49(1): Basics of the Lorentz Transformation.

Consider two inertial frames K and K' with a relative velocity $\underline{v}$. The time and space coordinates of a point in K are (ct, x, y, z) and (ct', x', y', z'). It is postulated that the speed of light is constant. Lorentz assumed that:

$$c^2t'^2 - (x'^2 + y'^2 + z'^2) = \lambda^2 \left(c^2t^2 - (x^2 + y^2 + z^2)\right). \quad -(1)$$

Now denote:

$$x_0 = ct, \; x_1 = Z, \; x_2 = X, \; x_3 = Y \quad -(2)$$

(Jackson, 3rd. ed., p. 525). Then:

$$x_0' = \gamma(x_0 - \beta x_1) \quad -(3)$$

$$x_1' = \gamma(x_1 - \beta x_0) \quad -(4)$$

$$x_2' = x_2 \quad -(5)$$

$$x_3' = x_3. \quad -(6)$$

Similarly:

$$x_0 = \gamma(x_0' + \beta x_1') \quad -(7)$$

$$x_1 = \gamma(x_1' + \beta x_0') \quad -(8)$$

$$x_2 = x_2' \quad -(9)$$

$$x_3 = x_3'. \quad -(10)$$

Eqs. (3) to (6) are the Lorentz transform, and eqs. (7) to (10) are the inverse Lorentz transform. Therefore:

$$ct' = \gamma(ct - \beta Z) \quad -(11)$$

$$Z' = \gamma(Z - \beta ct) \quad -(12)$$

Here:

2) $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad \beta = \frac{v}{c}$ — (13)

Consider the four vector x^μ:

$$x^\mu = (ct, 0, 0, Z) \quad -(14)$$

and the unit vector describing the basis:

$$e_\mu = (e_0, 0, 0, -e_3) \quad -(15)$$

then the scalar:

$$V = x^\mu e_\mu = x^{\mu'} e_{\mu'} \quad -(16)$$

is the complete vector field, which is constant under the Lorentz transform. From eqs. (14) and (15):

$$V = cte_0 - Z e_3 \quad -(17)$$

so

$$V' = ct' e_0' - Z' e_3' \quad -(18)$$

$$= \gamma(ct - \beta Z) e_0' - \gamma(Z - \beta ct) e_3'.$$

In general the rule for transforming x^μ is given by (Carroll in his page 44):

$$x^{\mu'} = \left(\frac{\partial x^{\mu'}}{\partial x^\mu}\right) x^\mu \quad -(19)$$

and

$$e_{\mu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right) e_\mu. \quad -(20)$$

So the transformation matrix in eq. (20) is the inverse of the transformation matrix in eq. (19). If the Lorentz transform matrix is used in eq. (19), the inverse Lorentz transform matrix is used in eq. (20).

3) Eq. (11) can be expressed as:

$$ct' = ct\cosh\phi - z\sinh\phi \qquad -(21)$$

$$z' = -ct\sinh\phi + z\cosh\phi \qquad -(22)$$

where:

$$v = \frac{z}{t} = \tanh\phi. \qquad -(23)$$

So:

$$\begin{bmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad -(24)$$

Therefore:

$$\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right) = \begin{bmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad -(25)$$

and

$$\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) = \begin{bmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad -(26)$$

Thus:

$$\begin{bmatrix} e_0' \\ -e_1' \\ -e_2' \\ -e_3' \end{bmatrix} = \begin{bmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_0 \\ -e_1 \\ -e_2 \\ -e_3 \end{bmatrix} \qquad -(27)$$

i.e.

$$\boxed{\begin{aligned} e_0' &= e_0\cosh\phi - e_1\sinh\phi & -(28) \\ e_1' &= -e_0\sinh\phi + e_1\cosh\phi & -(29) \end{aligned}}$$

4) From eq. (24):

$$x_0' = x_0 \cosh\phi - x_1 \sinh\phi \quad -(30)$$
$$x_1' = -x_0 \sinh\phi + x_1 \cosh\phi . \quad -(31)$$

Thus:

$$\underline{V} = x^\mu e_\mu = x_0 e_0 - x_1 e_1$$
$$= x^{\mu'} e_\mu' = x_0' e_0' - x_1' e_1'$$
$$= (x_0 \cosh\phi - x_1 \sinh\phi)(e_0 \cosh\phi - e_1 \sinh\phi)$$
$$- (-x_0 \sinh\phi + x_1 \cosh\phi)(-e_0 \sinh\phi + e_1 \cosh\phi)$$
$$= x_0 e_0 (\cosh^2\phi - \sinh^2\phi) - x_1 e_1 (\cosh^2\phi - \sinh^2\phi) \quad -(32)$$
$$= x_0 e_0 - x_1 e_1 \qquad QED$$

The complete vector field $\underline{V}$ is a constant. Eq. (25) is the Lorentz boost along Z, eq. (36) is the inverse Lorentz boost. Eq. (27) show that the components transform under the inverse Lorentz boost.

Vector Notation

The complete vector field is:

$$\underline{V} = ct\underline{e}_0 - Z\underline{k} \quad -(33)$$

From eq. (29)

$$\boxed{\underline{k}' = -\underline{e_0} \sinh\phi + \underline{k} \cosh\phi} \quad -(34)$$

from eq. (28)

5) $$\boxed{\underline{e}_0' = \cosh\phi\, \underline{e}_0 - \sinh\phi\, \underline{k}} \quad -(35)$$

In frame K:

$$\underline{k} = \underline{i} \times \underline{j} \quad -(36)$$

which is the usual Cartesian frame definition. Under the Lorentz boost in Z:

$$\left.\begin{aligned} &\underline{i} \rightarrow \underline{i}, \quad \underline{j} \rightarrow \underline{j}, \\ &\underline{k} \rightarrow -\underline{e}_0 \sinh\phi + \underline{k}\cosh\phi \end{aligned}\right\} -(37)$$

Therefore: $\cosh\phi = 1, \quad \sinh\phi = 0 \quad -(38)$

i.e. $v = 0. \quad -(39)$

Interpretation

The relation (36) is an $O(3)$ rotation generator relation which remains constant under the Lorentz boost. In the rotational Lorentz boost (Carroll (1.17)):

$$\begin{bmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad -(40)$$

and time is unchanged. We may therefore consider only the space relation (36)