INTERPRETATION OF THE FUNDAMENTAL EQUATIONS OF O(3) ELECTRODYNAMICS

ABSTRACT

The fundamental laws of O(3) electrodynamics are presented and interpreted on a classical level. These are the homogeneous and inhomogeneous field equations and the O(3) Lorentz force equation. They are based on a gauge field theory with an O(3) internal symmetry with complex basis ((1), (2), (3)) based on the existence of circular polarization. They reduce under well defined circumstances to the Maxwell-Heaviside and Lehnert equations, together with equations for $\boldsymbol{B}^{(3)}$, the longitudinal magnetic component of the O(3) symmetry gauge field theory.

INTRODUCTION

There are several severe flaws in U(1) gauge field theory applied to electrodynamics {1-5}. The most serious one is that an observable such as $\boldsymbol{A} \times \boldsymbol{A}^*$, the conjugate product of vector potentials in free space, is identically zero in the U(1) theory. In the complex basis ((1), (2), (3)), this conjugate product is denoted $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and is a fundamental observable in the third Stokes parameter (the archetypical signature of circular polarization {6}) and in magneto-optical effects such as the inverse Faraday effect {7-10}. Within the general structure of gauge field theory, there is no way in which this self-inconsistency can be remedied without using a gauge group of different symmetry from U(1). It is convenient to chose O(3), because it is the rotation group and the basis ((1), (2), (3)) {1-5} is a basis of the O(3) group. The indices (1) and (2) are complex conjugate indices which describe circular polarization. The (3) index is parallel to the $\boldsymbol{k}$ axis, the axis of propagation of the radiation. As an illustration of the self-inconsistencies of conventional theory, the third Stokes parameter and the inverse Faraday effect are described by the phenomenological introduction into Maxwell-Heaviside theory of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, which is however identically zero in the U(1) gauge theory, which is, in turn, the basis of Maxwell-Heaviside theory. This is a diametric self-inconsistency which can be remedied if and only if we work within the structure of a different gauge field theory such as O(3). In this paper, the fundamental field equations of O(3) electrodynamics are developed and interpreted.

In Section 2, they are given in condensed notation, and in Section 3, expanded out into a form which may be more transparent for interpretation. Under certain circumstances, they reduce to Maxwell-Heaviside type equations in indices (1) and (2), and to equations governing the extra component $\boldsymbol{B}^{(3)}$, a constant of motion. In Section 4, the Lorentz force equation is given in O(3) electrodynamics and similarly interpreted.

THE HOMOGENEOUS AND INHOMOGENEOUS FIELD EQUATIONS OF O(3) ELECTRODYNAMICS

Within a non-Abelian electrodynamic field theory, the classical field equations become structured with covariant derivatives, and the field tensors become vectors in the internal gauge space. In their most condensed form, the basic field equations are:

$$D_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{1}$$

$$D_\mu \boldsymbol{H}^{\mu\nu} = \boldsymbol{J}^\nu \tag{2}$$

where the four current components are defined by:

$$J^{\mu(i)} \equiv \left(\rho^{(i)}, \frac{1}{c} \boldsymbol{J}^{(i)} \right) \tag{3}$$

and where the field tensors are in S.I. units. The bold symbols in eqns (1) and (2) are vectors in the internal gauge space with O(3) symmetry whose algebra is developed as in ordinary vector algebra. Therefore, a symbol such as $\boldsymbol{J}^{\nu}$ in eqn. (3) is a three vector in the internal O(3) symmetry gauge space. Each component of the vector is itself a four-vector in Minkowski space-time. The algebra of the four-vector components is developed as in the ordinary algebra of special relativity, with contravariant-covariant components as usual.

The homogeneous eqn. (1) can be developed as {1-5}:

$$\left(\partial_{\mu} + g\boldsymbol{A}_{\mu} \times \right) \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{4}$$

using the definition of the covariant derivative in O(3):

$$D_{\mu} \equiv \partial_{\mu} + g\boldsymbol{A}_{\mu} \times \tag{5}$$

where g is a coefficient of proportionality. In the vacuum, it is given by $e/\hbar$, the ratio of the fundamental charge e to the Planck constant. The covariant derivative is Lorentz covariant in special relativity and the equations (1) and (2) are Lorentz covariant {1-5}.

