1) 121(1): Background to The Noether Theorem

The background to the Noether theorem is well described by Ryder ("Quantum Field Theory", chapter 3). This background is however given in terms of special relativity, but does provide useful insight to the meaning of the theorem and its construction in terms of symmetry. This is a theorem based on a lagrangian formalism. Ryder introduces the Noether Theorem in terms of the scalar field ϕ and the Klein Gordon equation. He uses the variational principle, and the action is unchanged under a change of x^μ and ϕ. The action is invariant under a group of transformations on x^μ and ϕ. As a consequence, there exist one or more conserved quantities which are invariant under the transformation. This is Noether's Theorem. Conservation of energy, momentum, angular momentum, charge, isospin, colour, etc. are based on this theorem.

Therefore it is important to define the generalized Noether theorem in the context of ECE theory. This would give the fundamental conservation theorem for generally covariant unified field theory, and this result would also be the first law of thermodynamics in GCUFT.

2)

It is natural to base such a theorem on the tetrad postulate:

$$D_\mu q^a_\nu = 0 \quad -(1)$$

which is invariant under the general coordinate transformation:

$$\left(D_\mu q^a_\nu\right)' = 0 \quad -(2)$$

in any frame of reference.

<u>Proof of the Tetrad Postulate</u>

The proof relies on the covariance of the complete vector field:

$$DX = D_\mu X^\nu dx^\mu \otimes \partial_\nu \quad -(3)$$

$$= \left(\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda\right) dx^\mu \otimes \partial_\nu$$

This is expressed in the mixed basis as:

$$DX = D_\mu X^a dx^\mu \otimes \hat{e}_a \quad -(4)$$

$$= \left(\partial_\mu X^a + \omega^a_{\mu b} X^b\right) dx^\mu \otimes \hat{e}_a .$$

Then use:

$$\hat{e}_a = q^\sigma_a \partial_\sigma \quad -(5)$$

$$X^a = q^a_\nu X^\nu . \quad -(6)$$

Therefore eq. (4) becomes:

$$DX = \left(\partial_\mu \left(q^a_\nu X^\nu\right) + \omega^a_{\mu b} q^b_\nu X^\nu\right) dx^\mu \otimes \left(q^\sigma_a \partial_\sigma\right) \quad -(7)$$

$$= \boxed{q^{\sigma}_{\ a}\left(\left(\partial_{\mu} q^{a}_{\ \nu}\right) X^{\nu} + q^{a}_{\ \nu}\left(\partial_{\mu} X^{\nu}\right) + \omega^{a}_{\ \mu b}\, q^{b}_{\ \lambda} X^{\lambda}\right) dx^{\mu} \otimes \partial_{\sigma}} \quad -(8)$$

Summation occurs over the repeated σ indices as indicated. Therefore:

$$DX = \left(q^{\sigma}_{\ a} q^{a}_{\ \nu}\, \partial_{\mu} X^{\nu} + q^{\sigma}_{\ a} X^{\nu} \partial_{\mu} q^{a}_{\ \nu} + q^{\sigma}_{\ a}\, \omega^{a}_{\ \mu b}\, q^{b}_{\ \lambda} X^{\lambda}\right) dx^{\mu} \otimes \partial_{\sigma} \quad -(9)$$

where $\quad q^{\sigma}_{\ a} q^{a}_{\ \nu} = \delta^{\sigma}_{\ \nu} \quad -(10)$

with:

$$\left.\begin{aligned} \delta^{\sigma}_{\ \nu} &= 1 \text{ if } \sigma = \nu \\ &= 0 \text{ if } \sigma \neq \nu \end{aligned}\right\} \quad -(11)$$

Now compare eqs. (3) and (9) with:

$$\sigma = \nu, \quad -(12)$$

so eq. (9) is:

$$DX = \left(\partial_{\mu} X^{\nu} + q^{\nu}_{\ a} X^{\lambda} \partial_{\mu} q^{a}_{\ \lambda} + q^{\nu}_{\ a}\, \omega^{a}_{\ \mu b}\, q^{b}_{\ \lambda} X^{\lambda}\right) dx^{\mu} \otimes \partial_{\nu} \quad -(13)$$

Therefore:

4)

$$\Gamma^{\nu}_{\mu\lambda} = q^{\nu}_{a} \partial_\mu q^{a}_{\lambda} + q^{\nu}_{a} \omega^{a}_{\mu b} q^{b}_{\lambda} \quad -(14)$$

Multiply both sides by q^{a}_{ν} to find:

$$\partial_\mu q^{a}_{\lambda} + \omega^{a}_{\mu b} q^{b}_{\lambda} = q^{a}_{\nu} \Gamma^{\nu}_{\mu\lambda} \quad -(15)$$

i.e.:

$$\boxed{\partial_\mu q^{a}_{\lambda} + \omega^{a}_{\mu b} q^{b}_{\lambda} - q^{a}_{\nu} \Gamma^{\nu}_{\mu\lambda} = 0} \quad -(16)$$

This is the tetrad postulate:

$$D_\mu q^{a}_{\lambda} = 0 \quad -(17)$$

Q.E.D.

Note: the correct rule for working out $q^{\nu}_{a} q^{a}_{\nu}$ is:

$$q^{\nu}_{a} q^{a}_{\nu} = \delta^{\nu}_{\nu} \quad -(18)$$

This convention means that the product is 1 because the two indices of the Kronecker Delta are the same.

Eq. (17) follows from eq. (16) using the rule for the covariant derivative of a mixed index tensor.

5) Eq. (18) is therefore eq. (10) (Carroll chapter 3) when $\sigma = \nu$!

The coordinate transformation of eq. (17) produces:

$$\boxed{\left(D_\mu q^a_\nu\right)' = \left(\Lambda^{a'}{}_a \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}}\right) D_\mu q^a_\nu = 0}$$

— (19)

where $\Lambda^{a'}{}_a$ is the Lorentz transform. Therefore the tetrad postulate is always true in any frame of reference.

It is therefore an invariant and useful property (19) to construct the generalized Noether Theorem, the generally covariant conservation theorem. The suggests for example that the energy-momentum density tensor is in general:

$$E^a_\mu = E^{(0)} q^a_\mu \quad -(20)$$

This form is also suggested by the fact that

6) in conventional Noether theorem theory (Ryder, chapter 3):

$$\theta^{\nu}_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \delta^{\mu}_{\nu} \mathcal{L} \quad -(21)$$

where $\mathcal{L}$ is the la gran. density and ϕ a scalar field. More generally there is a conserved current J^{μ}_{ν} whose existence follows from the invariance of the action under:

$$\Delta x^{\mu} = X^{\mu}_{\nu} \delta\omega^{\nu} \quad -(22)$$

$$\Delta \phi = \Phi_{\mu} \delta\omega^{\mu}. \quad -(23)$$

In flat spacetime:

$$\partial_\mu J^{\mu}_{\nu} = 0 \quad -(24)$$

but otherwise:

$$D_\mu J^{\mu}_{\nu} = 0. \quad -(25)$$

It is reasonable to suppose as in paper 116 that eq. (25) is a special case of:

$$D_\mu J^{a}_{\nu} = 0 \quad -(26)$$

where

$$J^{a}_{\nu} = J^{(0)} q^{a}_{\nu} \quad -(27)$$