because the molecular ensemble average $\langle \Xi_Z \rangle$ is nonzero in general. Therefore, there is no near forward scattering.

## V. CONCLUSION

The phenomenon of antisymmetric light scattering has been interpreted in terms of the novel incident and scattered magnetostatic flux density vectors $\mathbf{B}_{\Pi}$ and $\mathbf{B}_{\Pi S}$, respectively. This shows that antisymmetric scattering is a purely magneto-optic phenomenon, giving information on the nature of the scattered $\mathbf{B}_{\Pi S}$ vector. In magneto-photonics, the vector $\mathbf{B}_{\Pi}$ is replaced by the operator $\hat{B}_{\Pi}$, and the appropriate quantum theory must be employed.

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# *MANIFESTLY COVARIANT THEORY OF THE ELECTROMAGNETIC FIELD IN FREE SPACETIME, PART 1: ELECTRIC AND MAGNETIC FIELDS AND MAXWELL'S EQUATIONS*

## I. INTRODUCTION

It has recently been shown[1–5] that there exist longitudinal solutions of Maxwell's equations in free spacetime which are independent of the phase $\phi$ of the traveling plane wave. These longitudinal electric and magnetic fields, denoted $\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$, respectively, are consistent with the conclusion of quantum electrodynamics that there exist four photon polarizations in free spacetime, one timelike ((0)), two transverse spacelike ((1) and (2)), and one longitudinal spacelike ((3)).[6,7] However, the existence of four photon polarizations has to date been regarded[7] as being in conflict with the deduction that the photon can have only two helicities, $+1$ and $-1$. This in turn has led to the arbitrary assertion that only the two transverse spacelike polarizations (1) and (2) can be "physically meaningful" in free spacetime. The timelike ((0)) and longitudinal spacelike ((3)) are conventionally discarded as physically meaningless. This implies that the theory of the electromagnetic field in free spacetime loses manifest covariance.[7] This fundamental difficulty is well described by Ryder,[7] from whose Chapter 4 we quote the following: "the electromagnetic field, like any massless field, possesses only two independent components, but is covariantly described by a (potential) four vector $A_{\mu}$. In choosing two of these components as the physical ones, and thence quantizing them, we lose manifest covariance. Alternatively, if we wish to keep covariance, we have two redundant components."

Clearly, if the theory of the electromagnetic field in free spacetime is to be made manifestly covariant and therefore rigorously consistent with special relativity, then all four photon polarizations must be physically meaningful. This implies that electric and magnetic fields in vacuo must be four vectors, $E_\mu$ and $B_\mu$, respectively, in spacetime. The difficulty with this notion to date appears to have been the preconception that any longitudinal solution of Maxwell's equations in free spacetime must necessarily be phase dependent, so that the longitudinal spacelike solution cannot be solenoidal. This means that Gauss's theorem in differential form is violated by such a solution.[8–15] However, with the recent discovery[1–5] that the longitudinal solutions to Maxwell's equations in vacuo are not phase dependent, the conflict with Gauss's theorem disappears, and one of the most intractable difficulties of electromagnetic field theory is removed. In so doing, the very basis of electrodynamics is changed profoundly, because at present the subject is based on the existence in vacuo of a potential four vector $A_\mu$, whose four-curl gives the antisymmetric electromagnetic field tensor $F_{\mu\nu} = -F_{\nu\mu}$ in spacetime. The components of $F_{\mu\nu}$ contain no explicit reference to the timelike component of the four vectors $E_\mu$ and $B_\mu$, and the longitudinal components that appear in $F_{\mu\nu}$ are evidently discarded as unphysical. To maintain manifest covariance the timelike and longitudinal components must be retained, and must have physical meaning. In other words, the electric and magnetic parts of the electromagnetic plane wave in free space are treated conventionally as three vectors in Euclidean space, and not as manifestly covariant four vectors in pseudo-Euclidean spacetime. This reveals an internal inconsistency in electrodynamics in vacuo, in that the d'Alembert equation

$$\Box A_\mu = 0 \tag{1}$$

allows four photon polarizations, but the Maxwell equations

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = 0 \qquad \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\nu} = 0 \qquad (x \equiv (X, Y, Z, ict)) \tag{2}$$