The O(3) field tensor {1-5} contains a component:

$$\boldsymbol{B}^{(3)*} = -i \frac{e}{\hbar} \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{6}$$

which does not occur in Maxwell-Heaviside electrodynamics but which satisfactorily describes the third Stokes parameter through $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$. On this point alone, O(3) electrodynamics is superior to U(1) electrodynamics in which $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is zero, and so there is no circular polarization, a reduction to absurdity. Similar considerations apply in O(3) quantum electrodynamics {11}, which is more self-consistent than U(1) quantum electrodynamics. Classical O(3) electrodynamics also gives a self-consistent description of magneto-optical effects such as the inverse Faraday effect through the magnetization:

$$\boldsymbol{M}^{(3)}(\text{IFE}) = -\frac{i}{\mu_0} g' \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{7}$$

in which the g factor of the vacuum is replaced by a material property that can be determined through the Verdet constant {7-10}. In U(1) electrodynamics, the factor $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is zero and there is no inverse Faraday effect, contrary to observation. Therefore these are major advantages of the O(3) gauge symmetry applied to electrodynamics.

Special solutions of eqn. (1) result in Maxwellian type equations, together with equations governing the component $\boldsymbol{B}^{(3)}$. For example, the special solution:

$$\partial_{\mu} \tilde{\boldsymbol{G}}^{\mu\nu} = \boldsymbol{0} \tag{8}$$

$$\boldsymbol{A}_{\mu} \times \tilde{\boldsymbol{G}}^{\mu\nu} = \boldsymbol{0} \tag{9}$$

gives a complex conjugate pair of Maxwellian type equations together with the equation:

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \boldsymbol{0} \tag{10}$$

which shows that $\boldsymbol{B}^{(3)}$ is a constant of motion {1-5}. In general, however, eqns. (1) and (2) must be solved without approximation in the vacuum and in the presence of matter, giving a more self-consistent view of electrodynamics. This problem of solution can be addressed numerically.

The inhomogeneous Maxwellian equations in vacuo in indices (1) and (2) are recovered from eqn. (2) using the particular solution:

$$\partial_\mu \boldsymbol{G}^{\mu\nu} = \boldsymbol{0} \tag{11}$$

$$\boldsymbol{J}^\nu = g\varepsilon_0 \boldsymbol{A}_\mu \times \boldsymbol{G}^{\mu\nu} \tag{12}$$

which gives two Maxwellian equations and the irrotational equation for $\boldsymbol{B}^{(3)}$ {1-5}:

$$\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0}. \tag{13}$$

Therefore $\boldsymbol{B}^{(3)}$ is a fundamental spin of the electromagnetic field.

The additional cyclic equation (12) produces a finite energy term due to $\boldsymbol{B}^{(3)}$ which does not exist in U(1) electrodynamics:

$$En^{(3)} = -\int \boldsymbol{J}^\nu \bullet \boldsymbol{A}_\nu \, dV \tag{14}$$

where V is the radiation volume. This energy term can be developed as follows:

$$En^{(3)} = -\frac{1}{\mu_0} \int g\boldsymbol{A}_\mu \times \boldsymbol{G}^{\mu\nu} \bullet \boldsymbol{A}_\nu dV \tag{15}$$

$$En^{(3)} = \frac{g}{\mu_0} \int \boldsymbol{G}^{\mu\nu} \bullet \boldsymbol{A}_\mu \times \boldsymbol{A}_\nu \, dV \tag{16}$$

$$En^{(3)} = \frac{g^2}{\mu_0} \int \boldsymbol{A}^\mu \times \boldsymbol{A}^\nu \bullet \boldsymbol{A}_\mu \times \boldsymbol{A}_\nu \, dV \tag{17}$$

$$En^{(3)} = \frac{1}{\mu_0} \int \boldsymbol{B}^{(3)} \bullet \boldsymbol{B}^{(3)} dV. \tag{18}$$

and is, self-consistently, the electromagnetic energy in a volume V of radiation due to the $\boldsymbol{B}^{(3)}$ component of vacuum electrodynamics.

