1)

Note 119(5)

Some Solar System Data and Precessions due to Gravitomagnetism

These data will be useful in calculations of gravitomagnetic effects.

Sun

Equatorial radius $= 6.955 \times 10^{8}$ m
Mass $= 1.9891 \times 10^{30}$ kgm
Equatorial surface gravity $= 27.94g = 274.0$ m s^{-2}
Rotational linear velocity at equator
$= 7.284 \times 10^{3}$ km per hour
$= 7.284 \times 10^{6} / 3600$ m s^{-1}
$= 2.023 \times 10^{3}$ m s^{-1}

Sun's orbital vel. around centre of galaxy $\sim 2.20 \times 10^{5}$ m/s
Sun's vel. relative to other stars $= 2 \times 10^{4}$ m s^{-1}

Planetary Data : Table 1

Planet	R (m)	v (m s^{-1})	M (kgm)
Mercury	0.6×10^{11}	4.79×10^{4}	3.30×10^{23}
Venus	1.04×10^{11}	3.50×10^{4}	4.87×10^{24}
Earth	1.49×10^{11}	2.98×10^{4}	5.98×10^{24}
Mars	2.24×10^{11}	2.41×10^{4}	6.42×10^{23}
Jupiter	7.75×10^{11}	1.31×10^{4}	1.90×10^{27}
Saturn	1.42×10^{12}	9.69×10^{3}	5.68×10^{26}
Uranus	2.92×10^{12}	6.81×10^{3}	8.37×10^{25}
Neptune	4.47×10^{12}	5.43×10^{3}	1.02×10^{26}

R = distance from sun
v = orbital velocity around the sun
M = mass

Therefore

Spin velocity of sun $= 2023$ m s^{-1}
Spin velocity of earth $= 512$ m s^{-1}

Mass of moon $= 7.3 \times 10^{21}$ kgm
Earth moon distance $= 3.84 \times 10^{8}$ m

2)

Angular Momenta ($kgm\,m^2\,s^{-1}$)

Solar System	Orbital	Spin
Sun	—	1.12×10^{42}
Mercury	9.48×10^{38}	
Venus	1.77×10^{40}	
Earth	2.66×10^{40}	7.08×10^{33}
Mars	3.47×10^{40}	
Jupiter	1.93×10^{43}	
Saturn	7.82×10^{42}	
Uranus	1.66×10^{42}	
Neptune	2.48×10^{42}	

The orbital angular momentum is:

$$\underline{L}\,(\text{orbital}) = MRv\,(\text{orbital}) \quad -(1)$$

The spin angular momentum is:

$$\underline{L}\,(\text{spin}) = \frac{2}{5} Mrv\,(\text{spin}) \quad -(2)$$

In eq. (2) the earth and sun are regarded as spheres in the first approximation.

Gravitomagnetic Precession Angles

These can be calculated from the equation:

$$\boxed{\Omega = \frac{G}{c^2 r^3} L} \quad -(3)$$

where r is the distance between the centre of the reference frame of the angular momentum, and a reference point in space (e.g. a satellite).

Objects with Respect to the Earth

The earth spins on its own axis and orbits around the sun, so it generates spin and orbital angular momentum. In the case of gravity probe B, the satellite was in low orbit, so the spin angular momentum of the earth is the relevant one (paper 117). However, the orbital angular momentum of the earth is a million or so times larger than its spin angular momentum. So precessional angle generated by the orbital angular momentum of the earth in its orbit around the sun.

At present nothing is known experimentally about this phenomenon. In order to interpret it correctly the fundamental relation between $\underline{\Omega}$ and $\underline{g}$ must be used:

$$\underline{\Omega} = -\frac{1}{c^2}\,\underline{v} \times \underline{g} \quad -(4)$$

where:

$$\underline{g} = -MG\,\underset{\sim}{r} \,/\, r^3 \quad -(5)$$

These two equations give eq. (3), and provide the correct way to interpret eq. (3). In eq. (5) $\underline{g}$ is the Earth's acceleration due to gravity so r is the radius of the earth and M is the mass of the earth.

