1) 108(6): Motion of Two Particles

Following a textbook such as Marion and Thornton the problem of the motion of two particles can be reduced to an equivalent one body problem using the reduced mass:

$$\mu = m_1 m_2 / (m_1 + m_2). \quad — (1)$$

This describes the motion of a particle of mass μ in a central field described by μ. The lagrangian is:

$$L = \frac{1}{2} \mu |\dot{r}|^2 - U(r) \quad — (2)$$

where $U(r)$ is the potential. The problem is described by:

The translational motion of the system as a whole is uninteresting from the point of view of particle orbits with respect to each other, as in a binary pulsar. So:

$$\underline{R} = \underline{0}, \quad m_1 \underline{r_1} + m_2 \underline{r_2} = \underline{0} \quad — (3)$$

and

$$\underline{r} = \underline{r_1} - \underline{r_2} \quad — (4)$$

So:

$$\underline{r_1} = \frac{m_2}{m_1 + m_2} \underline{r}, \quad — (5)$$

$$\underline{r_2} = - \frac{m_1}{m_1 + m_2} \underline{r} \quad — (6)$$

The lagrangian (2) is found from substituting eqs. (5) and (6) into:

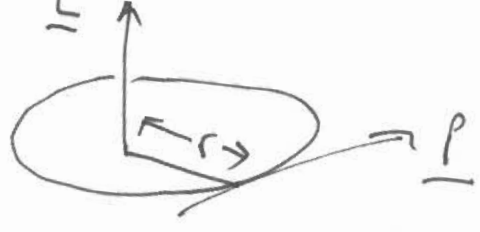
2)

$$L = \frac{1}{2} m_1 |\dot{\underline{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\underline{r}}_2|^2 - U(\underline{r}) \quad -(7)$$

The potential energy does not depend on orientation and its total angular momentum is conserved:

$$\underline{L} = \underline{r} \times \underline{p} = \text{constant} \quad -(8)$$

The Lagrangian can therefore be expressed in plane polar coordinates:



$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - u(r) \quad -(9)$$

The conjugate angular momentum is conserved:

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} \quad -(10).$$

i.e.

$$\boxed{p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = \text{constant}} \quad -(11)$$

The <u>areal</u> <u>velocity</u> of the orbit is:

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} := \frac{\ell}{2\mu} \quad -(12)$$

$$= \text{constant}$$

where:

$$\boxed{\ell = \mu r^2 \dot{\theta} = \text{constant}} \quad -(13)$$

This is <u>Kepler's</u> <u>second</u> <u>law</u> – <u>equal areas</u> <u>in</u> <u>equal</u> <u>times</u>.

3) The laws of conservation of energy is:

$$E = T + U = \text{constant} \quad -(14)$$

$$= \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r)$$

i.e.

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r) \quad -(15)$$

Thus:

$$\dot{r} = \frac{dr}{dt} = \left(\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}\right)^{1/2} \quad -(16)$$

Writing:

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}\,dr = \frac{\dot{\theta}}{\dot{r}}\,dr \quad -(17)$$

and using:

$$\dot{\theta} = \frac{\ell}{\mu r^2} \quad -(18)$$

$$\boxed{\theta(r) = \int \frac{\ell}{r^2}\left(2\mu\left(E - U - \frac{\ell^2}{2\mu r^2}\right)\right)^{-1/2} dr} \quad -(19)$$

This is the classical equation of the orbit.

Lagrangian Approach

The Lagrange equation is:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \quad -(20)$$

where L is given by eq. (9). So the force is given by:

$$F(r) = -\frac{\partial U}{\partial r} = \mu\left(\ddot{r} - r\dot{\theta}^2\right) \quad -(21)$$

4) Now use the change of variable:

$$u = \frac{1}{r} \quad -(22)$$

so:
$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} \quad -(23)$$

with:
$$\dot{\theta} = \frac{l}{\mu r^2} \quad -(24)$$

so:
$$\frac{du}{d\theta} = -\frac{\mu}{l}\dot{r} \quad -(25)$$

Also:
$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{\mu}{l}\dot{r}\right) = \frac{dt}{d\theta}\frac{d}{dt}\left(-\frac{\mu}{l}\dot{r}\right)$$

$$= -\frac{\mu}{l\dot{\theta}}\ddot{r} \quad -(26)$$

$$= -\frac{\mu^2}{l^2}r^2\ddot{r}$$

Thus:
$$\ddot{r} = -\frac{l^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}$$

$$r\dot{\theta}^2 = \frac{l^2}{\mu^2}u^3$$

and:
$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{l^2}\frac{1}{u^2}F(u)} \quad -(27)$$

This is very useful to find the force law that gives a particular orbit $r = r(\theta)$.

