Foundations of Physics, Vol. 30, No. 7, 2000

Operator Derivation of the Gauge-Invariant Proca and Lehnert Equations; Elimination of the Lorenz Condition

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Received January 30, 2000

Using covariant derivatives and the operator definitions of quantum mechanics, gauge invariant Proca and Lehnert equations are derived and the Lorenz condition is eliminated in $U(1)$ invariant electrodynamics. It is shown that the structure of the gauge invariant Lehnert equation is the same in an $O(3)$ invariant theory of electrodynamics.

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0015-9018/00/0700-1123$18.00/0

1. INTRODUCTION

A U(1) invariant theory of electrodynamics is used in the received opinion in the electromagnetic sector of the unified field theory, usually with the photon mass assumed to be identically zero, and using the Lorenz condition to obtain the wave equation from the field equation.[1–3] In this communication, it is argued, using covariant derivatives, that the wave equation is in general a gauge invariant Proca equation, in which the mass of the photon is identically non-zero,[4–6] and that the inhomogeneous field equation is in general a Lehnert equation.[7] The Lorenz condition of the received opinion is not needed to obtain these results, and the homogeneous field equation is obtained with a subsidiary condition with important physical implications. It is demonstrated, finally, that the structure of the Lehnert equation is the same in an O(3) invariant theory of electrodynamics by using a gauge theory with an internal physical space of O(3) symmetry described by a complex basis ((1), (2), (3)) based on the empirical evidence of circular polarization.

2. DERIVATION OF THE GAUGE-INVARIANT PROCA EQUATION

In momentum space, consider the Einstein equation of classical special relativity

$$p^{\mu}p_{\mu}=m_0c^2 \tag{1}$$

where m_0 is the mass particle, m_0c^2 is its rest energy, and p^{μ} is the four-momentum. If it is assumed that the mass of the particle is identically zero, then the Einstein equation becomes

$$p^{\mu}p_{\mu}=0 \tag{2}$$

Use of the operator identity of quantum mechanics,

$$p^{\mu}=i\hbar\partial^{\mu} \tag{3}$$

transforms Eq. (2) into an eigenvalue equation, where ψ is an eigenfunction:

$$\partial^{\mu}\partial_{\mu}\psi=0 \tag{4}$$

In gauge-covariant form, this equation is

$$(\partial_{\mu}-igA_{\mu})(\partial^{\mu}+igA^{\mu})\,\psi=0 \tag{5}$$

where

$$D_\mu \equiv \partial_\mu - igA_\mu \tag{6}$$

is the covariant derivative[1–3] necessitated by special relativity. Here g is a proportionality constant equal to $\kappa/A^{(0)}$, and A^μ is the four-potential of the electromagnetic field in the vacuum. In a U(1) invariant theory of electrodynamics, the internal gauge space is a scalar and A^μ is a four-vector in Minkowski spacetime. An expanding Eq. (5):

$$(\Box - igA_\mu \partial^\mu + ig\partial_\mu A^\mu + g^2 A_\mu A^\mu)\, \psi = 0 \tag{7}$$

and using the operator identity (3), the second and third terms can be eliminated:

$$-\frac{A_\mu p^\mu}{\hbar} + \frac{A^\mu p_\mu}{\hbar} = 0 \tag{8}$$

leaving the eigenvalue equation

$$(\Box + g^2 A_\mu A^\mu)\, \psi = 0 \tag{9}$$

on using

$$|A_\mu A^\mu| = A^{(0)\,2}, \qquad g = \frac{\kappa}{A^{(0)}} \tag{10}$$

Eq. (9) reduces to

$$(\Box + \kappa^2)\, \psi = 0 \tag{11}$$

which becomes the Proca equation

$$(\Box + \kappa^2)\, A^\nu = 0 \tag{12}$$

if the eigenfunction ψ is identified with A^ν. The latter is regarded as an eigenfunction of quantum mechanics. The gauge invariance of the Proca equation follows from the fact that a gauge transformation on the eigenfunction A^ν produces the eigenfunction:

$$A^{\nu'} = e^{i\Lambda} A^\nu \tag{13}$$

where Λ is arbitrary. The eigenfunctions on both sides of the eigenvalue equation (12) are multiplied by an arbitrary eigenfunction $e^{i\Lambda}$ and the eigenvalue $-\kappa^2$ is unchanged. Using the de Broglie guidance theorem

$$\hbar\omega = m_0 c^2 \tag{14}$$

the gauge invariant Proca equation is obtained in its usual form

$$\Box A^{\nu} = -\left(\frac{m_0 c^2}{\hbar}\right)^2 A^{\nu} \tag{15}$$

As soon as covariant derivatives are introduced, as in Eq. (5), the Proca equation follows without the use of the Lorenz condition. The received view is that the Proca equation implies the Lorenz condition and that the Proca equation is not gauge invariant.[1–3] We have argued to the contrary using the above straightforward method. The Lorenz condition does not appear in our derivation and is not needed. In momentum space, the Proca equation (15) is the Einstein equation for a particle with identically non-zero mass, Eq. (1). Therefore, the use of covariant derivatives in the vacuum gives the Einstein equation self-consistently. Photon mass is therefore a property of gauge covariance in a U(1) invariant electrodynamics, and indeed in electrodynamics for any internal group symmetry. The overall conclusion is that, in gauge theory, photon mass is identically a non-zero, a conclusion of major importance.

