1)

Notes 109(2): New Cyclic Identity for Torsion

This new cyclic identity is found from the internal structure of the Bianchi identity:

$$D \wedge T := R \wedge q \quad -(1)$$

In tensor notation:

$$D_\mu T^a_{\ \nu\sigma} + D_\sigma T^a_{\ \mu\nu} + D_\nu T^a_{\ \sigma\mu} := R^a_{\ \mu\nu\sigma} + R^a_{\ \sigma\mu\nu} + R^a_{\ \nu\sigma\mu} \quad -(2)$$

with:

$$T^a_{\ \nu\sigma} = T^\lambda_{\ \nu\sigma} q^a_{\ \lambda} \ ; \quad R^a_{\ \mu\nu\sigma} = R^\lambda_{\ \mu\nu\sigma} q^a_{\ \lambda} \quad -(3)$$

and

$$D_\mu q^a_{\ \nu} = 0. \quad -(4)$$

Thus:

$$D_\mu \left(T^\lambda_{\ \nu\sigma} q^a_{\ \lambda} \right) + \dots := R^\lambda_{\ \mu\nu\sigma} q^a_{\ \lambda} + \dots \quad -(5)$$

i.e.

$$\left(D_\mu T^\lambda_{\ \nu\sigma} + D_\sigma T^\lambda_{\ \mu\nu} + D_\nu T^\lambda_{\ \sigma\mu} \right) q^a_{\ \mu} = \left(R^\lambda_{\ \mu\nu\sigma} + R^\lambda_{\ \sigma\mu\nu} + R^\lambda_{\ \nu\sigma\mu} \right) q^a_{\ \mu} \quad -(6)$$

using the Leibnitz Theorem and eq. (4), the tetrad postulate.

A particular solution of eq (6) is:

2)

$$D_\mu T_{\nu\sigma}{}^{\kappa} + D_\sigma T_{\mu\nu}{}^{\kappa} + D_\nu T_{\sigma\mu}{}^{\kappa}$$
$$= R^{\kappa}{}_{\mu\nu\sigma} + R^{\kappa}{}_{\sigma\mu\nu} + R^{\kappa}{}_{\nu\sigma\mu} \quad -(7)$$

i.e.: $$D_\mu \tilde{T}^{\kappa\mu\nu} = \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(8)$$

Now use:

$$D_\sigma T^{\kappa}_{\mu\nu} = \partial_\sigma T^{\kappa}_{\mu\nu} + \Gamma^{\kappa}_{\sigma\lambda} T^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\sigma\mu} T^{\kappa}_{\lambda\nu} - \Gamma^{\lambda}_{\sigma\nu} \Gamma^{\kappa}_{\mu\lambda}$$
$$D_\mu T^{\kappa}_{\nu\sigma} = \partial_\mu T^{\kappa}_{\nu\sigma} + \Gamma^{\kappa}_{\mu\lambda} T^{\lambda}_{\nu\sigma} - \Gamma^{\lambda}_{\mu\nu} T^{\kappa}_{\lambda\sigma} - \Gamma^{\lambda}_{\mu\sigma} T^{\kappa}_{\nu\lambda}$$
$$D_\nu T^{\kappa}_{\sigma\mu} = \partial_\nu T^{\kappa}_{\sigma\mu} + \Gamma^{\kappa}_{\nu\lambda} T^{\lambda}_{\sigma\mu} - \Gamma^{\lambda}_{\nu\sigma} \Gamma^{\kappa}_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} \Gamma^{\kappa}_{\sigma\lambda}$$
$$-(9)$$

Eq (9) follows from the rule for the covariant derivative of a rank three tensor (Carroll chapter 3).

