116(7): ECE Continuity Equation

The continuity equation of the standard model is the law of conservation of charge current density. In Minkowski spacetime this law is:

$$\partial_\mu j^\mu = 0 \quad -(1)$$

where

$$j^\mu = (c\rho, \underline{J}), \quad -(2)$$

i.e.

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \underline{J} = 0. \quad -(3)$$

In the standard physics this is a direct consequence of the Noether conservation of charge / current density and the theorem. A decrease in charge inside a small volume with time must result in a flow of charge out through the surface of the small volume. Steady state magnetic phenomena correspond to:

$$\underline{\nabla} \cdot \underline{J} = 0. \quad -(4)$$

In ECE physics:

$$j^\mu = (J^0, \underline{J}) \quad -(5)$$

where:

$$J^0 = J^0{}_1{}^{10} + J^0{}_2{}^{20} + J^0{}_3{}^{30} \quad -(6)$$

and

$$\underline{J} = J_x \underline{i} + J_y \underline{j} + J_z \underline{k} \quad -(7)$$

2) where:

$$J_x = J^1{}_0{}^{01} + J^1{}_2{}^{21} + J^1{}_3{}^{31} \quad -(8)$$

$$J_y = J^2{}_0{}^{02} + J^2{}_1{}^{12} + J^2{}_3{}^{32} \quad -(9)$$

$$J_Z = J^3{}_0{}^{03} + J^3{}_1{}^{13} + J^3{}_2{}^{23}. \quad -(10)$$

In standard physics:

$$\partial_\mu F^{\mu\nu} = j^\nu \quad -(11)$$

in Minkowski spacetime, where:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_X & -E_Y & -E_Z \\ E_X & 0 & -cB_Z & cB_Y \\ E_Y & cB_Z & 0 & -cB_X \\ E_Z & -cB_Y & cB_X & 0 \end{bmatrix} \quad -(12)$$

In ECE physics:

$$\partial_\mu F^{0\mu 0} = J^0{}_\mu{}^{\mu 0} \quad -(13)$$

$$\partial_\mu F^{1\mu 1} = J^1{}_\mu{}^{\mu 1} \quad -(14)$$

$$\partial_\mu F^{2\mu 2} = J^2{}_\mu{}^{\mu 2} \quad -(15)$$

$$\partial_\mu F^{3\mu 3} = J^3{}_\mu{}^{\mu 3}. \quad -(16)$$

Eq (13) gives the Coulomb law and eqs. (14) to (16) give the Ampère Maxwell law.

Eqs. (13) to (16) are written in a spacetime with torsion and curvature both present, and are generally covariant, their basic structure being the Hodge dual structure of the Bianchi identity:

$$D_\mu T^{\kappa\mu\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(17)$$

In eq. (6), J^0 is a scalar valued sum of elements of rank four tensors. In eq. (7), J_X, J_Y and J_Z are also scalar valued elements of a vector.

With the definition of j^μ in eq. (5) there exists an ECE theorem of charge-current density conservation.

Electrostatics

From eq. (13):

$$\underline{\nabla} \cdot \underline{E} = \rho / \epsilon_0 \quad -(18)$$

where:

$$\underline{E} = E^{010}\,\underline{i} + E^{020}\,\underline{j} + E^{030}\,\underline{k} \quad -(19)$$

and

$$\rho = \epsilon_0 J^0 . \quad -(20)$$

Since charge is conserved:

$$\boxed{\frac{d\rho}{dt} = 0} \quad -(21)$$

4)

This is the equation of conservation of electric charge. The charge on the electron, $-e$, is conserved.

Magnetostatics

Steady-state magnetic phenomena are characterized by no change in net charge density anywhere in space. So:

$$\underline{\nabla} \cdot \underline{J} = 0 \quad - (22)$$

where $\underline{J}$ is given by eq. (7). It is noted that eqs. (21) and (22) are valid in general relativity. Here ρ and $\underline{J}$ are already defined covariantly in general relativity. Similarly, eqs. (13) to (16) are generally covariant because they are deduced from the structure (17).

Electrodynamics

The continuity equation of ECE physics is

$$\boxed{\partial_\mu j^\mu = 0} \quad - (23)$$

where j^μ is defined covariantly by eq. (5). In eq. (23), j^μ is a four-vector in a spacetime with torsion and curvature. This can be seen from the fact that j^μ is defined in terms of elements of the curvature and torsion.

5) The spin connection also enters into the definition of j^μ. The basic geometrical structure of the current is defined by:

$$\partial_\mu T^{\kappa\mu\nu} = J^{\kappa\ \mu\nu}_{\ \mu} = R^{\kappa\ \mu\nu}_{\ \mu} - 4\omega^\kappa_{\ \mu\lambda} T^{\lambda\mu\nu} \quad -(24)$$

so the current is a property of spacetime itself. In geometry, the covariant derivative of a scalar is the same as the ordinary four derivative of a scalar, so:

$$D_\mu J^0 = \partial_\mu J^0 \quad -(25)$$

$$D_\mu J_X = \partial_\mu J_X \quad -(26)$$

$$D_\mu J_Y = \partial_\mu J_Y \quad -(27)$$

$$D_\mu J_Z = \partial_\mu J_Z . \quad -(28)$$

Eq. (21) is therefore:

$$\boxed{D_0 J^0 = 0 = \partial_0 J^0} \quad -(29)$$

which is the ECE expression for conservation of charge. Similarly, eq. (22) is:

$$\boxed{\begin{aligned} &D_1 J^1 + D_2 J^2 + D_3 J^3 \\ &= \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3 = 0 \end{aligned}} \quad -(30)$$

which is the ECE conservation of current.