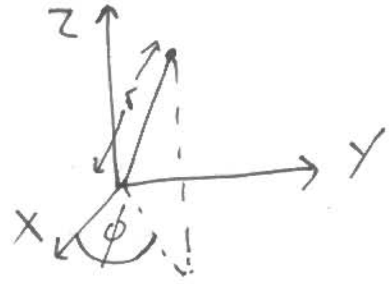
1) 110(7): Tetrads of the Thomas Precession.

The Thomas precession is caused by a rotating Minkowski spacetime. In cylindrical polar coordinates the static Minkowski spacetime is:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2 \quad -(1)$$

where:

$$x = r\cos\phi,$$
$$y = r\sin\phi,$$
$$z = z$$



The frame is rotated as follows:

$$\phi = \phi' - \omega t \quad -(2)$$

$$d\phi' = d\phi + \omega dt. \quad -(3)$$

The rotating metric is defined as:

$$ds^2 := c^2 dt^2 - dr^2 - r^2 d\phi'^2 - dz^2, \quad -(2)$$

$$d\phi'^2 = d\phi^2 + 2\omega\, d\phi\, dt + \omega^2 dt^2. \quad -(3)$$

So:

$$ds^2 = \left(c^2 - r^2\omega^2\right) dt^2 - dr^2 - r^2 d\phi^2 - dz^2 - 2r^2\omega\, d\phi\, dt \quad -(4)$$

$$= \left(1 - \frac{v^2}{c^2}\right) c^2 dt^2 - dr^2 - r^2 d\phi^2 - dz^2 - 2r^2 \left(\frac{\omega}{1 - \frac{v^2}{c^2}}\right)\left(1 - \frac{v^2}{c^2}\right) d\phi\, dt$$

$$-(5)$$
$$=(6)$$

where $v = r\omega$.

2) Therefore:

$$ds^2 = \left(1 - \frac{v^2}{c^2}\right)\left(c^2 dt^2 - 2r^2 \Omega \, d\phi \, dt\right) - dr^2 - r^2 d\phi^2 - dz^2 \quad -(7)$$

where the Thomas angle is:

$$\Omega := \omega \left(1 - \frac{v^2}{c^2}\right)^{-1} \quad -(8)$$

and where:

$$dt \rightarrow \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt. \quad -(9)$$

The Thomas precession is defined as:

$$\alpha = \Omega \, d\tau - 2\pi \quad -(10)$$

in a rotation of 2π. Here:

$$\Omega \, d\tau = \left(1 - \frac{v^2}{c^2}\right)^{-1} \left(1 - \frac{v^2}{c^2}\right)^{1/2} \omega \, dt \quad -(11)$$

So:

$$\boxed{\alpha = 2\pi \left(\left(1 - \frac{v}{c}\right)^{-1/2} - 1 \right)} \quad -(12)$$

Tetrads

Eq. (1) is written as:

$$ds^2 = (q^0_0)^2 c^2 dt^2 - (q^1_1)^2 dr^2 - (q^2_2)^2 r^2 d\phi^2 - (q^3_3)^2 dz^2 \quad -(13)$$

$$= \left(q^0_0 \, c\, dt \right)^2 - \left(q^1_1 \, dr \right)^2 - \left(q^2_2 \, r \, d\phi \right)^2 - \left(q^3_3 \, dz \right)^2 \quad -(14)$$

Now use:

$$\phi \rightarrow \phi' - \omega t \quad -(15)$$

and

$$q^{2\,\prime}_2 \, r \, d\phi = q^2_2 \, r \, d(\phi + \omega t) \quad -(16)$$

i.e.

$$q^{2\,\prime}_2 \, d\phi = d\phi + \omega \, dt \quad -(17)$$

because

$$q^2_2 = 1. \quad -(18)$$

So:

$$q^{2\,\prime}_2 = 1 + \omega \frac{dt}{d\phi} \quad -(19).$$

i.e.

$$\boxed{q^{2\,\prime}_2 = 1 + \frac{v}{r} \left(\frac{dt}{d\phi} \right)} \quad -(20)$$

Therefore:

$$q^0_0 = 1, \quad q^1_1 = 1,$$

$$q^{2\,\prime}_2 = 1 + \frac{v}{r} \left(\frac{dt}{d\phi} \right), \quad q^3_3 = 1 \quad -(20a)$$

These are the four tetrads of the Thomas precession. The torsion is calculated as usual from:

$$T = D \wedge q \quad -(21)$$

4) From eq. (7):

$$g_{00} = \left(1 - \frac{v^2}{c^2}\right), \; g_{11} = -1, \; g_{22} = -r^2, \; g_{33} = -1,$$

$$g_{02} = -2r^2\omega \quad -(22)$$

$$= g_{20}$$

In general, Christoffel symbols exist from eq. (2), but the Christoffel torsion is zero:

$$T^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} \quad -(23)$$

$$= 0$$

Cartan Torsion

This is in general non-zero, due to the rotation. The Christoffel torsion refers only to central motion and symmetric metrics. In general

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(24)$$

where: $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1) \quad -(25)$

and the tetrads of the Thomas precession may also be worked out from eqs. (22) and (24). The Cartan torsion is then:

$$T = D \wedge q \quad -(26)$$

This will be the subject of note 110(8).