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LAGRANGIAN THEORY FOR THE FIELD $\mathbf{B}^{(3)}$: SELF INTERACTION IN THE VACUUM ELECTROMAGNETIC FIELD

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By considering in vacuo the appropriate Lagrangian, it is shown that the conjugate product of vector potentials in the electromagnetic field generates in vacuo the Evans-Vigier field $\mathbf{B}^{(3)}$ which thereby quantizes the conjugate product. In other words, classical field-field interaction becomes a self-interaction which manifests itself experimentally in measurable magneto-optic effects. Thus $\mathbf{B}^{(3)}$ is a physical magnetic field which does not exist in Maxwell's linear theory of electromagnetic propagation in vacuo.

1. INTRODUCTION

There are different philosophical approaches to fields and particles, ably summarized for example by Barut [1] and

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Ryder [2]. Fields can be described through functions and particles through coordinates. In classical physics [1] the concepts of field and particle are distinct and different. In quantum field theory, wave/particle duality blurs this clear distinction: as suggested by de Broglie [3,4], there exist matter waves and wave functions, matter waves which can be detected experimentally [4]. In this communication the idea [1] is used of classical field-field interaction to derive the recently inferred Evans-Vigier field $\boldsymbol{B}^{(3)}$, whose classical source in vacuo is the conjugate product $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and which propagates at the speed of light in the limit of zero mass. Here $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ is the vector potential of the electromagnetic field. Finally, quantization of the conjugate product is demonstrated through the relation [5–9]

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = \hbar\left(\frac{i\boldsymbol{B}^{(3)*}}{e}\right), \tag{1}$$

where e is both the elementary charge and a gauge scaling constant of the O(3) gauge group. In analogy with the Planck-Einstein Law, therefore, $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is to $i\boldsymbol{B}^{(3)*}/e$ as energy is to angular frequency.

2. PRELIMINARY CONSIDERATIONS

We use the reduced units and covariant-contravariant notation of Barut [1], whose derivation we follow closely. The idea is to consider the interaction of a vector field $\Phi_\mu(\chi)$ with the electromagnetic field A_μ, as discussed by Barut [1], but to assert that the vector field is itself an electromagnetic field. Therefore, we consider the interaction of A_μ with A_μ in classical physics and it is shown that the appropriate Lagrangian [1] for this interaction leads consistently to the d'Alembert and vacuum Maxwell equations, the Lorentz condition, and finally, to the novel Evans-Vigier field.

Consider a four vector field $\Phi_\mu(\chi)$ and set its mass to zero for the sake of illustration only. More generally, the mass of the field is non-zero [1]. Consider now the covariant derivative of $\Phi_\mu(\chi)$,

$$D_\nu(\Phi_\mu(\chi)) = (i\partial_\nu - eA_\nu)\Phi_\mu(\chi), \tag{2}$$

and we immediately have a non-linear field theory in the vacuum, because of the term in $A_\nu\Phi_\mu(\chi)$. The use of the covariant derivative guarantees, as usual, gauge invariance with respect to the electromagnetic field A_ν. The gauge invariant Lagrangian for the interaction of Φ_μ with A_μ is therefore [1]

$$\mathcal{L} = A^{\mu\nu}_{\sigma\rho}\left(D^*_\mu \phi^{*\sigma}\right)\left(D_\nu \phi^\rho\right), \tag{3}$$

where $A^{\mu\nu}_{\sigma\rho} = \alpha g^{\mu\nu} g_{\sigma\rho} + \beta g^\mu_\sigma g^\nu_\rho + \gamma g^\mu_\rho g^\nu_\sigma$, and where α, β and γ are scalars [1]. Barut uses the following Lagrange equation of motion,

$$\frac{\partial \mathcal{L}}{\partial \phi^{*\sigma}} = D_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(D^*_\mu \phi^{*\sigma}\right)} \right), \tag{4}$$

to derive

$$A^{\mu\nu}_{\sigma\rho} D_\mu D_\nu \phi^\rho = 0, \tag{5}$$

which is an equation of motion for ϕ_μ in the presence of A_μ. If ϕ_μ is itself an electromagnetic field, then

$$A^{\mu\nu}_{\sigma\rho} D_\mu D_\nu A^\rho = 0, \tag{6}$$

which is the equation for interacting electromagnetic fields. (In quantum field theory this becomes a consideration of two interacting particles, photons.) Particles of matter (electrons) have not been considered in the development, which has been restricted to the interaction of two classical electromagnetic fields, an interaction that can take place in the absence of matter, in the absence of sources, and therefore in the vacuum. Therefore the factor e that enters into the covariant derivative originates in gauge invariance, which originates in turn [1,2,10] from charge conservation.