link only the spacelike components of $E_\mu$ and $B_\mu$. They make no explicit reference to their timelike components $E^{(0)}$ and $B^{(0)}$. (In Eq. (2), the Maxwell equations are stated in terms of the four divergence of $F_{\mu\nu}$ and of its dual, $\tilde{F}_{\mu\nu}$ (Ref. 7).) A consistent, manifestly covariant, and rigorous theory of electrodynamics in vacuo must link $E_\mu$ and $B_\mu$ to the tensor $F_{\mu\nu}$, which is the four-curl of $A_\mu$.



In Section II of this paper, a brief review is given of the phase-independent longitudinal components $E^{(3)}$ and $B^{(3)}$ of the electromagnetic plane wave in vacuo. These are identified with the conclusions of quantum field theory[7] that physical photon states in a manifestly covariant gauge such as the Lorentz gauge are described as admixtures of operators of the field, namely the creation and annihilation operators.

Section III links the four vectors $E_\mu$ and $B_\mu$ to the four tensor $F_{\mu\nu}$, and shows that $E_\mu$ and $B_\mu$ take the form of a Pauli Lubanski vector and pseudovector, respectively, in spacetime. These are well defined[7] within the inhomogeneous Lorentz group (or Poincaré group). This leads in turn to the conclusion that the two photon helicities, +1 and −1, can be reconciled rigorously with four physically meaningful photon polarizations, because the helicities can be described equally well in terms either of (1) and (2) polarizations or of (0) and (3) polarizations. This is consistent with our earlier[1,2] conclusion that one photon generates the longitudinal magnetic field component:

$$\mathbf{B}^{(3)} = \langle\psi|\hat{B}^{(3)}|\psi\rangle = \frac{B^{(0)}}{\hbar}\langle\psi|\hat{J}|\psi\rangle \tag{3}$$

where $|\psi\rangle$ is an eigenstate of the photon and where the eigenvalues of the operator $\hat{J}$ are $M_J\hbar$; $M_J = +1$ and $-1$, the photon helicities. The result (3) is generalized in Section III through the definition of $E_\mu$ and $B_\mu$ as Pauli-Lubanski types in spacetime.

Section IV deals with some consequences in vacuum electrodynamics of the existence of manifestly covariant $E_\mu$ and $B_\mu$, with four physically meaningful components. The Maxwell equations, in particular the differential form of Gauss's theorem, are developed covariantly in terms of $E_\mu$ and $B_\mu$. Specifically, Gauss's theorem in differential form becomes

$$\frac{\partial E_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{E} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t} = 0 \tag{4a}$$

and

$$\frac{\partial B_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{B} + \frac{1}{c}\frac{\partial B^{(0)}}{\partial t} = 0 \tag{4b}$$

The electromagnetic energy and energy flux densities in vacuo are expressed in terms of products of $E_\mu$ and $B_\mu$, showing that the (3) and (0) polarizations do not make explicit contributions to either on a time-averaged basis. The four Stokes parameters, however, are profoundly affected by the manifest covariance of $E_\mu$ and $B_\mu$, in that it is no longer sufficient to describe $S_0$, $S_1$, $S_2$, and $S_3$ in terms of Pauli matrices.[17] It is shown that the covariant description of the Stokes parameters in vacuo can be obtained through the use of Dirac matrices.[18] This description maintains the fundamental relation

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{5}$$

on the Poincaré sphere, while at the same time showing that $S_1$ and $S_2$ become different in a description based on $E_\mu$ and $B_\mu$ rather than on the usual transverse spacelike **E** and **B**. A new term appears in both $S_1$ and $S_2$ due to the existence of physically meaningful (0) and (3) states of the electromagnetic field. The parameters $S_0$ and $S_3$, on the other hand, are unaffected.

Section V is a discussion of the available experimental evidence for $\mathbf{B}^{(3)}$, and suggests several experimental tests of its physical existence when the electromagnetic field interacts with matter.

