Gamma^{\kappa\mu\nu} - \Gamma^{\kappa\nu\mu} \\ &= \Gamma^{\kappa\mu\nu} - \Gamma^{\kappa\nu\mu} \end{aligned}} \qquad -(15)$$

3)

This is because by metric compatibility:

$$\partial^\mu g^{\kappa\nu} = \partial^\nu g^{\kappa\mu} = 0 \qquad (16)$$

For electrodynamics:

$$F^{\kappa\mu\nu} = \partial^\mu A^{\kappa\nu} - \partial^\nu A^{\kappa\mu} + A^{(0)}\left(\Gamma^{\kappa\mu\nu} - \Gamma^{\kappa\nu\mu}\right) \qquad (17)$$

i.e.

$$\boxed{\begin{aligned} F^{\kappa\mu\nu} &= A^{(0)}\left(\Gamma^{\kappa\mu\nu} - \Gamma^{\kappa\nu\mu}\right) \\ &= A^{(0)}\left(\omega^{\kappa\mu\nu} - \omega^{\kappa\nu\mu}\right) \end{aligned}} \qquad (18)$$

<u>The Electric Field in the Coulomb Law</u>

This is:

$$\underline{E} = E^{010}\,\underline{i} + E^{020}\,\underline{j} + E^{030}\,\underline{k} \qquad (19)$$

where

$$\boxed{\begin{aligned} E^{010} &= cA^{(0)}\left(\Gamma^{010} - \Gamma^{001}\right) \\ E^{020} &= cA^{(0)}\left(\Gamma^{020} - \Gamma^{002}\right) \\ E^{030} &= cA^{(0)}\left(\Gamma^{030} - \Gamma^{003}\right) \end{aligned}} \qquad (20)$$

Therefore <u>the electric field of the Coulomb law</u> is <u>defined by the connection</u>.

4)

We have:

$$\phi = cA^{(0)} \quad -(21)$$

where ϕ is the electric scalar potential in volts.

Therefore:

$$E^{\kappa\mu\nu} = \phi\left(\Gamma^{\kappa\mu\nu} - \Gamma^{\kappa\nu\mu}\right) \quad -(22)$$

or

$$\boxed{\begin{aligned} E^{\kappa}_{\mu\nu} &= \phi\left(\Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu}\right) \\ &= \phi\, T^{\kappa}_{\mu\nu} \end{aligned}} \quad -(23)$$

i.e. the electric field is generated directly by the spacetime torsion multiplied by the background voltage ϕ observed in the radiative corrections.

ELECTRIC POWER MAY BE OBTAINED FROM SPACETIME.

Spin Connection Resonance in the Coulomb Law

The spin connection resonance structure emerges from the tetrad postulate in the form:

$$\Gamma^{\nu}_{\mu\lambda} = q^{\nu}_{a}\,\partial_{\mu} q^{a}_{\lambda} + q^{\nu}_{a}\, q^{b}_{\lambda}\, \omega^{a}_{\mu b} \quad -(24)$$

so that spin connection resonance is introduced by the Cartan tangent spacetime. This gives an internal structure to the gamma connection, i.e.

5) $\Gamma^{\nu}_{\mu\lambda} = q^{\nu}_{a}\left(\partial_\mu q^{a}_{\lambda} + \omega^{a}_{\mu\lambda}\right),$

$$\boxed{\Gamma^{\nu}_{\mu\lambda} = \omega^{\nu}_{\mu\lambda} + q^{\nu}_{a}\,\partial_\mu q^{a}_{\lambda}} \quad -(25)$$

The electric field is:

$$E^{a}_{\mu\nu} = \phi\, T^{a}_{\mu\nu}. \quad -(26)$$

From eq. (1):

$$\boxed{E^{a}_{\mu\nu} = \phi\left(\partial_\mu q^{a}_{\nu} - \partial_\nu q^{a}_{\mu} + \omega^{a}_{\mu\nu} - \omega^{a}_{\nu\mu}\right)} \quad -(27)$$

or:

$$E^{a\mu\nu} = \phi\left(\partial^\mu q^{a\nu} - \partial^\nu q^{a\mu} + \omega^{a\mu\nu} - \omega^{a\nu\mu}\right) \quad -(28)$$

In the Coulomb law: $a = 0$ $\quad -(29)$

so:

$$E^{010} = \phi\left(\partial^1 q^{00} - \partial^0 q^{01} + \left(\omega^{010} - \omega^{001}\right)\right) \quad -(30)$$

i.e.