EXPANDED FORM OF THE BASIC FIELD EQUATIONS

The four fundamental laws of O(3) electrodynamics are similar to the Gauss, Faraday, Coulomb and Ampère laws of U(1) electrodynamics and reduce to them in particular solutions. However, in general, O(3) electrodynamics is a non-Abelian gauge field theory which considerably enriches the content of electrodynamics. The four basic laws are written out in full in this section in the complex basis ((1), (2), (3)).

The O(3) Gauss Law

The O(3) Gauss law allows for the existence of a topological magnetic monopole and takes the form of three equations:

$$\nabla\cdot\boldsymbol{B}^{(1)*} \equiv ig\left(\boldsymbol{A}^{(2)}\cdot\boldsymbol{B}^{(3)} - \boldsymbol{B}^{(2)}\cdot\boldsymbol{A}^{(3)}\right) \tag{19}$$

$$\nabla\cdot\boldsymbol{B}^{(2)*} \equiv ig\left(\boldsymbol{A}^{(3)}\cdot\boldsymbol{B}^{(1)} - \boldsymbol{B}^{(3)}\cdot\boldsymbol{A}^{(1)}\right) \tag{20}$$

$$\nabla\cdot\boldsymbol{B}^{(3)*} \equiv ig\left(\boldsymbol{A}^{(1)}\cdot\boldsymbol{B}^{(2)} - \boldsymbol{B}^{(1)}\cdot\boldsymbol{A}^{(2)}\right). \tag{21}$$

The particular solutions discussed in Section 2 reduce these to:

$$\nabla\cdot\boldsymbol{B}^{(1)} = \nabla\cdot\boldsymbol{B}^{(2)} = \nabla\cdot\boldsymbol{B}^{(3)} = 0 \tag{22}$$

where we recover the Gauss law together with the law for the divergence of $\boldsymbol{B}^{(3)}$.

The O(3) Faraday Induction Law

The Faraday induction law can be developed as an O(3) gauge field theory described by the three equations:

$$\nabla\times\boldsymbol{E}^{(1)*} + \frac{\partial\boldsymbol{B}^{(1)*}}{\partial t} \equiv -ig\left(cA_0^{(3)}\boldsymbol{B}^{(2)} - cA_0^{(2)}\boldsymbol{B}^{(3)} + \boldsymbol{A}^{(2)}\times\boldsymbol{E}^{(3)} - \boldsymbol{A}^{(3)}\times\boldsymbol{E}^{(2)}\right) \tag{23}$$

$$\nabla\times\boldsymbol{E}^{(2)*} + \frac{\partial\boldsymbol{B}^{(2)*}}{\partial t} \equiv -ig\left(cA_0^{(1)}\boldsymbol{B}^{(3)} - cA_0^{(3)}\boldsymbol{B}^{(1)} + \boldsymbol{A}^{(3)}\times\boldsymbol{E}^{(1)} - \boldsymbol{A}^{(1)}\times\boldsymbol{E}^{(3)}\right) \tag{24}$$

$$\nabla\times\boldsymbol{E}^{(3)*} + \frac{\partial\boldsymbol{B}^{(3)*}}{\partial t} \equiv -ig\left(cA_0^{(2)}\boldsymbol{B}^{(1)} - cA_0^{(1)}\boldsymbol{B}^{(2)} + \boldsymbol{A}^{(1)}\times\boldsymbol{E}^{(2)} - \boldsymbol{A}^{(2)}\times\boldsymbol{E}^{(1)}\right) \tag{25}$$

The particular solution discussed in Section 2 reduces these to:

$$\nabla\times\boldsymbol{E}^{(1)*} + \frac{\partial\boldsymbol{B}^{(1)*}}{\partial t} = \boldsymbol{0} \tag{26}$$

$$\nabla\times\boldsymbol{E}^{(2)*} + \frac{\partial\boldsymbol{B}^{(2)*}}{\partial t} = \boldsymbol{0} \tag{27}$$

$$\frac{\partial\boldsymbol{B}^{(3)*}}{\partial t} = \boldsymbol{0}. \tag{28}$$

Eqns. (26) and (27) are complex conjugate Faraday laws of induction and eqn. (28) is the law for the constant of motion $\boldsymbol{B}^{(3)}$.