It can be checked [1] that the free field A_μ in equation (6) obeys the right equation of motion, the vacuum d'Alembert equation.

1) Firstly set $\alpha = 1 = -(\beta + \gamma)$, so $\alpha + \beta + \gamma = 0$.

2) Secondly use

$$\phi_\mu = A_\mu + \partial_\mu \zeta, \tag{7}$$

[1] where ζ is a scalar field and where A_μ is a divergentless four-vector field.

Clearly, if we impose the further condition

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\zeta = 0, \tag{8}$$

then ϕ_μ and A_μ become identical, which is the starting point of our argument, i.e., the identification of ϕ_μ with the electromagnetic field. This is because the condition (8) is none other than that used to define gauge invariance [1,2] of physical quantities dependent on A_μ, usually $F_{\mu\nu}$, the electromagnetic field four-tensor. The latter is usually described [1,2] as being invariant under the gauge transformation,

$$A_\mu \to A_\mu + \partial_\mu \chi . \tag{9}$$

Identifying Eqs. (7) and (9) gives

$$\phi_\mu := A_\mu , \qquad \zeta := \chi , \tag{10}$$

and so Eq. (7) becomes identifiable as the usual gauge transformation [1,2] that leaves $F_{\mu\nu}$ invariant where,

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{11}$$

and the required divergentless nature of A_μ becomes the Lorentz condition,

$$\frac{\partial A^\mu}{\partial x^\mu} = 0 . \tag{12}$$

The equation of motion for the free field under these conditions [1] becomes

$$\partial^\mu \partial_\mu A_\nu := \Box A_\nu = 0 , \tag{13}$$

which is the required d'Alembert equation. Furthermore, if we define the four-potential A_ν in terms of a scalar potential ϕ and a vector potential $\boldsymbol{A}$, i.e.,

$$A^{\nu} := (\phi, \boldsymbol{A}), \qquad A_{\nu} := (\phi, -\boldsymbol{A}), \tag{14}$$

it can be shown, following the Barut method [1] that the d'Alembert equation is accompanied by

$$\partial^{\mu}\partial_{\mu}\phi := \Box\phi = 0. \tag{15}$$

Converting finally to S.I. units gives

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = 0, \qquad \left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = 0, \tag{16}$$

which are the standard d'Alembert equations for the scalar and vector potentials in the absence of a source [11]. The solutions are the vacuum Liénard-Wiechert potentials [11]. In the Coulomb gauge and when there are no sources,

Liénard

$$\nabla\cdot\boldsymbol{A} = \phi = 0, \qquad \boldsymbol{B} = \nabla\times\boldsymbol{A}, \qquad \boldsymbol{E} = -\frac{\partial\boldsymbol{A}}{\partial t}, \tag{17}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are vacuum electric and magnetic field components of the electromagnetic field. It is well known that Eqs. (16) are equivalent to the vacuum Maxwell equations under these conditions, so the free field A_{μ} obeys the Maxwell equations. Equation (17) obeys the Lorentz condition in S.I. units,

$$\frac{\partial A^{\mu}}{\partial \chi^{\mu}} = \nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0. \qquad (18)$$

Therefore the interaction of A_μ with A_μ modifies the d'Alembert equations (16) or alternatively the Maxwell equations, to give equation (6). It is by no means obvious that electromagnetic field-field interaction occurs in this way, and this method shows that rigorous non-linear field theory in the vacuum is quite different in structure from linear field theory in the vacuum. Barut [1] has recognized that:

1) "It is remarkable that the starting Lagrangian, which was written down by relativistic and gauge invariance conditions only......" leads to Eqs. (13) and (15).

2) "The appearance of vector and scalar fields together is perhaps of some deep significance."