$$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t} + \underline{\omega}\,\phi \quad -(31)$$

where:

$$\underline{\omega} = \left(\omega^{010} - \omega^{001}\right)\underline{i} + \left(\omega^{020} - \omega^{002}\right)\underline{j} + \left(\omega^{030} - \omega^{003}\right)\underline{k}$$

6)

From eq. (20):

$$\underline{E} = \phi \, \underline{\Gamma} \quad - (33)$$

where
$$\underline{\Gamma} = \left(\Gamma^{010} - \Gamma^{001}\right) \underline{i} + \left(\Gamma^{020} - \Gamma^{002}\right) \underline{j} + \left(\Gamma^{030} - \Gamma^{003}\right) \underline{k} \quad - (34)$$

so:

$$\boxed{\begin{gathered} \underline{E} = \phi \, \underline{\Gamma} = -\underline{\nabla} \phi + \underline{\omega} \phi \\ \underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \end{gathered}} \quad - (35)$$

if $\underline{A} = \underline{0}$.

Therefore:

$$\phi \, \underline{\Gamma} = \phi \, \underline{\omega} - \underline{\nabla} \phi \quad - (36)$$

or
$$\boxed{\underline{\nabla} \phi = \phi \left(\underline{\omega} - \underline{\Gamma} \right)} \quad - (37)$$

122(3): Simplified Derivation of the Resonant Coulomb Law

The Cartan torsion is:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu \quad -(1)$$

$$= \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu\nu} - \omega^a_{\nu\mu} \quad -(2)$$

i.e.:

$$T^{a\mu\nu} = \partial^\mu q^{a\nu} - \partial^\nu q^{a\mu} + \omega^{a\mu\nu} - \omega^{a\nu\mu} \quad -(3)$$

The resonant Coulomb law is derived using:

$$a = 0, \; \mu = 1, \; \nu = 0 \quad -(4)$$

so:

$$T^{010} = \partial^1 q^{00} - \partial^0 q^{01} + \omega^{010} - \omega^{001} \quad -(5)$$

where:

$$\omega^{010} = -\omega^{001} = \omega^0{}_b q^{b0}. \quad -(6)$$

The electric field strength in volts per metre is:

$$E^{010} = cA^{(0)} T^{010} \quad -(7)$$

and the electric scalar potential is:

$$\phi = cA^{(0)} q^{00}. \quad -(8)$$

Denote:

$$\phi^{(0)} = cA^{(0)} \text{ in volts} \quad -(9)$$

The vector potential is:

$$\underline{A} = A^{(0)} \left(q^{01}\,\underline{i} + q^{02}\,\underline{j} + q^{03}\,\underline{k} \right) \quad -(10)$$

For the Coulomb law the vector potential is

, zero, so:

$$\omega^{010} = -\omega^{001} = \omega^{01}_{\ \ 0}\, q^{00}_{\ \ \cdot} \quad -(11)$$

The Cartan torsion simplifies to the vector:

$$\underline{T} = T^{010}\,\underline{i} + T^{020}\,\underline{j} + T^{030}\,\underline{k} \quad -(12)$$

and the spin connection simplifies to:

$$\underline{\omega} = \left(\omega^{010} - \omega^{001}\right)\underline{i} + \left(\omega^{020} - \omega^{002}\right)\underline{j} + \left(\omega^{030} - \omega^{003}\right)\underline{k} \quad -(13)$$

Therefore:

$$\underline{E} = -\underline{\nabla}\,\phi + \underline{\omega}\,\phi \quad -(14)$$

Finally transform to the base manifold using

$$T^{\kappa\mu\nu} = q^{\kappa}_{\ a}\, T^{a\mu\nu} \quad -(15)$$

and

$$\boxed{\begin{aligned} T^{010}(\text{base manifold}) &= q^{0}_{\ 0}\, T^{010}(\text{Cartan}) \\ &= \Gamma^{010} - \Gamma^{001} \end{aligned}} \quad -(16)$$

The connection vector is:

$$\underline{\Gamma} = \left(\Gamma^{010} - \Gamma^{001}\right)\underline{i} + \left(\Gamma^{020} - \Gamma^{002}\right)\underline{j} + \left(\Gamma^{030} - \Gamma^{003}\right)\underline{k} \quad -(17)$$

Finally:

$$\boxed{\underline{E} = cA^{(0)}\underline{\Gamma} = q^0_0 \left(-\underline{\nabla}\phi + \underline{\omega}\phi \right)} \quad -(18)$$

and:

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad -(19)$$

Eqs. (18) and (19) are self - consistently written in the same manifold.

