1) Note 109(5): Generalized Cartan / Bianchi Identity

Theorem

Define any two antisymmetric tensors A and B by:

$$[D_\mu, D_\nu] V^\kappa := A^\kappa{}_{\sigma\mu\nu} V^\sigma - B^\lambda{}_{\mu\nu} D_\lambda V^\kappa \quad -(1)$$

then:

$$D \wedge B := A \wedge q. \quad -(2)$$

Proof

Eq. (1) implies that:

$$A^\kappa{}_{\sigma\mu\nu} := \partial_\mu \Gamma^\kappa_{\nu\sigma} - \partial_\nu \Gamma^\kappa_{\mu\sigma} + \Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\kappa_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \quad -(3)$$

$$B^\kappa_{\mu\nu} := \Gamma^\kappa_{\mu\nu} - \Gamma^\kappa_{\nu\mu}. \quad -(4)$$

Eqs (3) and (4) are definitions that follow directly from the fundamental definitions of D_μ and D_ν. In general both A and B must be non-zero, and both must exist. This result is true for all connections Γ. The result is independent of the assumption of metric compatibility, and is true for all symmetries of the metric q and of the connection Γ. It depends only on the anti-symmetry:

$$[D_\mu, D_\nu] = -[D_\nu, D_\mu] \quad -(5)$$

of the commutator of covariant derivatives.

2)

In form notation eq. (2) is:

$$D_\mu B^a_{\nu\rho} + D_\rho B^a_{\mu\nu} + D_\nu B^a_{\rho\mu} = A^a_{\mu\nu\rho} + A^a_{\rho\mu\nu} + A^a_{\nu\rho\mu} \quad -(6)$$

As shown in previous work, eq. (6) is the cyclic sum of the definition (3) identically equal to the sum of A tensors. This result is true if same cyclic sum of the definition (4) is also true. Therefore and only if eq. (1) is true, Q.E.D. it is true if and only if

Discussion

Any two tensors defined as in eqs (1), (3) and (4) obey eq. (2). The only requirement is that the tensor be antisymmetric in μ and ν. A particular solution of eq. (6) is:

$$D_\mu B^\kappa_{\nu\rho} + D_\rho B^\kappa_{\mu\nu} + D_\nu B^\kappa_{\rho\mu} = A^\kappa_{\mu\nu\rho} + A^\kappa_{\rho\mu\nu} + A^\kappa_{\nu\rho\mu}. \quad -(7)$$

The computer algebra results of paper 93 showed that curvature tensors of the type $R^\kappa{}_{\mu\nu}$ are in general non-zero. These curvature tensors are found from exact solutions of the Einstein Hilbert equation, and all these solutions use the Christoffel convention for which the torsion tensor is zero by definition.

3)

So for example:

$$R^{\circ}{}_{\mu}{}^{\mu\circ} \neq 0 \quad — (8)$$

i.e., summing over the repeated indices μ:

$$R^{\circ}{}_{1}{}^{10} + R^{\circ}{}_{2}{}^{20} + R^{\circ}{}_{3}{}^{30} \neq 0. \quad — (9)$$

From antisymmetry

$$R^{\circ}{}_{1}{}^{01} + R^{\circ}{}_{2}{}^{02} + R^{\circ}{}_{2}{}^{03} \neq 0. \quad — (10)$$

By definition, the Hodge duals in 4-D of these tensors are also anti-symmetric in their last two indices:

$$\left.\begin{aligned} \tilde{R}^{\circ}{}_{123} &= \|g\|^{1/2} R^{\circ}{}_{1}{}^{01} \\ \tilde{R}^{\circ}{}_{231} &= \|g\|^{1/2} R^{\circ}{}_{2}{}^{02} \\ \tilde{R}^{\circ}{}_{312} &= \|g\|^{1/2} R^{\circ}{}_{3}{}^{03} \end{aligned}\right\} \quad — (11)$$

So eq (10) is:

$$\boxed{\tilde{R}^{\circ}{}_{123} + \tilde{R}^{\circ}{}_{231} + \tilde{R}_{312} \neq 0} \quad — (12)$$

in general for exact solutions of the EH field equation. Eq. (12) is an example of:

$$\tilde{R}^{\kappa}{}_{\mu\nu\rho} + \tilde{R}^{\kappa}{}_{\rho\mu\nu} + \tilde{R}^{\kappa}{}_{\nu\rho\mu} \neq 0. \quad — (13)$$

and <u>therefore</u> <u>by</u> <u>eq. (7)</u>: — (14)

$$D_{\mu}\tilde{T}^{\kappa}{}_{\nu\rho} + D_{\rho}\tilde{T}^{\kappa}{}_{\mu\nu} + D_{\nu}\tilde{T}^{\kappa}{}_{\rho\mu} = \tilde{R}^{\kappa}{}_{\mu\nu\rho} + \tilde{R}^{\kappa}{}_{\rho\mu\nu} + \tilde{R}^{\kappa}{}_{\nu\rho\mu}$$

4)

This equation may be rewritten as:

$$\boxed{D_\mu T^{\kappa\mu\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad — (15)$$

For all solutions of EH, the LHS is in general zero but the RHS is in general non-zero. So EH is not true in general. In shorthand notation eq. (15) is:

$$\boxed{D \wedge \tilde{T} := \tilde{R} \wedge q.} \quad — (16)$$

Computer Algebra Check

In paper 93 it was checked that for all exact solutions of EH:

$$R^{\kappa}{}_{\mu\nu\sigma} + R^{\kappa}{}_{\sigma\mu\nu} + R^{\kappa}{}_{\nu\sigma\mu} = 0. \quad — (17)$$

This is known in the standard model as "the first Bianchi identity". For example:

$$R^{0}{}_{123} + R^{0}{}_{312} + R^{0}{}_{231} = 0 \quad — (18)$$

Using eq. (11) this equation is:

$$\tilde{R}^{0}{}_{1}{}^{01} + \tilde{R}^{0}{}_{2}{}^{02} + \tilde{R}^{0}{}_{3}{}^{03} = 0 \quad — (19)$$

which is an example of, for EH:

$$\tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} = 0. \quad — (20)$$

So

$$D_\mu \tilde{T}^{\kappa\mu\nu} = \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} = 0 \quad — (21)$$

for EH, but eq. (15) is NOT ~~true for~~ obeyed by EH.