The velocity $\underline{v}$ in Gravity Probe B

Therefore $$\Omega = \frac{MG}{c^2 r^2}\,v \quad -(6)$$

In the Gravity Probe B experiment, v is the spin velocity of the earth:

$$v\,(\text{earth spin}) = 512\ m\,s^{-1}$$

and r is the distance of the satellite from the centre of the earth. The expression used in paper 117 was:

$$\Omega = \frac{2}{5}\frac{MGR}{c^2 r^3}\,v \quad -(7)$$

with:

$$L = \frac{2}{5}MRv \quad -(8)$$

and with R as the earth's radius. The factor $2/5$ entered from the assumption that the earth is a perfect sphere in the first approximation. So the interpretation of eq. (3) for Gravity Probe B was that L is given by the spin angular momentum of the earth, and that r was the distance between the satellite and the centre of the earth. There is therefore an influence of the earth's gravitomagnetic field on the satellite.

<u>The Velocity $\underline{v}$ in General</u>

The fundamental origin of $\underline{v}$ is given by the fact that eq. (4) is the precise analogue of:

$$\underline{B} = -\frac{1}{c^2}\underline{v}\times\underline{E} \quad -(9)$$

which is the inverse Lorentz transform when $v \ll c$. Therefore v is the velocity of one frame of reference with respect to another. For any acceleration due to gravity $\underline{g}$ in the presence of v, there is an observable $\underline{\Omega}$. Its magnitude is:

$$|\underline{\Omega}| = -\frac{1}{c^2}|\underline{v}\times\underline{g}| \quad -(10)$$

The earth's g is $9.81\ ms^{-2}$ directed from a point on the surface to the centre of the earth. So if $\underline{v}$ is not parallel to $\underline{g}$ there is an observable $\underline{\Omega}$. It is maximized when $\underline{v}\times\underline{g}$ is maximized, i.e. when v is perpendicular to $\underline{g}$. Its order of magnitude is:

$$\boxed{\Omega = -vg/c^2} \quad -(11)$$

and since g is negative by convention, Ω is positive:

$$|\Omega| = vg/c^2 \quad -(12)$$

in radians per second. Therefore:

$$|\Omega| = 6.51 \times 10^{12} \, vg / c^2 \quad - (13)$$

in arcseconds per year, i.e.

$$\boxed{|\Omega| = 7.10 \times 10^{-4} \, v} \quad - (14)$$

in arcseconds per year.

Equinoctial Procession

This is observed to be about 50 arcseconds per year and is due to a velocity $\underline{v}$ relative to the earth in $\underline{g}$. This is because the precession is measured on the surface of the earth. Therefore the observation measures:

$$gv = |\underline{v} \times \underline{g}|, \quad - (15)$$

the modulus of the vector product of $\underline{v}$ and $\underline{g}$. In general:

$$\underline{\Omega} = -\frac{1}{c^2} \underline{v} \times \underline{g} = -\frac{vg}{c^2} \sin\theta \, \underline{k} \quad - (16)$$

where $\underline{k}$ is in the plane perpendicular to $\underline{v}$ and $\underline{g}$, so:

$$|\underline{\Omega}| = \frac{vg}{c^2} \sin\theta \quad - (17)$$

so in general eq. (14) is:

$$\boxed{|\Omega| = 7.10 \times 10^{-4} \, v \sin\theta} \quad - (18)$$

in arcseconds per year. If $|\Omega|$ is 50 arcseconds a year then:

$$\boxed{v \sin\theta = 7.042 \times 10^{4} \text{ metres per second}} \quad - (19)$$

6)

Discussion

The type of precession depends on the magnitude and direction of $\underline{v}$ with respect to $\underline{g}$. So if v increases and changes direction, so will the observed precession. This is happening with the equinoctial precession. The precession for the earth's spin is:

$$\boxed{\begin{aligned}|\Omega| &= 7.10 \times 10^{-4} \times 512 \\ &= 0.36 \text{ arcseconds a year}\end{aligned}}$$

which is very close to the result of LAGEOS.

a) If v is interpreted as the earth's orbital velocity around the sun, then:

$$\boxed{|\Omega| = 21.2 \sin\theta \text{ arcseconds a year}}$$

If $\underline{v} \perp \underline{g}$ then this is 21.2 arcseconds a year.

b) The precession due to the basic solar motion is calculated with:

$$v = 2 \times 10^{4} \text{ ms}^{-1}$$

i.e

$$\boxed{\Omega = 14.2 \sin\theta \text{ arcseconds a year}}$$

c) The precession due to the sun's orbital velocity around the centre of the galaxy is given by:

$$v = 2.2 \times 10^{5} \text{ ms}^{-1}$$

i.e

$$\boxed{\Omega = 156.2 \sin\theta \text{ arcseconds a year}}$$

This is 50 arcseconds a year if:

$$\boxed{\begin{aligned}\sin\theta &= 0.32 \\ \theta &= 18.7^\circ\end{aligned}}$$