1) 117(8): Electric Field due to a Dipole

The ECE theory of gravitomagnetism is based on the theory of electrostatics. The scalar potential in electrostatics is used to define the electric field as follows:

$$\underline{E} = -\underline{\nabla}\phi \quad -(1)$$

For the sake of argument this standard theory will be used to define the range of validity of the multipole expansion used in electrostatics:

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|}\,d^3x' \quad -(2)$$

$$= \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\,q_{lm}\,\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

As discussed by Jackson on p. 145 (3rd. ed.) this choice of constant coefficients is a convention. A localized distribution of charge is described by the charge density $\rho(\underline{x}')$, which is non-vanishing only inside a sphere of radius R defined around the origin. The sphere is used only to divide space into regions with and without charge. The multipole expansion is valid if and only if the charge density falls off with distance faster than any power of r.

Under these assumptions:

$$\phi(\underline{x}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{\underline{p}\cdot\underline{x}}{r^3} + \ldots\right) \quad -(3)$$

where

$$q = \int \rho(\underline{x}')\,d^3x', \quad -(4)$$

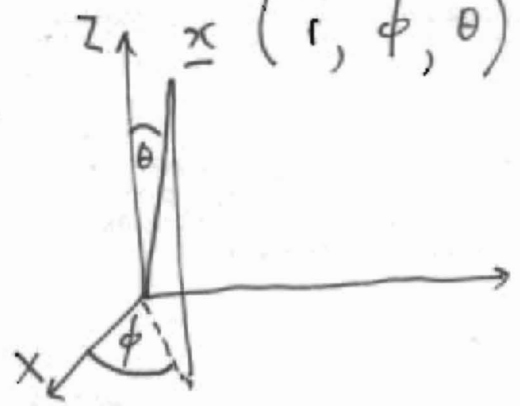
$$\underline{p} = \int \underline{x}'\rho(\underline{x}')\,d^3x' \quad -(5)$$

2) For a dipole along the z axis:

$$E_r = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}, \quad E_\theta = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}, \quad E_\phi = 0 \qquad -(6)$$

Where:

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta$$



Therefore:

$$\underline{E} = E_r\,\underline{e}_r + E_\phi\,\underline{e}_\phi + E_\theta\,\underline{e}_\theta \qquad -(7)$$

Where:

$$\underline{e}_r = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}$$
$$\underline{e}_\phi = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}$$
$$\underline{e}_\theta = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k}$$

So:

$$\underline{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(3\sin\theta\cos\theta\cos\phi\,\underline{i} + 3\sin\theta\cos\theta\sin\phi\,\underline{j} + \left(2\cos^2\theta - \sin^2\theta\right)\underline{k}\right) \qquad -(8)$$

Where:

$$\sin\theta\cos\phi = \frac{x}{r},$$
$$\sin\theta\sin\phi = \frac{y}{r}$$
$$\cos\theta = \frac{z}{r}, \quad \sin\theta = \left(1 - \frac{z^2}{r^2}\right)^{1/2}$$

3)

So:

$$\boxed{\underline{E}\,(\text{dipole}) = \frac{p}{4\pi\epsilon_0 r^3}\left(\frac{3z}{r^2}\left(x\underline{i} + y\underline{j} + z\underline{k}\right) - \underline{k}\right)}$$

— (9)

This result is denoted by Jackson, eq. (4.13), as:

$$\underline{E}\,(\underline{x}) = \frac{3\underline{n}\,(\underline{p}\cdot\underline{n}) - \underline{p}}{4\pi\epsilon_0\,|\underline{x} - \underline{x}_0|^3} \quad -(10)$$

where $\underline{n}$ is a unit vector directed from $\underline{x}_0$ to $\underline{x}$. Comparing eqs. (9) and (10) ($\underline{n} = \underline{r}/|\underline{r}|$):

$$r = |\underline{x} - \underline{x}_0| \quad -(11)$$

$$\underline{n}\,(\underline{p}\cdot\underline{n}) = \frac{pz}{r^2}\left(x\underline{i} + y\underline{j} + z\underline{k}\right) \quad -(12)$$