If Barut's field $\Phi_\mu(\chi)$ [1] is identified with A_μ itself, and if we derive Eq. (6) and then check for the free field behavior, we find self-consistently the d'Alembert equation (13) for A_μ, the free field Maxwell equations, and the Lorentz condition. There is no mystery therefore, in point (2) above, i.e., the vector and scalar fields occur together because they are parts of the same four-vector, A_μ. The idea of A_μ interacting with A_μ therefore leads to a new interaction equation in the vacuum, Eq. (6), from which we now derive the Evans-Vigier field from fundamental first principles in the classical theory of fields.

3. DERIVATION OF THE EVANS-VIGIER FIELD

We now use the Barut method [1] to consider the interaction of the two four-vector fields $A_\mu^{(1)}$ and $A_\mu^{(2)}$. In the usual radiation gauge [1,2],

$$A_\mu^{(1)} = (\boldsymbol{A}^{(1)}, 0), \qquad A_\mu^{(2)} = (\boldsymbol{A}^{(2)}, 0), \tag{19}$$

where $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)*}$ is a plane wave solution of d'Alembert's equation (16). The general Lagrangian method developed by Barut leads to the Evans-Vigier field $\boldsymbol{B}^{(3)}$ through the interaction of $\boldsymbol{A}^{(1)}$ and $\boldsymbol{A}^{(2)}$, specifically through the conjugate product [12–15] $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, whose existence has been verified experimentally. The cross product of conjugate solutions of d'Alembert's equation produces the field $\boldsymbol{B}^{(3)}$ in the vacuum. In the quantum field theory it is concluded that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is *quantized* through $i\boldsymbol{B}^{(3)*}/e$. Since $\boldsymbol{B}^{(3)}$ is a physical magnetic field, this conclusion is reached also in other gauges, in which $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ may be fully covariant [16].

In Barut's notation the interaction equations are

$$(i\partial^{\mu} - eA^{\mu(1)})F_{\mu\nu} = ie\gamma A^{\mu(2)}F_{\mu\nu}, \tag{20a}$$

$$F_{\mu\nu} = \left(i\partial^{\mu} - eA_{\mu}^{(1)}\right)A_{\nu}^{(2)} - \left(i\partial_{\nu} - eA_{\nu}^{(1)}\right)A_{\mu}^{(2)}, \tag{20b}$$

and it is already clear that the usual field four-tensor (11) is supplemented in vacuo by

$$F_{\mu\nu}^{(3)} = -e\left(A_{\mu}^{(1)}A_{\nu}^{(2)} - A_{\nu}^{(1)}A_{\mu}^{(2)}\right), \tag{21}$$

which in vector notation becomes

$$\mathbf{F}^{(3)} := -\mathbf{B}^{(3)} = -e\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}, \tag{22}$$

and is a magnetic field, because the conjugate product $\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}$ has magnetic symmetry [5–9], and produces magnetization [12–15] in the inverse Faraday effect. This magnetic field is the Evans-Vigier field [5]. Writing the covariant derivative in the standard form [2],

$$-iD_{\mu} = \partial_{\mu} + ieA_{\mu}, \tag{23}$$

it becomes clear that the magnetic field in this reduced ($\hbar = c = 1$) notation is a real quantity, a physical magnetic field in vacuo,

$$\boldsymbol{B}^{(3)} = -ie\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}. \tag{24}$$

Converting to S.I. units requires quantization [1,2] and the introduction of the elementary unit of action, $\hbar$, so that,

$$\boldsymbol{B}^{(3)}\underset{\text{units}}{(S.I.)} = -i\frac{e}{\hbar}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}, \tag{25}$$

and this is precisely the result obtained independently [5,6] from O(3) gauge theory [2] in vacuo.

The field $\boldsymbol{B}^{(3)}$ has been obtained from the Lagrangian method of Barut [1] by assuming *only* that $A_\mu^{(1)}$ interacts with $A_\mu^{(2)}$ in the vacuum as the electromagnetic field propagates. Such an interaction produces the well-known conjugate product from fundamental Lagrangian principles, without recourse to phenomenology [12–15]. As shown recently [12] (and a little more indirectly, by Pershan [13]) the conjugate product is the vacuum optical property responsible in semi-classical, phenomenological theory, for the inverse Faraday effect. The latter is therefore an effect of the Evans-Vigier field [5–9], which is gauge independent as is the conjugate product itself. Whenever the latter produces observable effects we observe $\boldsymbol{B}^{(3)}$ [5–9]. The vacuum $\boldsymbol{B}^{(3)}$

