1)

113(3): Orbital Torsion and Spin Connection

From the theorem of orbits it is found that the line element of all orbits is:

$$ds^2 = -\left(1+\frac{\mu}{r}\right)c^2 dt^2 + \left(1+\frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad -(1)$$

in spherical polar coordinates. The orbital torsion and spin connection can be evaluated using the results of paper 105, where the Hodge dual identity was given as:

$$\partial_\mu \tilde{T}^{\kappa\mu\nu} = \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(2)$$

In vector notation, eq. (2) is:

$$\underline{\nabla} \cdot \underline{T} = R_1 \quad -(3)$$

$$\underline{\nabla} \times \underline{T}_2 - \frac{1}{c}\frac{\partial \underline{T}_3}{\partial t} = \underline{R}_2 . \quad -(4)$$

Eq. (3) is the equivalent of the Coulomb law of electrostatics. Here:

$$\underline{T} = T^{010}\,\underline{e}_r + T^{020}\,\underline{e}_\theta + T^{030}\,\underline{e}_\phi \quad -(5)$$

$$R_1 = R^{0}{}_{1}{}^{10} + R^{0}{}_{2}{}^{20} + R^{0}{}_{3}{}^{30} . \quad -(6)$$

In the radial direction:

$$\underset{\sim}{T} = T^{010}\,\underline{e}_r . \quad -(7)$$

In general:

$$\underline{\nabla} \cdot \underline{g} = c^2 R_1 \quad -(8)$$

acceleration due to gravity $\quad \underline{g} = c^2 \underline{T} \quad -(9)$

mass density $\quad \rho_m = \frac{1}{k}\left(R^{0}{}_{1}{}^{10} + R^{0}{}_{2}{}^{20} + R^{0}{}_{3}{}^{30}\right) \quad -(10)$

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If T is radial as in eq. (7) then:

$$\rho_m = \frac{1}{k} R^0{}_1{}^{10} . \quad — (11)$$

In the weak field limit:

$$g \to -\frac{mG}{r^2} \quad — (12)$$

so:

$$T \to -\frac{mG}{c^2 r^2} \quad — (13)$$

Therefore:

$$\boxed{T^{010} \to -\frac{mG}{c^2 r^2}} \quad — (14)$$

torsion of Newtonian orbits

This is the orbital torsion in the Newtonian limit.

From antisymmetry:

$$T^{001} = \frac{mG}{c^2 r^2} . \quad — (15)$$

Now lower indices:

$$T^0{}_{01} = g_{00} \, g_{11} \, T^{001} \quad — (16)$$

In the weak field limit:

$$g_{00} \to -1 , \quad g_{11} \to 1 , \quad — (17)$$

so:

$$T^0{}_{01} = -\frac{mG}{c^2 r^2} . \quad — (18)$$

This torsion is defined by the first Cartan

3)

Structure equation:

$$T^a_{\mu\nu} = \partial_\mu q^a_\nu - \partial_\nu q^a_\mu + \omega^a_{\mu b} q^b_\nu - \omega^a_{\nu b} q^b_\mu, \quad -(19)$$

so:

$$T^0_{01} = -T^0_{10} = -\frac{mG}{c^2 r^2}, \quad -(20)$$

and

$$T^0_{10} = \partial_1 q^0_0 + \omega^0_{10} q^0_0. \quad -(21)$$

Now use the diagonal tetrad assumption:

$$q^0_0 = g_{00}^{1/2} = \left(1 + \frac{\mu}{r}\right)^{1/2} \quad -(22)$$

to find that:

$$T^0_{10} = -\frac{\mu}{2r^2} = -\frac{\mu}{2r^2}\left(1+\frac{\mu}{r}\right)^{-1/2} + \left(1+\frac{\mu}{r}\right)^{1/2} \omega^0_{10}$$

$$-(23)$$

So:

$$\boxed{\omega^0_{10} = \frac{\mu}{2r^2}\left(\left(1+\frac{\mu}{r}\right)^{-1} - \left(1+\frac{\mu}{r}\right)^{-1/2}\right)}$$

$$-(24)$$

with:

$$\mu = -\frac{2mG}{c^2} \quad -(25)$$

In the weak field limit ($\mu << r$):

Spin connection of Newtonian orbits

$$\boxed{\omega^0_{10} \longrightarrow -\frac{\mu}{4r^2} = \frac{mG}{2c^2 r^2}} \quad -(26)$$