The tetrad in eq. (18) may be found from:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(20)$$

so:

$$\boxed{g_{00} = \left(q^0_0 \right)^2} \quad -(21)$$

Therefore: $-(22)$

$$\boxed{\underline{E} = \phi^{(0)}\underline{\Gamma} = g_{00}^{1/2} \left(-\underline{\nabla}\phi + \underline{\omega}\phi \right)}$$

where g_{00} is the 00 element of the metric.

References

1) Paper 81 on www.aias.us.
2) "The Enigmatic Photon", vol 4, p. 74, eq. (3.73).

The vacuum in ECE theory is filled with the potential $A^{(0)}$ and primordial voltage $cA^{(0)}$. From ref. (2):

$$A^{(0)} = \left(\frac{e\mu_0}{4\pi\alpha} \right) \omega \quad -(1)$$

$$\boxed{\omega = \left(\frac{4\pi\alpha}{e\mu_0 \kappa} \right) B} \quad -(2)$$

Therefore an applied magnetic flux density $\underline{B}$ causes the phase of a circularly polarized e/m beam to be shifted by eq. (2). As in paper 81 this causes a change in polarization.

The charge e is defined by:

$$eA^{(0)} = \hbar \kappa \quad -(3)$$

and the vacuum wavenumber is:

$$\kappa = R_0^{1/2} \quad -(4)$$

Here R_0 is the zeroth eigenvalue of:

$$\Box A^a_\mu = R A^a_\mu \quad -(5)$$

with

$$A^a_\mu = A^{(0)} q^a_\mu \quad -(6)$$

So:

$$\boxed{\omega = \left(\left(\frac{4\pi\alpha}{\hbar\mu_0} \right) \frac{A^{(0)}}{R_0} \right) B} \quad -(7)$$

2)

This is the vacuum Faraday effect recently observed experimentally. Therefore $\underline{A^{(0)}/R_0}$ may be determined from these experiments.

The vacuum inverse Faraday effect is given by:

$$\underline{B}^{(3)*} = -ig\,\underline{A}^{(1)} \times \underline{A}^{(2)} \qquad -(8)$$

where:

$$g = \frac{e}{\hbar} = \frac{R_0^{1/2}}{A^{(0)}} \qquad -(9)$$

Therefore there should be a small magnetization of the vacuum:

$$\underline{M}^{(3)} = \frac{1}{\mu_0}\,\underline{B}^{(3)} \qquad -(10)$$

caused by a circularly polarized electromagnetic field. This should give $R_0^{1/2}/A^{(0)}$. So combining results from both experiments will give R_0 and $A^{(0)}$, the primordial curvature R_0 and the primordial potential $A^{(0)}$. These are both fundamental properties of physics.

122(5): Summary of Key Proofs

Cartan / Evans Dual Identity

Define the covariant derivative by:

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \quad -(1)$$

and define the commutator:

$$[D^\mu, D^\nu] = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\rho\sigma} [D_\rho, D_\sigma]_{HD} \quad -(2)$$

Eq (2) means that $[D^\mu, D^\nu]$ is the Hodge dual of $[D_\rho, D_\sigma]_{HD}$ in four dimensions. Similarly define:

$$T^{\lambda\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\rho\sigma} \tilde{T}^\lambda{}_{\rho\sigma} \quad -(3)$$

$$R^\kappa{}_\lambda{}^{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\rho\sigma} \tilde{R}^\kappa{}_{\lambda\rho\sigma} \quad -(4).$$

Thus:

$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(5)$$

and

$$[D^\mu, D^\nu] V^\rho = R^\rho{}_\sigma{}^{\mu\nu} V^\sigma - T^{\lambda\mu\nu} D_\lambda V^\rho \quad -(6)$$

Here:

$$\tilde{R}^\rho{}_{\sigma\mu\nu} = \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \right)_{HD} \quad -(7)$$

$$\tilde{T}^\lambda{}_{\mu\nu} = \left(\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \right)_{HD} \quad -(8)$$

Note that if $\tilde{R}^\rho{}_{\sigma\mu\nu}$ has the structure (7), then $\tilde{T}^\lambda{}_{\mu\nu}$ must have the structure (8). The Hodge duals of the curvature and torsion tensor co-exist.

2) Consider the cyclic sum of eq. (7):

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} + \tilde{R}^{\lambda}{}_{\mu\nu\rho} + \tilde{R}^{\lambda}{}_{\nu\rho\mu}$$

$$\begin{aligned}
:= & \left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho}\right)_{HD} \\
& + \left(\partial_\nu \Gamma^{\lambda}_{\rho\mu} - \partial_\rho \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\sigma}_{\nu\mu}\right)_{HD} \\
& + \left(\partial_\rho \Gamma^{\lambda}_{\mu\nu} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\rho\nu}\right)_{HD}
\end{aligned} \quad -(9)$$

Eq (9) is an identity which equates the cyclic sum on the left hand side to the sum of definitions on the right hand side.