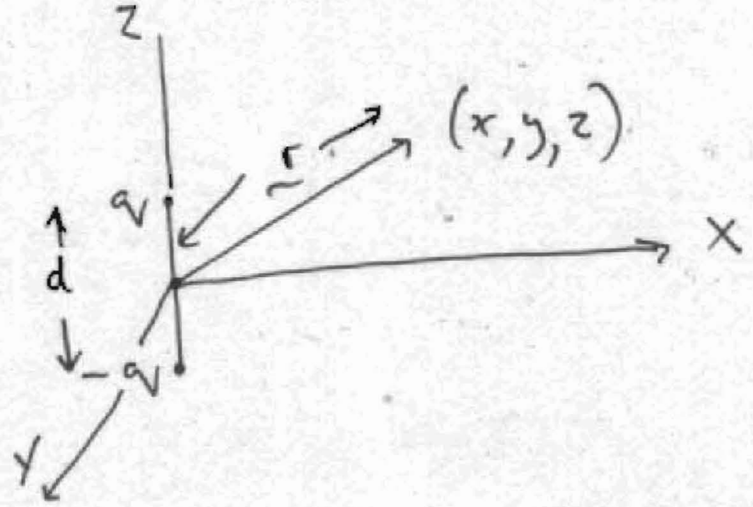
The electric dipole moment is defined by eq. (5). The range of validity of eq. (9) is:

$$\boxed{|\underline{x}'| \ll |\underline{x} - \underline{x}_0|,} \quad -(13)$$

where $|\underline{x}'|$ is the distance between the two charges of the dipole.

The dipole field is also developed in J. Mileski (ed.), "Vector Analysis Problem Solver", problem 11-30, p. 491.

4)



The exact solution is:

$$\phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\left(\left(z-\frac{d}{2}\right)^2+x^2+y^2\right)^{1/2}} - \frac{q}{\left(\left(z+\frac{d}{2}\right)^2+x^2+y^2\right)^{1/2}}\right) \quad -(14)$$

Eq. (9) is obtained if:

$$\boxed{d << |\underline{r}|} \quad -(15)$$

when:

$$\left(z-\frac{d}{2}\right)^2 \sim z^2 - zd \quad -(16)$$

and

$$\left(1-\frac{zd}{r^2}\right)^{-1/2} \sim 1 + \frac{1}{2}\frac{zd}{r^2} \quad -(17)$$

If:

$$p = qd \quad -(18)$$

then:

$$\boxed{\phi \sim \frac{1}{4\pi\epsilon_0}\frac{zqd}{r^3}} \quad -(19)$$

Now use:

$$\cos\theta = \frac{z}{r} \quad -(20)$$

5)

so:
$$\phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} \quad -(21)$$

The vector $\underline{p}$ is defined as along the z axis from $-q$ to q, with $|\underline{p}| = qd$. So:

$$\underline{p}\cdot\underline{r} = rp\cos\theta \quad -(22)$$

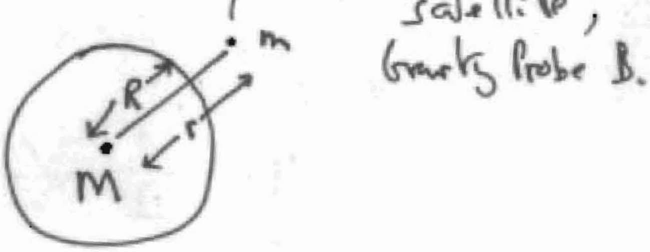
i.e.
$$p\cos\theta = \underline{p}\cdot\underline{r}/r \quad -(23)$$

and:
$$\boxed{\phi(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{\underline{p}\cdot\underline{r}}{r^3}} \quad -(24)$$

as in eq. (3).

Application to Gravity Probe B.

The situation is that the spacecraft orbits at a distance r from the centre of the earth of mass M and radius R.



Experimentally, it is known that:

$$\underline{F} = m\underline{g} = -\frac{mMG}{r^2}\underline{e}_r \quad -(25)$$

so the earth of volume $\frac{4}{3}\pi R^3$ and surface area $4\pi R^2$ can be treated as a mass M situated at the origin. The acceleration due to gravity is:

$$\underline{g} = -\frac{MG}{r^2}\underline{e}_r \quad -(26)$$

$$= -\underline{\nabla}\,\Phi \quad -(27)$$

6) So:

$$\Phi = -\frac{GM}{r} \quad -(28)$$

where:

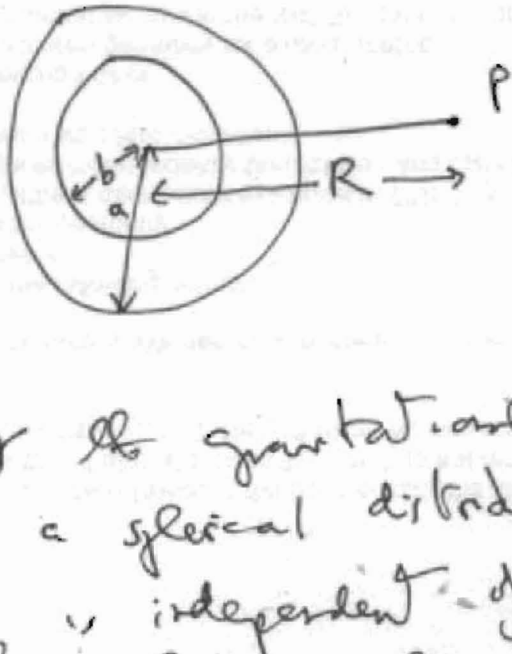
$$M = \int \rho_m(r')\,dV' \quad -(29)$$

This result is described by:

$$\boxed{\underline{\nabla} \cdot \underline{g} = 4\pi G \rho_m} \quad -(30)$$

Comparing eqs. (28) and (3) it is seen that only the first term in the multipole expansion of the gravitational potential Φ is needed. This is because the centre of mass of the earth is situated at the origin. This result is also obtained from the gravitational potential inside and outside a spherical shell of inner radius b and outer radius a.

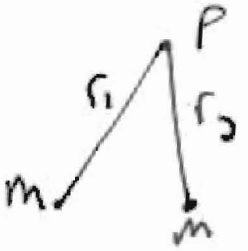
Marion & Thornton, p. 161 ff.



$$\boxed{\Phi(R > a) = -\frac{GM}{R}} \quad -(31)$$

Eq (31) states that the gravitational potential at any point outside a spherical distribution of matter, solid or shell, is independent of the size of the distribution. Therefore all the mass can be considered to be concentrated at a point, which is taken to be the origin. The field due to two masses may be calculated from:

$$\Phi = -GM\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$



Finally, to calculate the gravitomagnetic field, we use the equation:

$$\underline{\nabla}\times\underline{h} = \frac{4\pi G}{c}\,\underline{J}_m \quad \text{— (31)}$$

where

$$J^{\mu} = \left(c\rho_m,\ \underline{J}_m\right) \quad \text{— (32).}$$

The analogies are: — (33)

$$\boxed{\begin{aligned}\underline{\nabla}\cdot\underline{g} &= 4\pi G\rho_m\\ \underline{\nabla}\times\underline{h} &= \frac{4\pi G}{c}\,\underline{J}_m\end{aligned}} \longleftrightarrow \boxed{\begin{aligned}\underline{\nabla}\cdot\underline{E} &= \rho/\epsilon_0\\ \underline{\nabla}\times\underline{B} &= \mu_0\,\underline{J}\end{aligned}}$$

together with the Lorentz transform:

$$\underline{B}' = \gamma\left(\underline{B} - \frac{1}{c^2}\,\underline{v}\times\underline{E}\right) \quad \text{— (34)}$$

The inverse of eq. (34) means that in the frame K:

$$\boxed{\underline{B} = \frac{1}{c^2}\,\underline{v}\times\underline{E}} \quad \text{— (35).}$$

8)

Similarly:

$$\boxed{\underline{h} = \frac{1}{c} \underline{v} \times \underline{g}} \quad - (36)$$

From eq. (36):

$$\underline{g} = -\frac{MG}{r^3} \underline{r} \quad - (37)$$

so:

$$\underline{h} = -\frac{MG}{cr^3} \underline{v} \times \underline{r} \quad - (38)$$

The angular momentum is defined as:

$$\underline{L} = \underline{r} \times \underline{p} = M \underline{r} \times \underline{v} \quad - (39)$$

So:

$$\boxed{\underline{h} = \frac{G \underline{L}}{cr^3}} \quad - (40)$$

For a sphere:

$$\underline{L} = \frac{2}{5} M R^2 \underline{\omega} \quad - (41)$$

and:

$$\boxed{\Omega = \frac{h}{c} = \frac{2}{5} \frac{MR^2 \omega}{c^2 r^3}} \quad - (42)$$

as observed by Gravity Probe B.