APPENDIX THREE: THE O(3) EQUATIONS OF MOTION AND PLANE WAVE APPROXIMATIONS

The Gauss Law

$$\nabla \cdot \boldsymbol{B}^{(1)*} = ig\left(\boldsymbol{A}^{(2)} \cdot \boldsymbol{B}^{(3)} - \boldsymbol{B}^{(2)} \cdot \boldsymbol{A}^{(3)}\right) \quad \text{et cyclicum.} \tag{1}$$

In the plane wave approximation, this reduces to:

$$\nabla \cdot \boldsymbol{B}^{(i)*} = 0; \qquad i = 1, 2, 3 \tag{2}$$

The Coulomb Law

$$\nabla \cdot \boldsymbol{D}^{(1)*} = \rho^{(1)*} + ig\left(\boldsymbol{A}^{(2)} \cdot \boldsymbol{D}^{(3)} - \boldsymbol{D}^{(2)} \cdot \boldsymbol{A}^{(3)}\right) \quad \text{et cyclicum.} \tag{3}$$

$$\text{Lehnert charges } = ig\left(\boldsymbol{A}^{(2)} \cdot \boldsymbol{D}^{(3)} - \boldsymbol{D}^{(2)} \cdot \boldsymbol{A}^{(3)}\right) \quad \text{et cyclicum.} \tag{4}$$

The Coulomb Law in Vacuo

$$\nabla \cdot \mathrm{E}^{(1)*} = ig\left(\boldsymbol{A}^{(2)} \cdot \boldsymbol{E}^{(3)} - \boldsymbol{E}^{(2)} \cdot \boldsymbol{A}^{(3)}\right) \tag{5}$$

$$\text{Lehnert charges } = i\frac{g}{\varepsilon_0}\left(\boldsymbol{A}^{(2)} \cdot \boldsymbol{E}^{(3)} - \boldsymbol{E}^{(2)} \cdot \boldsymbol{A}^{(3)}\right) \tag{6}$$

In the plane wave approximation, this reduces to

$$\nabla \cdot \mathrm{E}^{(i)*} = 0; \qquad i = 1, 2 \tag{7}$$

because

$$\boldsymbol{E}^{(3)} = \mathbf{0} \tag{8}$$

The vector potential is:

$$A_\mu = A_\mu^{(1)} e^{(1)} + A_\mu^{(2)} e^{(2)} + A_\mu^{(3)} e^{(3)} \tag{9}$$

where:

$$A_\mu^{(3)} = \left(A^{(0)}, -\boldsymbol{A}^{(3)}\right) \equiv \left(A_0, -\boldsymbol{A}^{(3)}\right) \\ = A^{(0)}\left(1, -\boldsymbol{k}\right) \tag{10}$$

The Faraday Law

$$\nabla \times \boldsymbol{E}^{(1)*} + \frac{\partial \boldsymbol{B}^{(1)*}}{\partial t} = -ig\left(cA_0^{(3)}\boldsymbol{B}^{(2)} - cA_0^{(2)}\boldsymbol{B}^{(3)} + \boldsymbol{A}^{(2)} \times \boldsymbol{E}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{E}^{(2)}\right) \tag{11}$$

et cyclicum.

In the plane wave approximation, this reduces to:

$$\nabla \times \boldsymbol{E}^{(i)*} + \frac{\partial \boldsymbol{B}^{(i)*}}{\partial t} = \boldsymbol{0}; \qquad i - 1, 2 \\ \frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \boldsymbol{0} \tag{12}$$

These are the usual Faraday Equations for plane waves plus an equation for $B^{(3)}$.

The Ampère-Maxwell Law

$$\nabla \times \boldsymbol{H}^{(1)*} + \frac{\partial \boldsymbol{D}^{(1)*}}{\partial t} = \boldsymbol{J}^{(1)*} - ig\left(cA_0^{(2)}\boldsymbol{D}^{(3)} - cA_0^{(3)}\boldsymbol{D}^{(2)} + \boldsymbol{A}^{(2)} \times \boldsymbol{H}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{H}^{(2)}\right) \tag{13}$$

et cyclicum.

$$\text{Lehnert current } = -ig\left(cA_0^{(2)}\boldsymbol{D}^{(3)} - cA_0^{(3)}\boldsymbol{D}^{(2)} + \boldsymbol{A}^{(2)} \times \boldsymbol{H}^{(3)} - \boldsymbol{A}^{(3)} \times \boldsymbol{H}^{(2)}\right) \tag{14}$$

The Ampère-Maxwell Law in Vacuo

$$\nabla \times \boldsymbol{B}^{(1)*} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(1)*}}{\partial t} = -i\frac{g}{c}\left(A_0^{(2)}\boldsymbol{E}^{(3)} - A_0^{(3)}\boldsymbol{E}^{(2)} + c\boldsymbol{A}^{(2)} \times \boldsymbol{B}^{(3)} - c\boldsymbol{A}^{(3)} \times \boldsymbol{B}^{(2)}\right) \tag{15}$$

et cyclicum

$$\text{Lehnert currents } = -i\frac{g}{c}\left(A_0^{(2)}\boldsymbol{E}^{(3)} - A_0^{(3)}\boldsymbol{E}^{(2)} + c\boldsymbol{A}^{(2)} \times \boldsymbol{B}^{(3)} - c\boldsymbol{A}^{(3)} \times \boldsymbol{B}^{(2)}\right) \tag{16}$$

et cyclicum.

In tensor notation, eqn. (15) is part of:

$$\partial_\mu \boldsymbol{G}^{\mu\nu} + g\boldsymbol{A}_\nu \times \boldsymbol{G}^{\mu\nu} = \frac{\boldsymbol{J}^\mu_{(Vac)}}{\varepsilon_0} \tag{17}$$

where $\boldsymbol{J}^\mu_{(Vac)}$ is the **Lehnert current**.

We choose:

$$\partial_\mu \boldsymbol{G}^{\mu\nu} = \boldsymbol{0} \tag{18}$$

$$\boldsymbol{J}^\mu_{(Vac)} = g\varepsilon_0 \boldsymbol{A}_\mu \times \boldsymbol{G}^{\mu\nu} \tag{19}$$

Eqns. (18) give:

$$\nabla \times \boldsymbol{B}^{(i)} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(i)}}{\partial t} = 0; \quad i = 1,2 \tag{20}$$

$$\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0} \tag{21}$$

Eqn. (19) gives:

$$En = \frac{1}{\mu_0}\int \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} dV \tag{22}$$