Re-express eq. (9) as:

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu} + \tilde{R}^{\lambda}{}_{\mu\nu\rho} + \tilde{R}^{\lambda}{}_{\nu\rho\mu}$$

$$\begin{aligned}
:= & \left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} + \Gamma^{\lambda}_{\mu\sigma}\left(\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\rho\nu}\right)\right)_{HD} \\
& + \left(\partial_\rho \Gamma^{\lambda}_{\mu\nu} - \partial_\rho \Gamma^{\lambda}_{\nu\mu} + \Gamma^{\lambda}_{\rho\sigma}\left(\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\nu\mu}\right)\right)_{HD} \\
& + \left(\partial_\nu \Gamma^{\lambda}_{\rho\mu} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\nu\sigma}\left(\Gamma^{\sigma}_{\rho\mu} - \Gamma^{\sigma}_{\mu\rho}\right)\right)_{HD}
\end{aligned} \quad -(10)$$

Multiply each side by q^{a}_{λ}:

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu}\, q^{a}_{\lambda} + \ldots$$

$$\begin{aligned}
= & \left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu}\right)_{HD} q^{a}_{\lambda} \quad -(11) \\
& + \left(\Gamma^{\lambda}_{\mu\sigma}\left(\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\rho\nu}\right)\right)_{HD} q^{a}_{\lambda} + \ldots
\end{aligned}$$

3) The Hodge dual operation refers to the indices μ and ν, so:

$$\tilde{R}^{\lambda}_{\ \rho\mu\nu}\, q^{a}_{\ \lambda} + \dots = \left(\left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\rho\nu} \right) q^{a}_{\ \lambda} + \Gamma^{\lambda}_{\mu\sigma} \left(\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\rho\nu} \right) q^{a}_{\ \lambda} \right)_{HD} + \dots \quad -(12)$$

Now use the tetrad postulate:

$$\Gamma^{\lambda}_{\mu\sigma}\, q^{a}_{\ \lambda} = \partial_\mu q^{a}_{\ \sigma} + \omega^{a}_{\ \mu b}\, q^{b}_{\ \sigma} \quad -(13)$$

to obtain:

$$\tilde{R}^{\lambda}_{\ \rho\mu\nu}\, q^{a}_{\ \lambda} + \dots = \left(\left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} \right) q^{a}_{\ \lambda} + \left(\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\rho\nu} \right) \left(\partial_\mu q^{a}_{\ \sigma} + \omega^{a}_{\ \mu b}\, q^{b}_{\ \sigma} \right) \right)_{HD} + \dots \quad -(14)$$

Now relabel dummy indices in the second term on the right hand side:

$$\sigma \rightarrow \lambda. \quad -(15)$$

This is allowed because repeated indices are summed over. Therefore eq (14) becomes:

$$\tilde{R}^{\lambda}_{\ \rho\mu\nu}\, q^{a}_{\ \lambda} + \dots = \left(\left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu} \right) q^{a}_{\ \lambda} + \left(\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\lambda}_{\rho\nu} \right) \left(\partial_\mu q^{a}_{\ \lambda} + \omega^{a}_{\ \mu b}\, q^{b}_{\ \lambda} \right) \right)_{HD} + \dots \quad -(16)$$

4) By definition of the Hodge dual of a differential two-form:

$$\tilde{T}^{\kappa}{}_{\nu\rho} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\nu\rho\alpha\beta} T^{\kappa\alpha\beta} \quad -(17)$$

it follows that:

$$\tilde{T}^{a}{}_{\nu\rho} = q^{a}{}_{\kappa} \tilde{T}^{\kappa}{}_{\nu\rho} = \left(q^{a}{}_{\kappa} T^{\kappa}{}_{\nu\rho} \right)_{HD}$$
$$= \left(\left(\Gamma^{\lambda}{}_{\nu\rho} - \Gamma^{\lambda}{}_{\rho\nu} \right) q^{a}{}_{\lambda} \right)_{HD} \quad -(18)$$

i.e by definition, the Hodge dual operation always acts on a differential form and the indices of the differential form in the same manifold.

