00(18): Fundamental Origin of the Bianchi Identity

The fundamental origin of the Bianchi identity is the equation:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(1)$$

where the commutator is defined by:

$$[D_\mu, D_\nu] = D_\mu D_\nu - D_\nu D_\mu \quad -(2)$$

and where the covariant derivative is

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu{}_{\mu\lambda} V^\lambda. \quad -(3)$$

Eq (1) is true for any connection $\Gamma^\nu{}_{\mu\lambda}$ and is independent of metric compatibility (see Carroll, chapter 3). Therefore the fundamental curvature and torsion tensors are defined by the commutator of covariant derivatives (2) as shown in paper 99. Using the rules (2) and (3) they are defined as:

$$R^\lambda{}_{\rho\mu\nu} := \partial_\mu \Gamma^\lambda{}_{\nu\rho} - \partial_\nu \Gamma^\lambda{}_{\mu\rho} + \Gamma^\lambda{}_{\mu\sigma} \Gamma^\sigma{}_{\nu\rho} - \Gamma^\lambda{}_{\nu\sigma} \Gamma^\sigma{}_{\mu\rho} \quad -(4)$$

and

$$T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}. \quad -(5)$$

Textbook general relativity deals almost entirely with the Christoffel connection:

$$\Gamma^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu}, \quad -(6)$$

which implies:

$$T^\lambda{}_{\mu\nu} = 0. \quad -(7)$$

2)

However, there is no a priori reason for assuming that the connection is the Christoffel connection. The latter comes from metric compatibility and a symmetric metric:

$$g_{\mu\nu} = g_{\nu\mu} \quad — (8)$$

(see Carroll chapter 3). The rigorous Bianchi identity of differential geometry does not make any assumption about the nature of the connection. It assumes only the tetrad postulate

$$D_\mu q^a_{\ \nu} = 0 \quad — (9)$$

which is the basic property that a complete vector field is invariant under the general coordinate transformation. As in paper 101 the Bianchi identity is an exact identity consisting of a cyclic sum of the definitions (4). In order for this to be true, eq. (5) must also be used, so the Bianchi identity uses both definitions (4) and (5). These originate in eq. (1), so the latter is the origin of the Bianchi identity. In shorthand the latter is:

$$D \wedge T := R \wedge q \quad — (10)$$

If it is arbitrarily assumed that:

$$T = 0 \quad — (11)$$

the Bianchi identity cannot be true.

3)

Unfortunately, textbook general relativity almost always assumes eq. (11), and so almost always assumes:

$$R \wedge q = 0. \quad -(12)$$

Furthermore, standard model or textbook gr almost always refers to eq. (12) as "the first Bianchi identity". It is neither an identity nor first derived by Bianchi. Eq (12) is true if and only if eq. (8) is true, and so it is true only for a Christoffel connection and zero torsion.

Therefore the geometrical basis of textbook general relativity is wholly incorrect.

The "second Bianchi identity" used by Einstein is:

$$D \wedge R = 0 \quad -(13)$$

but the true second Bianchi identity is:

$$D \wedge (D \wedge T) := D \wedge (D \wedge R) \quad -(14)$$

as shown in paper 88. Eq. (13) is an approximation of eq (14) when eq. (11) is used.

The problems in textbook general relativity are shown very clearly when the Bianchi identity (10) is translated into tensor notation to give:

4)

$$D_\mu \tilde{T}^{\kappa\mu\nu} = \tilde{R}^{\kappa}{}_{\mu}{}^{\mu\nu} \quad -(15)$$

where the tilde denotes Hodge dual. The Hodge duals of eqs. (4) and (5) are defined by:

$$\tilde{T}^{\lambda}{}_{\alpha\beta} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\alpha\beta\mu\nu} g^{\mu\kappa} g^{\nu\sigma} T^{\lambda}{}_{\kappa\sigma} \quad -(16)$$

$$\tilde{R}^{\lambda}{}_{\rho\alpha\beta} = \frac{1}{2}\|g\|^{1/2}\epsilon_{\alpha\beta\mu\nu} g^{\mu\kappa} g^{\nu\sigma} R^{\lambda}{}_{\rho\kappa\sigma} \quad -(17)$$

and are defined on these two differential two-forms. The Hodge dual operates on the last two indices of the tensors (4) and (5) because these are the anti-symmetric indices defining a differential two-form:

$$R^{\lambda}{}_{\rho\mu\nu} = -R^{\lambda}{}_{\rho\nu\mu} \quad -(18)$$

$$T^{\lambda}{}_{\mu\nu} = -T^{\lambda}{}_{\nu\mu} \quad -(19)$$

The Hodge duals also have this property and are also two forms

$$\tilde{R}^{\lambda}{}_{\rho\alpha\beta} = -\tilde{R}^{\lambda}{}_{\rho\beta\alpha} \quad -(20)$$

$$\tilde{T}^{\lambda}{}_{\alpha\beta} = -\tilde{T}^{\lambda}{}_{\beta\alpha} \quad -(21)$$

This fundamental antisymmetry comes from eq. (1), because the commutator is anti-symmetric.

5) Multiplying left side of eq. (1) by:

$$\frac{1}{2}\|g\|^{1/2}\epsilon_{\alpha\beta\mu\nu}\,g^{\mu\kappa}g^{\nu\sigma} \quad -(22)$$

and using:

$$[D_\kappa, D_\sigma]V^\rho = R^\rho{}_{\delta\kappa\sigma}V^\delta - T^\lambda{}_{\kappa\sigma}D_\lambda V^\rho \quad -(23)$$

it is found that:

$$[D_\alpha, D_\beta]V^\rho = \tilde{R}^\rho{}_{\delta\alpha\beta}V^\delta - \tilde{T}^\lambda{}_{\alpha\beta}D_\lambda V^\rho \quad -(24)$$

where the connections appearing in $\tilde{R}^\rho{}_{\delta\alpha\beta}$ and $\tilde{T}^\lambda{}_{\alpha\beta}$ are the same as the connections appearing in $R^\rho{}_{\delta\kappa\sigma}$ and $T^\lambda{}_{\kappa\sigma}$. Eq. (24) implies the Hodge dual Bianchi identity:

$$\boxed{D\wedge\tilde{T} := \tilde{R}\wedge q} \quad -(25)$$

whose tensor formulation is:

$$\boxed{D_\mu T^{\kappa\mu\nu} := R^{\kappa}{}_{\mu}{}^{\mu\nu}} \quad -(26)$$

By computer algebra in papers 93 onwards it has been shown that $R^{\kappa}{}_{\mu}{}^{\mu\nu}$ is zero if and

6)

only if all elements of the Ricci tensor vanish, i.e. in a Ricci flat spacetime. Otherwise, even for a Christoffel connection, $R^{\kappa}{}_{\mu}{}^{\mu\nu}$ is not zero. However, a Christoffel connection always implies a zero $T^{\kappa\mu\nu}$. So a Christoffel connection is incompatible with the Bianchi identity (26).

Therefore the basics of textbook general relativity are not internally consistent. The only internally consistent theory is ECE theory, which is not restricted to the Christoffel connection.
