113(6): Derivation of the Orbital Equation for the Theorem of Orbits.

The theorem of orbits:

$$n r = \frac{r}{n} = \int dr = r + \mu \quad -(1)$$

gives the line element of a spherically symmetric spacetime as:

$$ds^2 = -\left(1+\frac{\mu}{r}\right)c^2 dt^2 + \left(1+\frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad -(2)$$

in spherical polar coordinates. The orbital equation is obtained by considering the special case:

$$ds^2 = -\left(1+\frac{\mu}{r}\right)c^2 dt^2 + \left(1+\frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\phi^2 \quad -(3)$$

for motion in a plane. The following constant of motion is defined:

$$-\epsilon = -\left(\frac{ds}{d\lambda}\right)^2 = -\left(1+\frac{\mu}{r}\right)c^2\left(\frac{dt}{d\lambda}\right)^2 + \left(1+\frac{\mu}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

$$= -c^2\left(\frac{d\tau}{d\lambda}\right)^2 \quad -(4)$$

where τ is the proper time. Now make the choice:

$$\lambda = \tau \quad -(5)$$

to find:

$$-c^2 = -\left(1+\frac{\mu}{r}\right)c^2\left(\frac{dt}{d\tau}\right)^2 + \left(1+\frac{\mu}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2$$

$$-(6)$$

To convert to S.I. units multiply through by $\frac{1}{2}m$, where m is to be determined:

2)

$$\frac{1}{2}mr^2\left(\frac{d\phi}{d\tau}\right)^2 - \frac{1}{2}m\left(1+\frac{\mu}{r}\right)c^2\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}m\left(1+\frac{\mu}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2$$

$$= -\frac{1}{2}mc^2 \quad -(7)$$

Now multiply through by $(1+\mu/r)$:

$$\frac{1}{2}mr^2\left(\frac{d\phi}{d\tau}\right)^2\left(1+\frac{\mu}{r}\right) - \frac{1}{2}m\left(1+\frac{\mu}{r}\right)^2c^2\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2$$

$$= -\frac{1}{2}mc^2\left(1+\frac{\mu}{r}\right), \quad -(8)$$

i.e.

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}mc^2\left(1+\frac{\mu}{r}\right) + \frac{1}{2}mr^2\left(1+\frac{\mu}{r}\right)\left(\frac{d\phi}{d\tau}\right)^2 \quad -(9)$$

$$= \frac{1}{2}\left(1+\frac{\mu}{r}\right)^2 mc^2\left(\frac{dt}{d\tau}\right)^2$$

This is <u>the</u> <u>orbital equation</u>:

$$\frac{1}{2}m\left(\frac{dr}{d\tau}\right)^2 + V = E. \quad -(10)$$

The <u>energy</u> in S.I units is:

$$E = \frac{1}{2}mc^2\left(1+\frac{\mu}{r}\right)^2\left(\frac{dt}{d\tau}\right)^2. \quad -(11)$$

<u>The</u> <u>potential</u> <u>energy</u> is:

$$V = \frac{1}{2}m\left(1+\frac{\mu}{r}\right)\left(c^2 + \frac{L^2}{r^2}\right) \quad -(12)$$

3) where:

$$L = r^2 \frac{d\phi}{d\tau} \quad - (13)$$

is a constant of motion, proportional to angular momentum. The factor $\frac{1}{2}$ in eq (10) was introduced to write the equation in standard dynamical form. The potential energy is:

$$V = \frac{1}{2} m c^2 \left(1 + \frac{\mu}{r}\right) + \frac{1}{2} m \frac{L^2}{r^2} \left(1 + \frac{\mu}{r}\right) \quad - (14)$$

$$= \frac{1}{2} m c^2 + \frac{1}{2} m c^2 \frac{\mu}{r} + \frac{1}{2} m \frac{L^2}{r^2} + \frac{1}{2} \frac{m L^2 \mu}{r^3} \quad - (15)$$

Experimentally:

$$\mu = - \frac{2 M G}{c^2} \quad - (16)$$

so:

$$\boxed{V = \frac{1}{2} m c^2 - \frac{m M G}{r} + \frac{1}{2} m \frac{L^2}{r^2} - \frac{L^2 m M G}{r^3}} \quad - (17)$$

for the vast majority of orbits. In binary pulsars:

$$\mu = - \left(\frac{2 M G}{c^2} + \frac{a}{r} \right) \quad - (18)$$

and there appears an inverse fourth term in the potential, together with another inverse square term

4)

The terms in eq. (17) are as follows:

1) A constant energy $\frac{1}{2}mc^2$.

2) The Newtonian attraction $-mMG/r$.

3) The centripetal repulsion $\frac{1}{2}m\frac{L^2}{r^2}$.

4) The relativistic attraction $-L^2 mMG/r^3$.

So the factor m is identifiable as the mass of an object attracted by an object of mass M. The theory of orbits (1) is the "geometrical control" over the way m and M interact.

<u>The Newtonian Limit</u>

This is defined by:

$$r \rightarrow \infty \quad - (19)$$

and the familiar Newtonian terms (2) and (3) dominate. The Newtonian force of attraction is:

$$F = -\frac{\partial V}{\partial r} = -\frac{mMG}{r^2} \quad - (20)$$

<u>which</u> is the <u>famous</u> <u>inverse</u> square <u>law</u> of <u>Newton</u>.

From eqs. (17) and (20) the total force

5) between m and M is:

$$F = -\frac{mMG}{r^2} + \frac{mL^2}{r^3} - 3L^2\frac{mMG}{r^4} \quad -(21)$$

This force law is sufficient to describe all orbital phenomena except binary pulsars. It describes perihelion precession, deflection of light by gravity, the Shapiro time delay, rotational frame dragging, and so on.

As can be seen, the force law is the result of the theorem of orbits (1), and not of the Einstein field equation. The masses m and M are introduced following the data.

The Newtonian force is eq. (20) and Newton did not realize the existence of the centripetal force; neither of course did he realize the existence of the inverse fourth relativistic force. An orbit is a balance between the Newtonian attraction and the centripetal repulsion. Relativistic corrections to orbits are as described in the third term of eq. (21).