Therefore:

$$D_\sigma T^{\kappa}_{\mu\nu} + D_\mu T^{\kappa}_{\nu\sigma} + D_\nu T^{\kappa}_{\sigma\mu}$$
$$= \left(\partial_\sigma T^{\kappa}_{\mu\nu} + \partial_\mu T^{\kappa}_{\nu\sigma} + \partial_\nu T^{\kappa}_{\sigma\mu} + \Gamma^{\kappa}_{\sigma\lambda} T^{\lambda}_{\mu\nu} + \Gamma^{\kappa}_{\mu\lambda} T^{\lambda}_{\nu\sigma} + \Gamma^{\kappa}_{\nu\lambda} T^{\lambda}_{\sigma\mu} \right)$$
$$- \left(\Gamma^{\lambda}_{\sigma\mu} T^{\kappa}_{\lambda\nu} + \Gamma^{\lambda}_{\mu\sigma} T^{\kappa}_{\nu\lambda} + \Gamma^{\lambda}_{\sigma\nu} T^{\kappa}_{\mu\lambda} + \Gamma^{\lambda}_{\nu\sigma} T^{\kappa}_{\lambda\mu} + \Gamma^{\lambda}_{\mu\nu} T^{\kappa}_{\lambda\sigma} + \Gamma^{\lambda}_{\nu\mu} T^{\kappa}_{\sigma\lambda} \right) \quad -(10)$$

3) Using the definition of the torsion:

$$T^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} \quad -(11)$$

the first bracket in eq. (10) gives:

$$R^{\kappa}{}_{\sigma\mu\nu} + R^{\kappa}{}_{\mu\nu\sigma} + R^{\kappa}{}_{\nu\sigma\mu}$$

$$:= \partial_\mu \Gamma^{\kappa}_{\nu\sigma} - \partial_\nu \Gamma^{\kappa}_{\mu\sigma} + \Gamma^{\kappa}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\kappa}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

$$+ \partial_\nu \Gamma^{\kappa}_{\sigma\mu} - \partial_\sigma \Gamma^{\kappa}_{\nu\mu} + \Gamma^{\kappa}_{\nu\lambda}\Gamma^{\lambda}_{\sigma\mu} - \Gamma^{\kappa}_{\sigma\lambda}\Gamma^{\lambda}_{\nu\mu}$$

$$+ \partial_\sigma \Gamma^{\kappa}_{\mu\nu} - \partial_\mu \Gamma^{\kappa}_{\sigma\nu} + \Gamma^{\kappa}_{\sigma\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\kappa}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\nu}$$

$$-(12)$$

This is the right hand side of eq. (7). Therefore the second bracket in eq. (11) must be identically zero, and so:

$$\Gamma^{\lambda}_{\sigma\mu} T^{\kappa}_{\lambda\nu} + \Gamma^{\lambda}_{\nu\sigma} T^{\kappa}_{\lambda\mu} + \Gamma^{\lambda}_{\mu\nu} T^{\kappa}_{\lambda\sigma}$$

$$+ \Gamma^{\lambda}_{\sigma\nu} T^{\kappa}_{\mu\lambda} + \Gamma^{\lambda}_{\mu\sigma} T^{\kappa}_{\nu\lambda} + \Gamma^{\lambda}_{\nu\mu} T^{\kappa}_{\sigma\lambda} := 0$$

$$-(13)$$

Now use:

$$T^{\kappa}_{\lambda\mu} = -T^{\kappa}_{\mu\lambda} \quad -(14)$$

etc. to find: $\quad -(15)$

$$\boxed{T^{\kappa}_{\lambda\nu} T^{\lambda}_{\sigma\mu} + T^{\kappa}_{\lambda\mu} T^{\lambda}_{\nu\sigma} + T^{\kappa}_{\lambda\sigma} T^{\lambda}_{\mu\nu} := 0}$$

4)

Eq (15) is a new identity of Riemann / Cartan geometry. It is the wedge product:

$$T^{\kappa}{}_{\lambda} \wedge T^{\lambda} := 0 \quad -(16)$$

or in shorthand:

$$T \wedge T := 0. \quad -(17)$$

in which $T^{\kappa}{}_{\lambda}$ is regarded as a tensor-valued one form with index ν, and T^{λ} as a vector-valued two-form of index $\sigma\mu$.

In note 109(3) it will be shown that the computer algebra result $R^{\kappa}{}_{\mu}{}^{\mu\nu} \neq 0$ implies

$$D_{\mu} T^{\kappa\mu\nu} = R^{\kappa}{}_{\mu}{}^{\mu\nu}, \quad -(18)$$

demonstrating in another way that EH geometry is fundamentally incorrect.