3. DERIVATION OF THE GAUGE INVARIANT LEHNERT EQUATION

The received view in a U(1) invariant electrodynamics is that the inhomogeneous field equation in the vacuum is

$$\partial_\mu F^{\mu\nu} = 0 \tag{16}$$

where $F^{\mu\nu}$ is the field tensor defined by the four-curl

$$F^{\mu\nu} = \partial^{\nu} A^{\nu} - \partial^{\mu} A^{\nu} \tag{17}$$

In gauge theory, however, the covariant derivative is required in vacuum, and Eq. (18) becomes:

$$(\partial_\mu + igA_\mu^*)\, F^{\mu\nu} = 0 \tag{18}$$

In general, A_μ and $F^{\mu\nu}$ are complex quantities necessitating the use of complex number algebra as in Eq. (18). Equation (18) is an eigenvalue whose eigenfunction is $F^{\mu\nu}$. The Proca equation (12) can be identified with Eq. (18) as follows:

$$\kappa^2 A^\nu = igA_\mu^* F^{\mu\nu}, \qquad \nu = 0 \tag{19}$$

and identity which can be developed as

$$\kappa^2 A^{(0)} = i\,\frac{g}{c}\,\mathbf{A}^* \cdot \mathbf{E} \tag{20}$$

and which, for plane waves, gives the self-consistent result

$$\kappa A^{(0)} = \frac{E^{(0)}}{c} = B^{(0)} \tag{21}$$

By identifying Eq. (18) with Eq. (12), the former becomes an equation of quantum mechanics, and the field tensor $F^{\mu\nu}$ in Eq. (18) becomes an eigenfunction of quantum mechanics. This inference implies that the operator ∂_μ acts on the eigenfunction $F^{\mu\nu}$ to give the eigenvalue $-igA_\mu^*$, which is therefore a four-momentum.

Equation (18) is the Lehnert equation

$$\partial_\mu F^{\mu\nu} = J^\nu(\text{vac}) = -igA_\mu^* F^{\mu\nu} \tag{22}$$

where the vacuum charge current four-vector J^ν (vac) is introduced through gauge theory without the need for empiricism. Under a gauge transformation, $F^{\mu\nu}$ is invariant.[1–3] The four-potential A_μ^* is changed to $A_\mu^{*\prime}$ after gauge transformation. The overall result is therefore

$$(\partial_\mu + igA_\mu^*)\, F^{\mu\nu} = 0 \to (\partial_\mu + igA_\mu^{*\prime})\, F^{\mu\nu} = 0 \tag{23}$$

and so

$$A_\mu^{*\prime} = A_\mu^* \tag{24}$$

In general, however, it is known that, on gauge transformation in a U(1) invariant theory,

$$A^\mu \to A^\mu + \partial^\mu \chi \tag{25}$$

so the only possibility is that

$$\partial^\mu \chi = 0 \tag{26}$$

If the scalar χ is identified with $\partial_\mu A^\mu$, we obtain

$$\partial^\nu \partial_\mu A^\mu = 0 \tag{27}$$

Therefore gauge fixing or gauge freedom is not used in our derivation, which shows that the result (27) follows from the introduction of covariant derivatives in the vacuum.

Similarly, the homogeneous field equation with covariant derivatives becomes

$$(\partial_\mu - igA_\mu)\,\tilde{F}^{\mu\nu} \equiv 0 \tag{28}$$

and is a Jacobi identity of U(1) invariant electrodynamics. The equation

$$\partial_\mu \tilde{F}^{\mu\nu} \equiv 0 \tag{29}$$

is also a Jacobi identity of a U(1) invariant electrodynamics, so the important subsidiary result

$$A_\mu \tilde{F}^{\mu\nu} \equiv 0 \tag{30}$$

is obtained. This result in a U(1) invariant electrodynamics is consistent with the fact that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{31}$$

4. IDENTIFICATION WITH AN O(3) INVARIANT ELECTRODYNAMICS

In the O(3) invariant electrodynamics[4–10] already defined, the vacuum inhomogenous field equation can be shown to be

$$D_\mu \mathbf{G}^{\mu\nu} = 0 \tag{32}$$

$$\mathbf{G}^{\mu\nu} = G^{\mu\nu(1)}\mathbf{e}^{(1)} + G^{\mu\nu(2)}\mathbf{e}^{(2)} + G^{\mu\nu(3)}\mathbf{e}^{(3)} \tag{33}$$

and, using the relations:

$$\begin{aligned} \mathbf{A} &= \mathbf{A}^{(1)} + \mathbf{A}^{(2)} + \mathbf{A}^{(3)} \\ \mathbf{B} &= \mathbf{B}^{(1)} + \mathbf{B}^{(2)} + \mathbf{B}^{(3)} \\ \mathbf{E} &= \mathbf{E}^{(1)} + \mathbf{E}^{(2)} + \mathbf{E}^{(3)} \end{aligned} \tag{34}$$

it is seen that the structure of Eq. (32), is the same as that of Eq. (22), but the meaning of the symbols of Eq. (22) is changed on the O(3) level, which has recently been shown to be capable of describing empirical data where a U(1) invariant theory fails.

5. CONCLUSION

The gauge invariant Proca equation (15) and Lehnert equation (23) have been derived without assuming the Lorenz condition, and the structure of the Lehnert equation was shown to be the same in an O(3) invariant electrodynamics. More generally, the Proca and Lehnert equations are invariant in any internal space symmetry of gauge theory.

ACKNOWLEDGMENTS

The U.S. Department of Energy is thanked for supercomputer time and the web site http://www.ott.doe.gov/electromagnetic/ and various sources of finding are acknowledged gratefully.

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