field cannot under any circumstances *self-induce* an electric field because the symmetry of the fundamental Lagrangian interaction term between $A_\mu^{(1)}$ and $A_\mu^{(2)}$ forbids this. In other words the interaction part of $F_{\mu\nu}$ (Eq. (21)) *contains only a magnetic field*. For Faraday induction to occur, $F_{\mu\nu}$ must contain both a physical magnetic and physical electric component, giving [1,2] the vacuum Maxwell equations, e.g., in Barut's notation [1],

$$F^{\mu\nu}{}_{,\nu} = 0. \tag{26}$$

Therefore $\boldsymbol{B}^{(3)}$ does not produce Faraday induction in the vacuum, e.g. by modulating a circularly polarized laser beam in an evacuated induction coil [17]. This conclusion has been verified experimentally on at least two occasions [17]. The Evans-Vigier field does not obey the Faraday law of induction, and this conclusion does not violate first principles. On the contrary, it is *deduced* from fundamental Lagrangian principles as just described. Related to this is the fact that the conjugate product of a circularly polarized electromagnetic field does not produce electric polarization effects experimentally, only magnetization and magnetic effects. The interaction of the conjugate plane wave solutions of the d'Alembert wave equation in vacuo always produces a magnetic field, never an electric field.

The constant γ in Eq. (20a) must be a scalar by construction, and in the vacuum, Eq. (20a) must reduce to a Maxwell equation in the hypothetical absence of interaction. (*Hypothetical* because experiment shows [12–15] that there is always an interaction between $A_\mu^{(1)}$ and $A_\mu^{(2)}$, which manifests itself in magneto-optics.) These considerations suggest that $\gamma = 0$, and that, from Eq. (20a)

$$\partial^\mu F_{\mu\nu} = -ieA^\mu F_{\mu\nu} = 0. \tag{27}$$

This equation is satisfied by $\partial^\mu = -ieA^\mu$ and by $(\partial^\mu\partial^{\mu*})^{1/2} = e\left(A_\mu A_\mu^*\right)^{1/2}$.

Denoting

$$\kappa = (\partial^\mu\partial^{\mu*})^{1/2}, \qquad A^{(0)} = (A^\mu A^{\mu*})^{1/2}, \tag{28}$$

we obtain, in S.I. units,

$$\kappa = \frac{e}{\hbar}A^{(0)}, \tag{29}$$

or [5],

$$eA^{(0)} = \hbar\kappa. \tag{30}$$

The photon momentum in vacuo, $\hbar\kappa$, has been identified with

$eA^{(0)}$ and charge has been quantized through $e = \hbar\left(\frac{\kappa}{A^{(0)}}\right)$.

These conclusions can be revealed self-consistently, moreover, by comparing [5] Eq. (25) with the classical result,

$$\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} = iA^{(0)2}\boldsymbol{e}^{(3)} = i\frac{A^{(0)}}{\kappa}\boldsymbol{B}^{(3)}, \tag{31}$$

obtained by direct evaluation of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. The result $\gamma = 0$ leads to the physical inference [1] that the magnetic moment generated in vacuo by the interaction of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is not zero. The interaction contributes a term [1],

$$j^{\mu}(int) = ie\partial_{\nu}(A^{\mu(1)}A^{\nu(2)} - A^{\nu(1)}A^{\mu(2)}) = \nabla\cdot\boldsymbol{B}^{(3)} = 0, \tag{32}$$

to the vacuum current density, so that in Barut's reduced units the magnetic moment of the vector particle associated with $A_{\mu}^{(2)}$ is unity multiplied by $\boldsymbol{B}^{(3)}$. In S.I. units it is $\mu_0\boldsymbol{B}^{(3)}$ where μ_0 is the vacuum permeability. The particle is the *photon* in the quantum field theory and we conclude that the photon has a magnetic moment although conventionally described as uncharged.

Finally, the above analysis can be re-worked readily for a field whose associated mass is not zero, whereby we obtain the Proca equation rather than the d'Alembert equation [5–9]. It is inconsistent to assert that the electromagnetic field has zero mass because $\boldsymbol{B}^{(3)}$ is longitudinal: a zero mass field must have transverse components, only, from Wigner's theory [1,2,5–9]. We conclude that magneto-optic effects imply that the mass of the photon is finite.

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