The O(3) Coulomb Law in Vacuo

The O(3) Coulomb law in vacuo is

$$\nabla \cdot \boldsymbol{E}^{(1)*} - \frac{\rho^{(1)*}}{\varepsilon_0} = ig\left(\boldsymbol{A}^{(2)} \cdot \boldsymbol{E}^{(3)} - \boldsymbol{E}^{(2)} \cdot \boldsymbol{A}^{(3)}\right) \tag{29}$$

$$\nabla \cdot \boldsymbol{E}^{(2)*} - \frac{\rho^{(2)*}}{\varepsilon_0} = ig\left(\boldsymbol{A}^{(3)} \cdot \boldsymbol{E}^{(1)} - \boldsymbol{E}^{(3)} \cdot \boldsymbol{A}^{(1)}\right) \tag{30}$$

$$\nabla \cdot \boldsymbol{E}^{(3)*} - \frac{\rho^{(3)*}}{\varepsilon_0} = ig\left(\boldsymbol{A}^{(1)} \cdot \boldsymbol{E}^{(2)} - \boldsymbol{E}^{(1)} \cdot \boldsymbol{A}^{(2)}\right) \tag{31}$$

and the particular solution of Section 2 reduces it to:

$$\nabla \cdot \boldsymbol{E}^{(1)} = \nabla \cdot \boldsymbol{E}^{(2)} = \nabla \cdot \boldsymbol{E}^{(3)} = 0. \tag{32}$$

The O(3) Ampere Maxwell Law in Vacuo

Similarly there are three O(3) Ampère-Maxwell equations:

$$\nabla \times \boldsymbol{B}^{(1)*} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(1)*}}{\partial t} - \frac{1}{c^2}\frac{\boldsymbol{J}^{(1)*}}{\varepsilon_0} = -\frac{ig}{c}\left(A_0^{(2)}\boldsymbol{E}^{(3)} - A_0^{(3)}\boldsymbol{E}^{(2)} + c\boldsymbol{A}^{(2)} \times \boldsymbol{B}^{(3)} - c\boldsymbol{A}^{(3)} \times \boldsymbol{B}^{(2)}\right) \tag{33}$$

$$\nabla \times \boldsymbol{B}^{(2)*} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(2)*}}{\partial t} - \frac{1}{c^2}\frac{\boldsymbol{J}^{(2)*}}{\varepsilon_0} = -\frac{ig}{c}\left(A_0^{(3)}\boldsymbol{E}^{(1)} - A_0^{(1)}\boldsymbol{E}^{(3)} + c\boldsymbol{A}^{(3)} \times \boldsymbol{B}^{(1)} - c\boldsymbol{A}^{(1)} \times \boldsymbol{B}^{(3)}\right) \tag{34}$$

$$\nabla \times \boldsymbol{B}^{(3)*} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(3)*}}{\partial t} - \frac{1}{c^2}\frac{\boldsymbol{J}^{(3)*}}{\varepsilon_0} = -\frac{ig}{c}\left(A_0^{(1)}\boldsymbol{E}^{(2)} - A_0^{(2)}\boldsymbol{E}^{(1)} + c\boldsymbol{A}^{(1)} \times \boldsymbol{B}^{(2)} - c\boldsymbol{A}^{(2)} \times \boldsymbol{B}^{(1)}\right). \tag{35}$$

and the particular solution of Section 2 reduces them to:

Eqns. (36) and (37) are complex conjugate equations corresponding to the original Ampère-Maxwell laws and eqn. (38) is the law for the constant of motion $\boldsymbol{B}^{(3)}$.

$$\nabla \times \boldsymbol{B}^{(1)*} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(1)*}}{\partial t} = \boldsymbol{0} \tag{36}$$

$$\nabla \times \boldsymbol{B}^{(2)*} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}^{(2)*}}{\partial t} = \boldsymbol{0} \tag{37}$$

$$\nabla \times \boldsymbol{B}^{(3)*} = \boldsymbol{0}. \tag{38}$$

In reducing the O(3) laws to the U(1) laws plus equations for $\boldsymbol{B}^{(3)}$, we obtain the new energy and momentum laws discussed in Section 2.

