1) 116(2): Summary of the Geometry that Leads to the ECE Field Equations.

Homogeneous Field Equations

These are based directly on the Bianchi identity:

$$D \wedge T^a := R^a{}_b \wedge q^b \quad -(1)$$

which is a logical consequence of the Cartan structure equations:

$$T^a = D \wedge q^a \quad -(2)$$

$$R^a{}_b = D \wedge \omega^a{}_b . \quad -(3)$$

Eq (1) is:

$$D_\mu T^a{}_{\nu\rho} + D_\rho T^a{}_{\mu\nu} + D_\nu T^a{}_{\rho\mu} := R^a{}_{\mu\nu\rho} + R^a{}_{\rho\mu\nu} + R^a{}_{\nu\rho\mu} \quad -(4)$$

The Hodge dual transform is:

$$\tilde{T}^{a\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\rho\sigma} T^a{}_{\rho\sigma} \quad -(5)$$

$$\tilde{R}^a{}_\lambda{}^{\mu\nu} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\mu\nu\rho\sigma} R^a{}_{\lambda\rho\sigma} . \quad -(6)$$

An example of eq. (4) is:

$$D_1 T^a{}_{23} + D_3 T^a{}_{12} + D_2 T^a{}_{31} := R^a{}_{123} + R^a{}_{312} + R^a{}_{231} \quad -(7)$$

From eqs. (5) and (6):

2)

$$\left.\begin{array}{ll} \tilde{T}^{a01} = \|g\|^{1/2} T^a{}_{23}, & \tilde{R}^a{}_1{}^{01} = \|g\|^{1/2} R^a{}_{123}, \\ \tilde{T}^{a02} = \|g\|^{1/2} T^a{}_{31}, & \tilde{R}^a{}_2{}^{02} = \|g\|^{1/2} R^a{}_{231}, \\ \tilde{T}^{a03} = \|g\|^{1/2} T^a{}_{12}, & \tilde{R}^a{}_3{}^{03} = \|g\|^{1/2} R^a{}_{312} \end{array}\right\} \quad -(8)$$

So:

$$D_1 \tilde{T}^{a01} + D_2 \tilde{T}^{a02} + D_3 \tilde{T}^{a03} = R^a{}_1{}^{01} + R^a{}_2{}^{02} + R^a{}_3{}^{03} \quad -(9)$$

i.e.

$$\boxed{D_\mu \tilde{T}^{a\mu\nu} = \tilde{R}^a{}_\mu{}^{\mu\nu}} \quad -(10)$$

The left hand side is:

$$\partial_\mu \tilde{T}^{a\mu\nu} + \omega^a{}_{\mu b} \tilde{T}^{b\mu\nu} = \tilde{R}^a{}_\mu{}^{\mu\nu}. \quad -(11)$$

By definition:

$$\tilde{T}^{\kappa\mu\nu} := q^\kappa{}_a \tilde{T}^{a\mu\nu} \quad -(12)$$

$$\tilde{R}^\kappa{}_\mu{}^{\mu\nu} := q^\kappa{}_a \tilde{R}^a{}_\mu{}^{\mu\nu}. \quad -(13)$$

Therefore:

$$D_\mu \tilde{T}^{\kappa\mu\nu} = D_\mu \left(q^\kappa{}_a \tilde{T}^{a\mu\nu} \right) \quad -(14)$$

$$= \tilde{T}^{a\mu\nu} D_\mu q^\kappa{}_a + q^\kappa{}_a D_\mu \tilde{T}^{a\mu\nu} \quad -(15)$$

3)

Now use the tetrad postulate:

$$D_\mu q^\kappa{}_a = 0 \quad -(16)$$

to find:

$$D_\mu \tilde{T}^{\kappa\mu\nu} = q^\kappa{}_a D_\mu \tilde{T}^{a\mu\nu} \quad -(17)$$

$$= q^\kappa{}_a \tilde{R}^a{}_\mu{}^{\mu\nu}$$

$$= \tilde{R}^\kappa{}_\mu{}^{\mu\nu}$$

Therefore:

$$\boxed{D_\mu \tilde{T}^{\kappa\mu\nu} = \tilde{R}^\kappa{}_\mu{}^{\mu\nu}} \quad -(18)$$

This equation is written in the base manifold and removes the Cartan tangent spacetime, thus simplifying the interpretation.

Eq. (18) is:

$$\partial_\mu \tilde{T}^{\kappa\mu\nu} + \omega^\kappa{}_{\mu b} \tilde{T}^{b\mu\nu} = \tilde{R}^\kappa{}_\mu{}^{\mu\nu} \quad -(19)$$

where:

$$\omega^\kappa{}_{\mu b} = q^\kappa{}_a \omega^a{}_{\mu b} \quad -(20)$$

Finally use:

4)

$$\omega^{\kappa}{}_{\mu b}\tilde{T}^{b\mu\nu} = q^{\lambda}{}_{b}\, q^{b}{}_{\lambda}\, \omega^{\kappa}{}_{\mu\lambda}\tilde{T}^{\lambda\mu\nu}$$

$$= 4\,\omega^{\kappa}{}_{\mu\lambda}\tilde{T}^{\lambda\mu\nu} \quad -(21)$$

to find:

$$\boxed{\partial_\mu \tilde{T}^{\kappa\mu\nu} := \tilde{j}^{\kappa\nu} = \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} - 4\,\omega^{\kappa}{}_{\mu\lambda}\tilde{T}^{\lambda\mu\nu}.} \quad -(22)$$

