100(40): On the Structure of the Hodge Dual of the Bianchi Identity

Theorem

If, in indexless notation:

$$D \wedge T := R \wedge q \quad -(1)$$

then:

$$D \wedge \tilde{T} := \tilde{R} \wedge q. \quad -(2)$$

The definition of the Hodge dual in these identities is determined by the basic commutator equation:

$$[D_\mu, D_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T^\lambda{}_{\mu\nu} D_\lambda V^\rho \quad -(3)$$

which is antisymmetric in μ and ν on both sides. In general the Hodge dual of an antisymmetric tensor in general relativity is:

$$\tilde{X}^{\rho\sigma} = \frac{1}{2} \|g\|^{1/2} \epsilon^{\rho\sigma\mu\nu} X_{\mu\nu} \quad -(4)$$

where $|g|$ is the determinant of the metric and $\epsilon^{\rho\sigma\mu\nu}$ the Minkowski space Levi-Civita tensor. In order to lower indices the metric is used as follows:

$$X_{\rho\sigma} = g_{\rho\kappa} g_{\sigma\lambda} X^{\kappa\lambda} \quad -(5)$$

Thus

$$\tilde{X}_{\rho\sigma} = \left(\frac{1}{2} \|g\|^{1/2} g_{\rho\kappa} g_{\sigma\lambda} \right) \epsilon^{\kappa\lambda\mu\nu} X_{\mu\nu} \quad -(6)$$

$$= \frac{1}{2} \|g\|^{1/2} \epsilon_{\rho\sigma}{}^{\mu\nu} X_{\mu\nu}$$

Now apply the rule (6) to each side of eq (3)

2) to obtain the Hodge dual of eq. (3):

$$[D_\alpha, D_\beta]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma\alpha\beta} V^\sigma - \tilde{T}^\lambda{}_{\alpha\beta} D_\lambda V^\rho \quad -(7)$$

where indices have been lowered as in eq. (6). Also:

$$[D^\alpha, D^\beta]_{HD} V^\rho = \tilde{R}^\rho{}_\sigma{}^{\alpha\beta} V^\sigma - \tilde{T}^{\lambda\alpha\beta} D_\lambda V^\rho \quad -(8)$$

In eq. (8):

$$\tilde{R}^\rho{}_\sigma{}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} R^\rho{}_{\sigma\mu\nu} \quad -(9)$$

$$\tilde{T}^{\lambda\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} T^\lambda{}_{\mu\nu} \quad -(10)$$

where $\epsilon^{\alpha\beta\mu\nu}$ is defined in Minkowski spacetime. The complication of $||g||^{1/2}$ has been removed. From eq. (3):

$$R^\lambda{}_{\rho\mu\nu} := \partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\rho\mu} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma} \Gamma^\sigma_{\mu\rho} \quad -(11)$$

$$T^\lambda{}_{\mu\nu} := \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \quad -(12)$$

The Hodge duals of eqs (9) and (10) are therefore defined in terms of the same Γ connections. In general, these are not the Christoffel connection.

In order for eq. (1) to be true, eqs. (11) and (12) must be true, and eqs (11) and (12) follow from eq. (3). Therefore, given

3) the tetrad postulate:

$$D_\mu q^a_\nu = 0 \quad -(13)$$

eq. (1) is equivalent to eq. (3). As shown elsewhere, eq. (1) is an exact identity consisting of cyclic sums of Riemann tensors.

By the same reasoning, eq. (2) is equivalent to eq. (8), Q.E.D. This proves the theorem that given the Bianchi identity (1), the Hodge dual (2) is also true.

Field Equations

In order to explain how this result is applied to the field equation of electrodynamics, some guidelines are needed from experimental data, and we build a the basis of the standard model. In the latter it is known that if:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -cB_z & cB_y \\ E_y & cB_z & 0 & -cB_x \\ E_z & -cB_y & cB_x & 0 \end{bmatrix} ; \tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & -E_z & E_y \\ -cB_y & E_z & 0 & -E_x \\ -cB_z & -E_y & E_x & 0 \end{bmatrix} \quad -(14)$$

then the field equations are:

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu F^{\mu\nu} = J^\nu / \epsilon_0 \quad -(15)$$