## II. THE LONGITUDINAL SOLUTIONS OF MAXWELL'S EQUATIONS IN FREE SPACETIME: (0) AND (3) POLARIZATIONS

Longitudinal solutions of Maxwell's equations in vacuo appear not to have been considered as physically meaningful in the great majority of standard texts. Jackson[8] simply states that the differential form of Gauss's theorem demands that phase-dependent solutions are transverse. The possibility of phase-independent solutions appears not to be considered. It is frequently considered[8–15] that a plane, monochromatic, electromagnetic wave traveling in $Z$ (the propagation axis) in vacuo is simply the sum of two coherent waves linearly polarized in the orthogonal axes $X$ and $Y$. Atkins[11] and Landau and Lifshitz[12] similarly consider only transverse fields, and thus transverse polarizations, in a Cartesian or circular basis. Whitner,[9] however, mentions briefly and without further development that "Plane waves are an important example but they do constitute a special case; we must not conclude that all electromagnetic waves are transverse." Similarly, other authors[8–15] in classical and quantum electrodynamics in vacuo make little or no mention of longitudinal solutions.

Recently, however, Evans[1–4] and Farahi and Evans[5] have systematically considered the theory of phase-independent longitudinal electric and magnetic fields, which are solutions to the free spacetime Maxwell equations and thus obey the differential form of Gauss's theorem in free spacetime. This work has developed rapidly from the observation[1] that spacelike components of the plane wave in vacuo are interrelated by

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{\mathrm{i}cE_0} \tag{6}$$

where

$$\mathbf{E}^{(1)} \equiv E_0 \hat{\mathbf{e}}^{(1)} \mathrm{e}^{\mathrm{i}\phi} \tag{7a}$$

$$\mathbf{E}^{(2)} \equiv E_0 \hat{\mathbf{e}}^{(2)} \mathrm{e}^{-\mathrm{i}\phi} \tag{7b}$$

are the transverse electric field components. Here

$$\hat{\mathbf{e}}^{(1)} = \frac{\mathbf{i} - \mathrm{i}\mathbf{j}}{\sqrt{2}}$$

$$\hat{\mathbf{e}}^{(2)} = \frac{\mathbf{i} + \mathrm{i}\mathbf{j}}{\sqrt{2}}$$

are unit vectors in the circular basis,[19] where **i** and **j** are unit cartesian vectors in $X$ and $Y$, orthogonal to the propagation axis $Z$. Here $\phi$ is the phase of the traveling monochromatic plane wave, defined by

$$\phi = \omega t - \boldsymbol{\kappa} \cdot \mathbf{r}$$

where $\omega$ is its angular frequency at instant $t$ and $\boldsymbol{\kappa}$ its wave vector at position **r** in Euclidean space. We have from Eqs. (6) and (7)

$$\mathbf{B}^{(3)} = B_0 \hat{\mathbf{e}}^{(3)} \tag{8}$$

with

$$\hat{\mathbf{e}}^{(1)} \times \hat{\mathbf{e}}^{(2)} = \mathrm{i}\hat{\mathbf{e}}^{(3)} \tag{9}$$

and the well-known free spacetime relation[8–15]

$$E_0 = cB_0 \tag{10}$$

With $B_0 \equiv B^{(0)}$ we write Eq. (8) as

$$B_0 - |\mathbf{B}^{(3)}| = 0 \tag{11}$$

where $|\mathbf{B}^{(3)}|$ c

From Eqs. (6), (7), and (11) it is clear that field polarizations (0) and (3) are not independent of polarizations (1) and (2). Furthermore, by considering the results of quantum field theory,[7] polarization (0) can be identified as being timelike in a manifestly covariant description, and (3) as longitudinal spacelike. The field $\mathbf{B}^{(3)}$ is consistent with Maxwell's equations in vacuo and therefore with Gauss's theorem. The key to this result is that the phase $\phi$ has been removed in the conjugate product (6). The electric counterpart of Eq. (11) is, in general:

$$E^{(0)} - |\mathbf{E}^{(3)}| = 0; \; E^{(0)} \propto E_0 \tag{12a}$$

$$\mathbf{E}^{(3)} = \mathbf{E}^{(0)} \hat{\mathbf{e}}^{(3)} \tag{12b}$$

Equations (12) also represent solutions of Maxwell's equations in vacuo.