5) The last term is the radical of eq. (19), i.e.

$$\left(E - U - \frac{\ell^2}{2\mu r^2}\right)^{1/2} \text{ is:}$$

$$U_c = \frac{\ell^2}{2\mu r^2} = \frac{1}{2}\mu r^2 \dot{\theta}^2 \quad -(28)$$

which generates the centrifugal force:

$$F_c = -\frac{\partial U_c}{\partial r} = \frac{\ell^2}{\mu r^3} = \mu r \dot{\theta}^2 \quad -(29)$$

The effective potential energy is therefore:

$$V(r) = U(r) + \frac{\ell^2}{2\mu r^2} \quad -(30)$$

For the inverse square law:

$$F(r) = -\frac{k}{r^2}, \quad k = mMG \quad -(31)$$

so:

$$U(r) = -\int F(r)\, dr = -\frac{k}{r} \quad -(32)$$

The effective potential function for gravitational attraction is therefore:

$$V(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2} \quad -(33)$$

Eq. (19) can be integrated to give:

$$\boxed{\frac{\alpha}{r} = 1 + \epsilon \cos\theta} \quad -(34)$$

where:

6)

$$\alpha = \frac{\ell^2}{\mu k}, \quad \epsilon = \left(1 + \frac{2E\ell^2}{\mu k^2}\right)^{1/2} \quad -(34)$$

This is the equation of a conic section with one focus at the origin. The quantity ϵ is the eccentricity.

Advance of the Perihelion

For an inverse square law, all elliptical orbits must be exactly closed. Any advance in the perihelion of a planet or a binary pulsar shows that the force law is not exactly inverse squared. For the inverse square law, eq. (27) is:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2}\frac{1}{u^2}F(u) \quad -(35a)$$

$$= \frac{Gm^2M}{\ell^2}$$

so μ has been replaced by m. Eq. (35a) describes the motion of a particle of mass m in the gravitational field of a particle of mass M. In general relativity, if:

$$r_s = \frac{2GM}{c^2} \quad -(36)$$

7) then:

$$\frac{d^2u}{d\theta^2} + u = \frac{Gm^2M}{\ell^2} + \frac{3GM}{c^2}u^2 \quad -(37)$$

If:

$$\frac{1}{\alpha} = \frac{Gm^2M}{\ell^2}, \quad \delta = \frac{3GM}{c^2} \quad -(38)$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{1}{\alpha} + \delta u^2} \quad -(39)$$

ECE Theory

Replace $\frac{2GM}{c^2}$ by T/R, i.e:

$$\boxed{GM = \frac{c^2}{2}\frac{T}{R}} \quad -(40)$$

in eq. (39).

Orbit of a Binary Pulsar

1) Find $u = 1/r$ from eq. (39)
2) Use eqs (5) and (6) to find the individual motions of the particles, i.e to find $\underline{r}_1$ and $\underline{r}_2$ given $\underline{r}$.
3) Use $m_1 = m$, $m_2 = M$.

8) The advance of the perihelion is usually found by using a method of successive approximations to eq. (39). It can be shown that the perihelion advance is:

$$\Delta \sim 6\pi \left(\frac{G m M}{c h} \right)^2 \quad - (41)$$

Numerical Method

Integrate eq. (39) numerically to produce a graph of:

$$u(\theta) := u(\theta, T/R) \quad - (42).$$

The motion is an ellipse for a binary pulsar. with advancing perihelion and decreasing orbit, i.e. r decreases for the HT binary by 3.1 mm a revolution and the pulsar perihelion advances by 4° a revolution. Find T/R that reproduces these data from eq. (39). Find the individual r_1 and r_2 by using the known masses of the two stars.