Inhomogeneous Equation

This is derived from the dual identity:

$$D\wedge\tilde{T}^{a} := \tilde{R}^{a}{}_{b}\wedge q^{b} \quad -(23)$$

which is:

$$D_\mu \tilde{T}^{a}{}_{\nu\rho} + D_\rho \tilde{T}^{a}{}_{\mu\nu} + D_\nu \tilde{T}^{a}{}_{\rho\mu} := \tilde{R}^{a}{}_{\mu\nu\rho} + \tilde{R}^{a}{}_{\rho\mu\nu} + \tilde{R}^{a}{}_{\nu\rho\mu}. \quad -(24)$$

Defining:

$$T^{a\mu\nu} := \frac{1}{2}\|g\|^{1/2}\epsilon^{\mu\nu\rho\sigma}\tilde{T}^{a}{}_{\rho\sigma} \quad -(25)$$

$$R^{a}{}_{\lambda}{}^{\mu\nu} := \frac{1}{2}\|g\|^{1/2}\epsilon^{\mu\nu\rho\sigma}\tilde{R}^{a}{}_{\lambda\rho\sigma} \quad -(26)$$

then:

$$\boxed{D_\mu T^{a\mu\nu} = R^{a}{}_{\mu}{}^{\mu\nu}} \quad -(27)$$

from which:

$$\boxed{\partial_\mu T^{\kappa\mu\nu} = j^{\kappa\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu} - 4\omega^{\kappa}{}_{\mu\lambda} T^{\lambda\mu\nu}} \quad -(28)$$

<u>Dynamical Field Equations</u>

These are:

$$\partial_\mu \tilde{T}^{\kappa\mu\nu} = \tilde{j}^{\kappa\nu} \quad -(29)$$

$$\partial_\mu T^{\kappa\mu\nu} = j^{\kappa\nu} \quad -(30)$$

<u>Electrodynamical Field Equations</u>

These are:

$$\partial_\mu \tilde{F}^{\kappa\mu\nu} = A^{(0)} \tilde{j}^{\kappa\nu} \quad -(31)$$

$$\partial_\mu F^{\kappa\mu\nu} = A^{(0)} j^{\kappa\nu} \quad -(32)$$

<u>Maxwell Heaviside Equations</u>

These are:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \quad -(33)$$

$$\partial_\mu F^{\mu\nu} = J^{\nu} / \epsilon_0 \quad -(34)$$

<u>The Orbital Torsion Equations</u>

These are defined by:

$$\kappa = 0. \quad -(35)$$

6) <u>Inhomogeneous Orbital Torsion Equations</u>

For dynamics:

$$\partial_\mu T^{0\mu\nu} = j^{0\nu} \qquad -(36)$$

and for electrodynamics:

$$\partial_\mu F^{0\mu\nu} = A^{(0)} j^{0\nu} = J^{0\nu} / \epsilon_0 \qquad -(37)$$

From eq. (37) the Coulomb law is obtained if:

$$\nu = 0, \ \mu = 1, 2, 3. \qquad -(38)$$

In the absence of polarization:

$$\mathbf{\nabla} \cdot \mathbf{E} = \rho / \epsilon_0 \qquad -(39)$$

and in the presence of polarization:

$$\mathbf{\nabla} \cdot \mathbf{D} = \rho. \qquad -(40)$$

Here:

$$\mathbf{E} = E^{010}\, \mathbf{i} + E^{020}\, \mathbf{j} + E^{030}\, \mathbf{k} \qquad -(41)$$

and:

$$\rho = J^{00}. \qquad -(42)$$

The static electric field is therefore an orbital torsion. The charge density of the Coulomb law is $\epsilon_0 A^{(0)}$ multiplied by:

7)

$$j^{00} = R^{0}{}_{1}{}^{10} + R^{0}{}_{2}{}^{20} + R^{0}{}_{3}{}^{30} - 4\left(\omega^{0}{}_{1\lambda}T^{\lambda 10} + \omega^{0}{}_{2\lambda}T^{\lambda 20} + \omega^{0}{}_{3\lambda}T^{\lambda 30}\right) \quad -(43)$$

The vector form of eq. (36) is:

$$\boxed{\underline{\nabla} \cdot \underset{\sim}{T} = j^{00}} \quad -(44)$$

where:

$$\underline{T} = T^{010}\,\underline{i} + T^{020}\,\underline{j} + T^{030}\,\underline{k}. \quad -(45)$$

The Newton law is a weak field limit of:

$$\boxed{\underline{\nabla} \cdot \underline{g} = 4\pi G\rho} \quad -(46)$$

where:

$$\boxed{\begin{aligned} \underline{g} &= c^2 \underset{\sim}{T}, & -(47) \\ 4\pi G\rho &= c^2 j^{00}. & -(48) \end{aligned}}$$

The acceleration due to gravity $\underline{g}$ is also an orbital torsion. The mass density ρ emerges from:

$$j^{00} = \frac{4\pi G}{c^2}\rho. \quad -(49)$$