Eq. (16) is therefore:

$$\left(\partial_\mu T^{a}{}_{\nu\rho} + \omega^{a}{}_{\mu b} T^{b}{}_{\nu\rho} \right)_{HD} + \left(\partial_\rho T^{a}{}_{\mu\nu} + \omega^{a}{}_{\rho b} T^{b}{}_{\mu\nu} \right)_{HD}$$
$$+ \left(\partial_\nu T^{a}{}_{\rho\mu} + \omega^{a}{}_{\nu b} T^{b}{}_{\rho\mu} \right)_{HD} \quad -(19)$$
$$= \tilde{R}^{a}{}_{\mu\nu\rho} + \tilde{R}^{a}{}_{\rho\mu\nu} + \tilde{R}^{a}{}_{\nu\rho\mu}$$
$$= \left(D_\mu T^{a}{}_{\nu\rho} \right)_{HD} + \left(D_\rho T^{a}{}_{\mu\nu} \right)_{HD} + \left(D_\nu T^{a}{}_{\rho\mu} \right)_{HD}$$
$$= D_\mu \tilde{T}^{a}{}_{\nu\rho} + D_\rho \tilde{T}^{a}{}_{\mu\nu} + D_\nu \tilde{T}^{a}{}_{\rho\mu} \quad -(20)$$

i.e $$D \wedge \tilde{T}^{a} = \tilde{R}^{a}{}_{b} \wedge q^{b} \quad -(21)$$

where we have used

5)

$$D_\mu \tilde{T}^a{}_{\nu\rho} = \left(D_\mu T^a{}_{\nu\rho} \right)_{HD} \quad -(22)$$

and so on.

Eq. (22) follows from:

$$\tilde{T}^a{}_{\nu\rho} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\nu\rho\alpha\beta} T^{a\alpha\beta} \quad -(23)$$

Therefore:

$$D_\mu \tilde{T}^a{}_{\nu\rho} = \frac{1}{2} \|g\|^{1/2} \epsilon_{\nu\rho\alpha\beta} D_\mu T^{a\alpha\beta}$$

$$+ D_\mu \left(\frac{1}{2} \|g\|^{1/2} \epsilon_{\nu\rho\alpha\beta} \right) T^{a\alpha\beta}$$

$$= \frac{1}{2} \|g\|^{1/2} \epsilon_{\nu\rho\alpha\beta} D_\mu T^{a\alpha\beta} \quad -(24)$$

$$= \left(D_\mu T^a{}_{\nu\rho} \right)_{HD} \quad \text{QED.}$$

because by metric compatibility:

$$D_\mu \left(\frac{1}{2} \|g\|^{1/2} \epsilon_{\nu\rho\alpha\beta} \right) = 0. \quad -(25)$$

Here $|g|$ is the determinant of the metric and each element of the metric is metric compatible. Therefore the covariant derivative of each element vanishes. The covariant derivative of the unit tensor $\epsilon_{\nu\rho\alpha\beta}$ vanishes by definition.

Therefore eq. (21) is:

6)

$$\boxed{D_\mu T^{a\mu\nu} = R^{a\ \mu\nu}_{\ \mu}} \quad -(26)$$

which is the Cartan Evans dual identity Q.E.D.

The original Cartan identity is:

$$\boxed{D_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^{a\ \mu\nu}_{\ \mu}} \quad -(27)$$

i.e.:

$$D_\mu T^a_{\ \nu\rho} + D_\rho T^a_{\ \mu\nu} + D_\nu T^a_{\ \rho\mu}$$
$$= R^a_{\ \mu\nu\rho} + R^a_{\ \rho\mu\nu} + R^a_{\ \nu\rho\mu} \quad -(28)$$

Using the fact that:

$$D_\mu \tilde{T}^{a\mu\nu} = \left(D_\mu T^{a\mu\nu}\right)_{HD} \quad -(29)$$

and

$$\tilde{R}^{a\ \mu\nu}_{\ \mu} = \left(R^{a\ \mu\nu}_{\ \mu}\right)_{HD} \quad -(30)$$

it is seen that the Cartan identity (27) is the Hodge dual of the Cartan Evans dual identity (26).

This is a profoundly important result which shows the central importance of torsion.