This is the result of taking a Hodge dual in

Minkowski spacetime and raising indices with metrics of Minkowski spacetime. Thus:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -F^{01} & -F^{02} & -F^{03} \\ F^{10} & 0 & -F^{12} & F^{13} \\ F^{20} & F^{21} & 0 & -F^{23} \\ F^{30} & -F^{31} & F^{32} & 0 \end{bmatrix} \rightarrow \tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & F^{32} & F^{13} & F^{21} \\ -F^{23} & 0 & -F^{03} & F^{20} \\ -F^{31} & F^{30} & 0 & -F^{01} \\ -F^{12} & -F^{02} & F^{10} & 0 \end{bmatrix} \quad -(16)$$

Thus:

$$\left.\begin{aligned} \tilde{F}^{01} &= F^{32}, \; \tilde{F}^{02} = F^{13}, \; \tilde{F}^{03} = F^{21}, \\ \tilde{F}^{10} &= F^{23}, \; \tilde{F}^{12} = F^{03}, \; \tilde{F}^{13} = F^{20}, \\ \tilde{F}^{20} &= F^{31}, \; \tilde{F}^{21} = F^{30}, \; \tilde{F}^{31} = F^{01}, \\ \tilde{F}^{30} &= F^{12}, \; \tilde{F}^{31} = F^{02}, \; \tilde{F}^{32} = F^{10}. \end{aligned}\right\} \quad -(17)$$

The homogeneous and inhomogeneous field equations of the standard model are therefore inter-related by eq. (17), in which indices are re-arranged in a well defined way, determined by the underlying Minkowski line-element and metric. In differential form notation this process is all represented in an elegant way by:

$$d \wedge F = 0, \quad d \wedge \tilde{F} = \tilde{J}/\epsilon_0. \quad -(18)$$

<u>The ECE Field Equations</u>

These are:

$$d \wedge F^a = 0, \quad d \wedge \tilde{F}^a = \tilde{J}^a/\epsilon_0. \quad -(19)$$

5) A particular solution of eq. (19) is the base manifold equation:

$$d \wedge F^{\kappa} = 0, \quad d \wedge \tilde{F}^{\kappa} = \tilde{J}^{\kappa} / \epsilon_0 \quad -(20)$$

which in tensor notation are:

$$\partial_\mu \tilde{F}^{\kappa\mu\nu} = 0, \quad \partial_\mu F^{\kappa\mu\nu} = J^{\kappa\nu} / \epsilon_0 \quad -(21)$$

The origin of eqns. (19) to (21) is the Bianchi identity (1), which is equivalent to eq. (3) given the tetrad postulate. For example:

$$[D_0, D_1] V^\rho = R^\rho{}_{\sigma 0 1} V^\sigma - T^\lambda{}_{01} D_\lambda V^\rho \quad -(22)$$

This equation can be written as:

$$[D_3, D_2]_{HD} V^\rho = \tilde{R}^\rho{}_{\sigma 3 2} V^\sigma - \tilde{T}^\lambda{}_{32} V^\rho. \quad -(23)$$

Therefore in forming the Hodge duals in the field equations (21), the rule is that indices are rearranged as in eq. (17) of the standard model, without the complication of non-Minkowski metrics, but with the addition of a κ index. Thus for example:

$$\boxed{\tilde{F}^{\kappa 01} = F^{\kappa 32}} \quad -(24)$$

and so on as in eq. (17).

In shorthand notation this rule means that the homogeneous field equation is defined as:

$$D \wedge F := R \wedge A \quad -(25)$$

and the inhomogeneous field equation is defined as:

$$D \wedge \tilde{F} := \tilde{R} \wedge A \quad -(26)$$

where the relation between $\tilde{F}$ and F is given by eq. (24). The relation between $\tilde{R}$ and R is given by:

$$\boxed{\tilde{R}^{\rho}{}_{\sigma}{}^{01} = R^{\rho}{}_{\sigma}{}^{32}} \quad -(27)$$

and so on.

This means that the complication of raising and lowering indices with $g_{\mu\nu}$ is eliminated, and ensures that in vector notation the ECE field equations are:

$$\left.\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} &= \underline{0} \\ \underline{\nabla} \cdot \underline{E} &= \rho / \epsilon_0 \\ \underline{\nabla} \times \underline{B} - \frac{1}{c^2}\frac{\partial \underline{E}}{\partial t} &= \mu_0 \underline{J} \end{aligned}\right] \quad -(28)$$

which are known to describe a considerable amount of experimental data.