1) Note 110(2): ECE Lemma for the General Metric in a Spherically Symmetric Spacetime.

In this case the metric is:

$$g_{00} = e^{2\alpha}, \quad g_{11} = -e^{-2\beta}, \quad -(1)$$

$$g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta$$

and is defined by:

$$g_{\mu\nu} = q^a_\mu q^b_\nu \eta_{ab} \quad -(2)$$

where the Minkowski metric η_{ab} is defined by:

$$\eta_{00} = 1, \quad \eta_{11} = -1, \quad \eta_{22} = -r^2, \quad \eta_{33} = -r^2 \sin^2\theta \quad -(3)$$

in spherical polar coordinates.

Therefore:

$$g_{\mu\nu} = q^0_\mu q^0_\nu \eta_{00} + q^1_\mu q^1_\nu \eta_{11} + q^2_\mu q^2_\nu \eta_{22} + q^3_\mu q^3_\nu \eta_{33} \quad -(4)$$

Diagonal Tetrad Assumption

Assume that the tetrad matrix is diagonal, i.e.

$$g_{00} = (q^0_0)^2, \quad g_{11} = -(q^1_1)^2,$$

$$g_{22} = -(q^2_2)^2, \quad g_{33} = -(q^3_3)^2 \quad -(5)$$

 Therefore:

$$q^0_0 = e^{\alpha}, \quad q^1_1 = e^{-\beta}, \qquad -(6)$$
$$q^2_2 = r, \quad q^3_3 = r \sin\theta.$$

The tetrad postulate is:

$$D_\mu q^a_\sigma = \partial_\mu q^a_\sigma + \omega^a_{\mu b} q^b_\sigma - \Gamma^\lambda_{\mu\sigma} q^a_\lambda = 0 \qquad -(7)$$

and the ECE Lemma is:

$$\square q^a_\mu = R\, q^a_\mu \qquad -(8)$$

where:

$$R = \frac{1}{4} q^\sigma_a \partial^\mu \left(\Gamma^\lambda_{\mu\sigma} q^a_\lambda - \omega^a_{\mu b} q^b_\sigma \right). \qquad -(9)$$

Therefore, for example:

$$\square q^0_0 = R\, q^0_0 \qquad -(10)$$

Assume that:

$$\alpha = \alpha' + i\alpha'' \qquad -(11)$$

i.e. that α is complex valued. For the sake of exemplification assume that:

$$\alpha' = 0. \qquad -(12)$$

Therefore

$$\square e^{i\alpha''} = R\, e^{i\alpha''} \qquad -(13)$$

3)

For propagation in the Z axis:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial Z^2} \quad -(14)$$

and for a plane wave:

$$\alpha'' = \omega t - \kappa Z \quad -(15)$$

Therefore:

$$\boxed{R = -\left(\frac{\omega^2}{c^2} + \kappa^2\right)} \quad -(16)$$

In this example the tetrad postulate (7) simplifies to:

$$\partial_\mu q^0_0 + \omega^0_{\mu 0} q^0_0 - \Gamma^0_{\mu 0} q^0_0 = 0 \quad -(17)$$

The only relevant Christoffel symbol is:

$$\Gamma^0_{00} = \partial_0 \alpha = \frac{i}{c}\frac{\partial}{\partial t}(\omega_0 t - \kappa_0 Z)$$

$$= \frac{i\omega_0}{c} \quad -(18)$$

Therefore eq. (17) simplifies to:

$$\partial_0 q^0_0 + \left(\omega^0_{00} - \Gamma^0_{00}\right) q^0_0 = 0 \quad -(19)$$

i.e.

$$\frac{1}{c}\frac{\partial}{\partial t} e^{i\alpha''} + \left(\omega^0_{00} - i\frac{\omega_0}{c}\right) e^{i\alpha''} = 0 \quad -(20)$$

4)

i.e. $\frac{i\omega_0}{c} + \omega^0_{00} - \frac{i\omega_0}{c} = 0$

or $\omega^0_{00} = 0$ — (21)

Therefore: $\boxed{\partial_0 q^0_0 = \Gamma^0_{00} q^0_0}$ — (22)

where $q^0_0 = \exp\left(i(\omega t - \kappa Z)\right)$ — (23)

$\Gamma^0_{00} = \frac{i\omega}{c} = i\kappa$

for propagation at c.

The d'Alembertian for

$\mu = 0$ — (24)

is $\Box = \partial^0 \partial_0 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$ — (25)

Therefore: $\boxed{\Box q^0_0 = \left(\Gamma^0_{00}\right)^2 q^0_0}$ — (26)

This is a mathematical example of the ECE Lemma for the general, spherically symmetric, spacetime. This example does not use the Einstein Hilbert field equation.

5)

This example is over-simplified however because it results in an imaginary Christoffel connection and zero spin connection, and has also assumed a diagonal tetrad.

Example 2

Assume that α is real-valued. For example:

$$\exp \alpha = \cos(\omega t - \kappa Z), \quad -(27)$$

then in eq. (19):

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\cos(\omega t - \kappa Z)\right) + \left(\omega^0_{\ 00} - \Gamma^0_{\ 00}\right)\cos(\omega t - \kappa Z) = 0 \quad -(28)$$

where

$$\Gamma^0_{\ 00} = \frac{1}{c}\frac{\partial}{\partial t}\alpha \quad -(29)$$

The scalar curvature is again found from:

$$\Box q^0_{\ 0} = R q^0_{\ 0} \quad -(30)$$

where

$$q^0_{\ 0} = e^{\alpha} = \cos(\omega t - \kappa Z) - (31)$$

Restricting attention to $\mu = 0$ in the d'Alembertian it is found that

$$\boxed{R = \frac{\omega^2}{c^2}} \quad -(32)$$

6)

From eq. (29)

$$\Gamma^{0}_{00} = \frac{1}{c}\frac{\partial}{\partial t}\left(\log_e\left(\cos(\omega t - \kappa Z)\right)\right) \quad - (33)$$

Therefore from eq. (28):

$$\frac{\omega}{c} + \omega^{0}_{00} - \frac{1}{c}\frac{\partial}{\partial t}\left(\log_e\left(\cos(\omega t - \kappa Z)\right)\right) = 0 \quad - (34)$$

and

$$\omega^{0}_{00} = \frac{1}{c}\frac{\partial}{\partial t}\left(\log_e\left(\cos(\omega t - \kappa Z)\right)\right) - \frac{\omega}{c}$$

$$\boxed{\omega^{0}_{00} = -\frac{\omega}{c}\left(\tan(\omega t - \kappa Z) - 1\right)} \quad - (35)$$

Results

If $e^{\alpha} = \cos(\omega t - \kappa Z)$, ~~then~~:

$$R = \frac{\omega}{c},$$

and: $$\Gamma^{0}_{00} = \partial_0 \alpha = \frac{1}{c}\frac{\partial}{\partial t}\left(\log_e\left(\cos(\omega t - \kappa Z)\right)\right)$$

$$\omega^{0}_{00} = -\frac{\omega}{c}\left(\tan(\omega t - \kappa Z) - 1\right) \quad - (36)$$