)

<u>Proof of Eq. (29)</u>

By definition of the Hodge dual operation:

$$D_\mu \tilde{T}^{a\mu\nu} = D_\mu \left(\frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} T^a_{\alpha\beta} \right) \quad -(31)$$

$$= \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} D_\mu T^a_{\alpha\beta}$$

by metric compatibility. The Hodge dual operation in the base manifold, applies to the indices of $T^{a\mu\nu}$. Therefore: i.e. to the indices μ and ν of $T^{a\mu\nu}$.

$$\left(D_\mu T^{a\mu\nu} \right)_{HD} := \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\alpha\beta} D_\mu T^a_{\alpha\beta} \quad -(32)$$

so:

$$\boxed{D_\mu \tilde{T}^{a\mu\nu} = \left(D_\mu T^{a\mu\nu} \right)_{HD}} \quad -(33)$$

<u>Q.E.D.</u>

<u>Cartan-Evans</u> <u>Identity</u> in the <u>Base Manifold</u>

In eq. (26):

$$T^{a\mu\nu} = q^a_{\ \kappa} T^{\kappa\mu\nu} \quad -(34)$$

$$R^{a\ \mu\nu}_{\ \mu} = q^a_{\ \kappa} R^{\kappa\ \mu\nu}_{\ \mu} \quad -(35)$$

Using the Leibnitz Theorem and tetrad postulate:

$$\boxed{D_\mu T^{\kappa\mu\nu} = R^{\kappa\ \mu\nu}_{\ \mu}} \quad -(36)$$

<u>which is the equation</u> that <u>proves the incorrectness of the Einstein</u> <u>field equation.</u>

8) <u>Details of Eq. (16)</u>

Written out in full the equation is:

$$\tilde{R}^{\lambda}{}_{\rho\mu\nu}\, q^{a}_{\lambda} + \tilde{R}^{\lambda}{}_{\nu\rho\mu}\, q^{a}_{\lambda} + \tilde{R}^{\lambda}{}_{\mu\nu\rho}\, q^{a}_{\lambda}$$

$$= \left(\left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu}\right) q^{a}_{\lambda} + \left(\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\lambda}_{\rho\nu}\right)\left(\partial_\mu q^{a}_{\lambda} + \omega^{a}_{\mu b}\, q^{b}_{\lambda}\right)\right)_{HD}$$

$$+ \left(\left(\partial_\rho \Gamma^{\lambda}_{\mu\nu} - \partial_\rho \Gamma^{\lambda}_{\nu\mu}\right) q^{a}_{\lambda} + \left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right)\left(\partial_\rho q^{a}_{\lambda} + \omega^{a}_{\rho b}\, q^{b}_{\lambda}\right)\right)_{HD}$$

$$+ \left(\left(\partial_\nu \Gamma^{\lambda}_{\rho\mu} - \partial_\nu \Gamma^{\lambda}_{\mu\rho}\right) q^{a}_{\lambda} + \left(\Gamma^{\lambda}_{\rho\mu} - \Gamma^{\lambda}_{\mu\rho}\right)\left(\partial_\nu q^{a}_{\lambda} + \omega^{a}_{\nu b}\, q^{b}_{\lambda}\right)\right)_{HD} \quad -(37)$$

In which:

$$\left(\partial_\mu T^{a}_{\nu\rho} + \omega^{a}_{\mu b} T^{b}_{\nu\rho}\right)_{HD} \quad -(38)$$

$$= \left(\left(\partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\mu \Gamma^{\lambda}_{\rho\nu}\right) q^{a}_{\lambda} + \left(\Gamma^{\lambda}_{\nu\rho} - \Gamma^{\lambda}_{\rho\nu}\right)\left(\partial_\mu q^{a}_{\lambda} + \omega^{a}_{\mu b}\, q^{b}_{\lambda}\right)\right)_{HD}$$

$$= \left(D_\mu T^{a}_{\nu\rho}\right)_{HD}$$

$$= D_\mu \tilde{T}^{a}_{\nu\rho}$$

and so on in cyclic permutation.

Notes 122(6): Second Proof of the Cartan Evans Dual Identity.

A differential p form is an antisymmetric $(0, p)$ tensor. The differential 2-form is therefore:

$$F_{\mu\nu} = - F_{\nu\mu}. \quad -(1)$$

The commutator of covariant derivatives is a 2-form operator. In four dimensions its Hodge dual is another 2-form operator. Denote this by $[D_\mu, D_\nu]_{HD}$. Thus:

$$[D_\mu, D_\nu] = - [D_\nu, D_\mu] \quad -(2)$$

and

$$[D_\mu, D_\nu]_{HD} = - [D_\nu, D_\mu]_{HD}. \quad -(3)$$

The Hodge dual operator is antisymmetric, and may operate on any tensor.

