1) 104(2) : Illustrating the Meaning of Connection for Rotation in a Plane

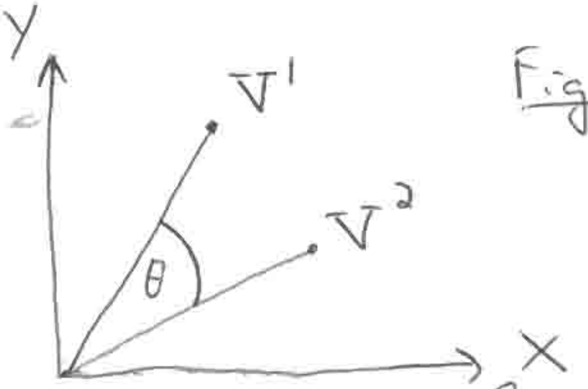

Fig (1)

Consider the clockwise rotation in a plane of a vector V^1 to V^2. This rotation is carried out by moving the vector and keeping the frame of reference fixed. This is equivalent to keeping the vector fixed and rotating the frame of reference anti-clockwise by an equal amount θ. In cartesian coordinates:

$$\underline{V}^1 = V^1_x \underline{i} + V^1_y \underline{j} \quad -(1)$$

$$\underline{V}^2 = V^2_x \underline{i} + V^2_y \underline{j} \quad -(2)$$

and

$$|\underline{V}^1| = |\underline{V}^2| \quad -(3)$$

where

$$|\underline{V}^1| = \left(V^{1\,2}_x + V^{1\,2}_y\right)^{1/2} \quad -(4)$$

$$|\underline{V}^2| = \left(V^{2\,2}_x + V^{2\,2}_y\right)^{1/2} \quad -(5)$$

This is a rotation in which the frame is fixed, i.e. the cartesian unit vectors $\underline{i}$ and $\underline{j}$ do not change. The rotation could equally well be represented by:

$$\underline{V}^1 = V_x \underline{i}_1 + V_y \underline{j}_1 \quad -(6)$$

$$\underline{V}^2 = V_x \underline{i}_2 + V_y \underline{j}_2 \quad -(7)$$

2) and in this case the vector is fixed and the frame rotated anti-clockwise. We now have:

$$|\underline{V}^1| = |\underline{V}^2| \qquad -(8)$$

$$= \left(V_x^{2\,2} + V_y^{2\,2}\right)^{1/2}$$

because:

$$\left.\begin{aligned} \underline{i}_1 \cdot \underline{i}_1 &= \underline{i}_2 \cdot \underline{i}_2 = 1 \\ \underline{j}_1 \cdot \underline{j}_1 &= \underline{j}_2 \cdot \underline{j}_2 = 1 \end{aligned}\right\} \qquad -(9)$$

The invariance under rotation of the complete vector field is true in both cases:

a) $V^{1\,2} = V_x^{1\,2} + V_y^{1\,2} = V_x^{2\,2} + V_y^{2\,2} = V^{2\,2}$

b) $V^{1\,2} = V_x^{2} + V_y^{2} = V^{2\,2}$ $\qquad -(10)$

The rotation can also be represented by:

$$\begin{bmatrix} V_x^1 \\ V_y^1 \\ V_z^1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x^2 \\ V_y^2 \\ V_z^2 \end{bmatrix} \qquad -(11)$$

i.e.

$$V_x^1 = V_x^2 \cos\theta + V_y^2 \sin\theta \qquad -(12)$$

$$V_y^1 = -V_x^2 \sin\theta + V_y^2 \cos\theta \qquad -(13)$$

$$V_z^1 = V_z^2 \qquad -(14)$$

These equations are usually interpreted as the vector rotated clockwise with fixed frame.