The laws of SU(2) classical electrodynamics are homomorphic with those of O(3) electrodynamics and have been given by Barrett {12}. In general, these laws replace those of the Maxwell-Heaviside equations and must be solved self-consistently without particular solution. This is probably best done on a computer. Recall that they are the direct logical consequence of the fact that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is non-zero in electrodynamics and is an empirical observable. In gauge theory, this means that the U(1) internal gauge group, which sets $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ to zero, must be replaced by the O(3) internal gauge group, in which $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ defines the $\boldsymbol{B}^{(3)}$ field.

THE O(3) LORENTZ FORCE EQUATION

In analogy with U(1) electrodynamics, we complete O(3) electrodynamics with the O(3) Lorentz force equation. The Lorentz force density (force per cubic meter) in O(3) electrodynamics is:

$$f^{\mu} = \boldsymbol{G}^{\mu\nu} \cdot \boldsymbol{J}_{\nu} \tag{39}$$

and is a scalar in the internal O(3) symmetry gauge space. Expanding eqn. (39), we obtain:

$$f^{\mu} = G^{\mu\nu(1)} J_{\nu}^{(2)} + G^{\mu\nu(2)} J_{\nu}^{(1)} + G^{\mu\nu(3)} J_{\nu}^{(3)}. \tag{40}$$

The Lorentz force due to $\boldsymbol{B}^{(3)}$ comes from the final term on the right hand side of eqn. (40):

$$\boldsymbol{f} = \boldsymbol{J}^{(3)} \times \boldsymbol{B}^{(3)} = \boldsymbol{0}. \tag{41}$$

There is in consequence no extra Lorentz force due to $\boldsymbol{B}^{(3)}$ as observed empirically {13, 14}. There is however a magnetization (transfer of angular momentum) due to $\boldsymbol{B}^{(3)}$ in the inverse Faraday effect as observed empirically. This means that the trajectory of an electron in an electromagnetic field is described by the first two terms on the right hand side of eqn. (40), as in the U(1) Lorentz force equation. There appears to be no observable real $\boldsymbol{E}^{(3)}$ field, but the imaginary quantity $-i\boldsymbol{E}^{(3)}$ may be physically significant.

There is however a linear momentum transfer due to $eA^{(3)} = \hbar\kappa$ from the field to an electron in O(3) electrodynamics, but force transfer is due to the transverse components as in U(1) electrodynamics. The Lorentz force density f^{μ} is the same in U(1) and O(3) electrodynamics and is defined as {15}:

$$f^{\mu} = -\partial_{\nu} T^{\mu\nu} \tag{42}$$

where $T^{\mu\nu}$ is the electromagnetic stress energy momentum tensor. The latter is a scalar in the internal gauge space of O(3) electrodynamics because it contains the scalar energy density:

$$U = \frac{1}{\mu_0} \boldsymbol{B} \cdot \boldsymbol{B} = \frac{1}{\mu_0} \left(\boldsymbol{B}^{(1)} \cdot \boldsymbol{B}^{(2)} + \boldsymbol{B}^{(2)} \cdot \boldsymbol{B}^{(1)} + \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} \right) \tag{43}$$

The extra energy density due to $\boldsymbol{B}^{(3)}$ is the third term on the right hand side and so the contribution of $\boldsymbol{B}^{(3)}$ to the diagonal of the stress tensor of the field vanishes:

$$\begin{aligned} T_{33} &= \frac{1}{\mu_0} \left(\boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} - \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} \right) \\ &= 0. \end{aligned} \tag{44}$$

DISCUSSION

In general, if we are to account self-consistently for the existence of circular polarization in gauge theory applied to electrodynamics, the field equations become structured as in Sections 2 and 3, and must be solved numerically. Some particular solutions give Maxwellian type equations for indices (1) and (2) and equations for the extra magnetic field $\boldsymbol{B}^{(3)}$ which is defined by O(3) symmetry gauge field theory. The subject of O(3) electrodynamics is considerably richer in structure than U(1) electrodynamics, and the same is true for O(3) quantum electrodynamics {16}. It has recently been shown {1-5} that U(1) electrodynamics is severely self-inconsistent, and if we are to accept gauge theory as a basic structure for field theory, the next step results in the equations developed in this Letter.

ACKNOWLEDGMENTS

Many colleagues worldwide are thanked for internet discussions, and the constituent institutions of AIAS (Alpha Foundation's Institute for Advanced Study) for funding to individual authors.

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