Let the Hodge dual operator be an operator on the four vector V^ρ. Furthermore, define the covariant derivative of the Hodge dual operator by:

$$D_\mu V^\rho = \partial_\mu V^\rho + \Lambda^\rho_{\mu\lambda} V^\lambda \quad -(4)$$

where $\Lambda^\rho_{\mu\lambda}$ is the connection. Therefore:

$$[D_\mu, D_\nu]_{HD} V^\rho = D_\mu (D_\nu V^\rho) - D_\nu (D_\mu V^\rho) \quad -(5)$$

Eq. (5) is necessary and sufficient to

2) define the Hodge dual of the curvature tensor and the Hodge dual of the torsion tensor. From eq. (5):

$$[D_\mu, D_\nu]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma\mu\nu} V^\sigma - \tilde{T}^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(6)$$

where:

$$\tilde{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \Lambda^\rho_{\nu\sigma} - \partial_\nu \Lambda^\rho_{\mu\sigma} + \Lambda^\rho_{\mu\lambda}\Lambda^\lambda_{\nu\sigma} - \Lambda^\rho_{\nu\lambda}\Lambda^\lambda_{\mu\sigma} \quad -(7)$$

and

$$\tilde{T}^\lambda{}_{\mu\nu} = \Lambda^\lambda_{\mu\nu} - \Lambda^\lambda_{\nu\mu} . \quad -(8)$$

Note that the definitions (7) and (8) follow directly from eq. (4). No other assumption is needed, for example no assumption about the symmetry of the connection has been made, and no assumption has been made about metric compatibility. As soon as we define the covariant derivative we define both curvature and torsion.

It is incorrect to eliminate torsion.

Now consider the cyclic sum:

$$\sum_{\text{cyclic}} := \tilde{R}^\rho{}_{\sigma\mu\nu} + \tilde{R}^\rho{}_{\nu\sigma\mu} + \tilde{R}_{\mu\nu\sigma} \quad -(9)$$

This cyclic sum is by definition identically equal to the cyclic sum of the right hand sides of eq. (7). Rearranging terms and re-labelling indices:

$$
\begin{aligned}
\tilde{R}^{\lambda}{}_{\rho\mu\nu} + \tilde{R}^{\lambda}{}_{\mu\nu\rho} + \tilde{R}^{\lambda}{}_{\nu\rho\mu}
:= {} & \partial_\mu \Lambda^{\lambda}_{\nu\rho} - \partial_\mu \Lambda^{\lambda}_{\rho\nu} + \Lambda^{\lambda}_{\mu\sigma}\left(\Lambda^{\sigma}_{\nu\rho} - \Lambda^{\sigma}_{\rho\nu}\right) \\
& + \partial_\rho \Lambda^{\lambda}_{\mu\nu} - \partial_\rho \Lambda^{\lambda}_{\nu\mu} + \Lambda^{\lambda}_{\rho\sigma}\left(\Lambda^{\sigma}_{\mu\nu} - \Lambda^{\sigma}_{\nu\mu}\right) \quad -(10) \\
& + \partial_\nu \Lambda^{\lambda}_{\rho\mu} - \partial_\nu \Lambda^{\lambda}_{\mu\rho} + \Lambda^{\lambda}_{\nu\sigma}\left(\Lambda^{\sigma}_{\rho\mu} - \Lambda^{\sigma}_{\mu\rho}\right)
\end{aligned}
$$

This is the tensorial format of the Cartan-Evans dual identity.

It is seen that the identity is proven directly from the definition of the covariant derivative in eq. (4). No other assumption is used. It is therefore a rigorously correct identity. The rest of the proof develops eq. (10) into the elegant differential geometry. This is done by multiplying both sides of eq. (10) by the Cartan tetrad q^a_λ and using the tetrad postulate. The latter is always true because it follows from the fact that a complete vector field is a constant under coordinate transformation.

4)

Therefore we obtain:

$$\left(\tilde{R}^{\lambda}_{\ \rho\mu\nu} + \tilde{R}^{\lambda}_{\ \mu\nu\rho} + \tilde{R}^{\lambda}_{\ \nu\rho\mu}\right) q^{a}_{\ \lambda}$$

$$:= \left(\partial_\mu \Lambda^{\lambda}_{\ \nu\rho} - \partial_\mu \Lambda^{\lambda}_{\ \rho\nu}\right) q^{a}_{\ \lambda} + \left(\Lambda^{\lambda}_{\ \nu\rho} - \Lambda^{\lambda}_{\ \rho\nu}\right)\left(\partial_\mu q^{a}_{\ \lambda} + \omega^{a}_{\ \mu b}\, q^{b}_{\ \lambda}\right)$$
$$+ \left(\partial_\rho \Lambda^{\lambda}_{\ \mu\nu} - \partial_\rho \Lambda^{\lambda}_{\ \nu\mu}\right) q^{a}_{\ \lambda} + \left(\Lambda^{\lambda}_{\ \mu\nu} - \Lambda^{\lambda}_{\ \nu\mu}\right)\left(\partial_\rho q^{a}_{\ \lambda} + \omega^{a}_{\ \rho b}\, q^{b}_{\ \lambda}\right)$$
$$+ \left(\partial_\nu \Lambda^{\lambda}_{\ \rho\mu} - \partial_\nu \Lambda^{\lambda}_{\ \mu\rho}\right) q^{a}_{\ \lambda} + \left(\Lambda^{\lambda}_{\ \rho\mu} - \Lambda^{\lambda}_{\ \mu\rho}\right)\left(\partial_\nu q^{a}_{\ \lambda} + \omega^{a}_{\ \nu b}\, q^{b}_{\ \lambda}\right)$$
$$\text{— (11)}$$