3) However, they are also true for a fixed vector and frame rotated anti-clockwise. This is an example of the frame itself moving. Therefore the rules governing the covariant derivative and connection apply. The general rule for covariant derivative is:

$$D_\nu \bar{V}^\mu = \partial_\nu \bar{V}^\mu + \Gamma^\mu_{\lambda\nu} \bar{V}^\lambda \quad -(15)$$

This equation means that D_ν acting on $\bar{V}^\mu$ is the ordinary derivative ∂_ν plus the term $\Gamma^\mu_{\lambda\nu} \bar{V}^\lambda$. The three index symbol $\Gamma^\mu_{\lambda\nu}$ is referred to as the connection, and describes the movement of the frame. The latter produces, for a given ν:

$$\bar{U}^\mu = \Gamma^\mu_\lambda \bar{V}^\lambda \quad -(16)$$

It is seen that eq. (11) is an example of eq (16) in three dimensions, X, Y and Z. So for a rotation of the frame anti-clockwise in 3-D about the Z axis the matrix Γ^μ_λ is the rotation matrix:

$$\Gamma^\mu_\lambda = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad -(17)$$

4) Thus:

$$\left.\begin{array}{lll} \Gamma^1{}_1 = \cos\theta, & \Gamma^1{}_2 = \sin\theta, & \Gamma^1{}_3 = 0, \\ \Gamma^2{}_1 = -\sin\theta, & \Gamma^2{}_2 = \cos\theta, & \Gamma^2{}_3 = 0, \\ \Gamma^3{}_1 = 0, & \Gamma^3{}_2 = 0, & \Gamma^3{}_3 = 1 \end{array}\right] - (18)$$

for each $\sim$. Summation over repeated indices is used in eq. (16), so:

$$\left.\begin{array}{l} U^1 = \Gamma^1{}_1 V^1 + \Gamma^1{}_2 V^2 + \Gamma^1{}_3 V^3 \\ U^2 = \Gamma^2{}_1 V^1 + \Gamma^2{}_2 V^2 + \Gamma^2{}_3 V^3 \\ U^3 = \Gamma^3{}_1 V^1 + \Gamma^3{}_2 V^2 + \Gamma^3{}_3 V^3 \end{array}\right\} - (19)$$

for each $\sim$. These equations (19) are the same as eqns. (12) to (14).

The covariant derivative of eq. (15) in this case is therefore:

$$D_\sim V^\mu = (\partial + \Gamma^\mu{}_\lambda)_\sim V^\lambda. \quad - (20)$$

For example:

$$D_\sim V^1 = (\partial + \Gamma^1{}_1)_\sim V^1 + \Gamma^1{}_{2\sim} V^2$$

$$D_\sim V^1 = (\partial + \cos\theta)_\sim V^1 + \sin\theta_\sim V^2$$

$$\boxed{D_\sim V^1 = \partial_\sim V^1 + (\cos\theta)_\sim V^1 + (\sin\theta)_\sim V^2}$$

$$- (21)$$

5) Thus:

$$\Gamma^1{}_{1\nu} = (\cos\theta)_\nu \; ; \; \Gamma^1{}_{2\nu} = (\sin\theta)_\nu \quad -(22)$$

These connections must have the units of inverse metres and must operate in the same way as the partial derivative ∂_ν. So it is reasonable to assume that:

$$\Gamma^1{}_{1\nu} = \frac{1}{2}\cos\theta \, \partial_\nu \; ; \; \Gamma^1{}_{2\nu} = \frac{1}{2}\sin\theta \, \partial_\nu \quad -(23)$$

and:

$$\boxed{D_\nu V^1 = \frac{1}{2}\left((1 + \cos\theta)\,\partial_\nu V^1 + \sin\theta\,\partial_\nu V^2\right)} \quad -(24)$$

If there is no frame rotation:

$$\theta = 0 \quad -(25)$$

and

$$D_\nu V^1 = \partial_\nu V^1. \quad -(26)$$

This method regards the connection as an operator. It is well known that the set $\{\partial_\mu\}$ is a basis set in Riemann geometry. Other possibilities are:

$$(\cos\theta)_\nu = \frac{\cos\theta}{r} \; ; \; (\sin\theta)_\nu = \frac{\sin\theta}{r}. \quad -(27)$$