To arrive at this equation the tetrad postulate has been used as follows:

$$D_\mu q^{a}_{\ \lambda} = \partial_\mu q^{a}_{\ \lambda} + \omega^{a}_{\ \mu b}\, q^{b}_{\ \lambda} - \Lambda^{\sigma}_{\ \mu\lambda}\, q^{a}_{\ \sigma}$$
$$= 0 \qquad \text{— (12)}$$

Note carefully that the connection $\Lambda^{\sigma}_{\ \mu\lambda}$ appears in the tetrad postulate (12) because the same connection appears in the covariant derivative (4) by definition.

Now define the curvature and torsion forms by:

$$\tilde{R}^{a}_{\ \mu\nu\rho} = \tilde{R}^{\lambda}_{\ \mu\nu\rho}\, q^{a}_{\ \lambda} \qquad \text{— (13)}$$
$$\tilde{T}^{a}_{\ \mu\nu} = \tilde{T}^{\lambda}_{\ \mu\nu}\, q^{a}_{\ \lambda} \qquad \text{— (14)}$$

5) These are basic definitions of differential or Cartan geometry. Furthermore, the covariant derivative of the torsion form is defined as:

$$D_\mu \tilde{T}^a{}_{\nu\rho} = \partial_\mu \tilde{T}^a{}_{\nu\rho} + \omega^a{}_{\mu b} \tilde{T}^b{}_{\nu\rho}. \quad -(15)$$

where:

$$\tilde{T}^a{}_{\nu\rho} = \Lambda^a{}_{\nu\rho} - \Lambda^a{}_{\rho\nu}. \quad -(16)$$

From eq. (14) in eq. (15):

$$D_\mu \tilde{T}^a{}_{\nu\rho} = D_\mu \left(q^a{}_\lambda \tilde{T}^\lambda{}_{\nu\rho} \right)$$

$$= \partial_\mu \left(q^a{}_\lambda \tilde{T}^\lambda{}_{\nu\rho} \right) + \omega^a{}_{\mu b} q^b{}_\lambda \tilde{T}^\lambda{}_{\nu\rho}$$

$$= \tilde{T}^\lambda{}_{\nu\rho} \partial_\mu q^a{}_\lambda + q^a{}_\lambda \partial_\mu \tilde{T}^\lambda{}_{\nu\rho} + \omega^a{}_{\mu b} q^b{}_\lambda \tilde{T}^\lambda{}_{\nu\rho}$$

$$-(17)$$

where the Leibnitz theorem has been used.

Using eq. (14), (16) and (17) in eq. (11) we obtain:

$$D_\mu \tilde{T}^a{}_{\nu\rho} + D_\rho \tilde{T}^a{}_{\mu\nu} + D_\nu \tilde{T}^a{}_{\rho\mu}$$

$$:= \tilde{R}^a{}_{\mu\nu\rho} + \tilde{R}^a{}_{\rho\mu\nu} + \tilde{R}^a{}_{\nu\rho\mu}$$

$$-(18)$$

6)

In the notation of differential geometry this is:

$$\boxed{D \wedge \tilde{T}^a := \tilde{R}^a{}_b \wedge q^b} \quad -(19)$$

which is the Evans - Cartan dual identity, Q.E.D.

A particularly useful form of eq. (19) is:

$$\boxed{D_\mu \tilde{T}^{a\mu\nu} := \tilde{R}^{a}{}_{\mu}{}^{\mu\nu}} \quad -(20)$$

Note that the original torsion form and curvature form appear in eq. (20). By use of the tetrad postulate, the index a may be eliminated from eq. (20) to give:

$$\boxed{D_\mu \tilde{T}^{\kappa\mu\nu} = \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad -(21)$$

In eq. (21), the connections to be used are $\Gamma^\kappa_{\mu\nu}$ connections, because:

$$T^\kappa_{\mu\nu} = \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu}, \quad -(22)$$

$$R^\lambda{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$-(23)$$