Equations (11) and (12) are, furthermore, related[5] through conservation of electromagnetic energy by the Euclidean space equation:

$$\mathbf{E}^{(3)} \times \mathbf{B}^{(2)} = \mathbf{B}^{(3)} \times \mathbf{E}^{(2)} \tag{13}$$

showing that if $\mathbf{B}^{(3)}$ is real, as in Eq. (6), $\mathbf{E}^{(3)}$ is imaginary. Relations such as (6) and (11) to (13) show that there are only two independent states for **E** and **B** because from Eq. (6) either of states (1) and (2) can be expressed in terms of (3); and the latter can be expressed in terms of (0) through Eq. (11) for $\mathbf{B}^{(3)}$ and Eq. (12) for $\mathbf{E}^{(3)}$. Finally, states (0) for **E** and **B** are related by Eq. (10) and states (3) by Eq. (13). This result, that there are only two independent states out of the four possible, (0) to (3), is evidently the classical expression of the fact[7] that the massless gauge field possesses only two independent components, but is at the same time covariantly described by a four vector, $A_\mu$ made up of four physically meaningful polarization states (0), (1), (2), and (3).

Thus far, we have used a conventional, classical description in terms of spacelike vectors **E** and **B**, but have introduced the novel $\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$. By using the relativistic quantum description of the electromagnetic field[7] we now introduce the concept of electric and magnetic field four vectors $E_\mu$ and $B_\mu$, respectively.

It is well known[7] that the quantization of Eq. (1) in the Lorentz gauge proceeds through a condition derived by Gupta and Bleuler in the early days[6] of quantum field theory:

$$\frac{\partial \hat{A}_\mu^{(+)}}{\partial x_\mu} |\boldsymbol{\psi}\rangle = 0 \tag{14}$$

where $\hat{A}_\mu^{(+)}$ is the operator equivalent of $A_\mu$ and acts on a photon eigenstate $|\boldsymbol{\psi}\rangle$. Equation (14) leads to the result[7] that physical photon states are admixtures of (0) and (3) photon polarizations in such a way that

$$(\hat{a}^{(0)} - \hat{a}^{(3)})|\psi\rangle = 0 \tag{15a}$$

$$\langle\psi|\hat{a}^{(0)+}\hat{a}^{(0)}|\psi\rangle = \langle\psi|\hat{a}^{(3)+}\hat{a}^{(3)}|\psi\rangle \tag{15b}$$

where $\hat{a}$ and $\hat{a}^+$ are annihilation and creation operators, respectively. Furthermore, the energy density of the quantized field is proportional[7] to the sum

$$\sum_{\lambda=0}^{3} (\hat{a}^{(\lambda)+}\hat{a}^{(\lambda)} - \hat{a}^{(0)+}\hat{a}^{(0)}) \tag{16}$$

and from Eq. (15b)[7] the contribution of the longitudinal ((3)) and timelike ((0)) states cancel. It will be shown in this section that the classical but manifestly covariant equivalents of Eqs. (15) and (16) are obtained from the four vectors $\mathbf{E}_\mu$ and $\mathbf{B}_\mu$.

We use the well-known relations[19,20] (in S.I. units),

$$\hat{E}^{(0)} = \left(\frac{2\hbar\omega}{\varepsilon_0 V}\right)^{1/2} \hat{a}^{(0)} \qquad \hat{E}^{(3)} = \left(\frac{2\hbar\omega}{\varepsilon_0 V}\right)^{1/2} \hat{a}^{(3)}$$
$$\hat{B}^{(0)} = \left(\frac{2\mu_0\hbar\omega}{V}\right)^{1/2} \hat{a}^{(0)} \qquad \hat{B}^{(3)} = \left(\frac{2\mu_0\hbar\omega}{V}\right)^{1/2} \hat{a}^{(3)} \tag{17}$$

to link the annihilation operators in states (0) and (3) to the equivalent field operators. Here $\varepsilon_0$ is the permittivity and $\mu_0$ the permeability of the vacuum state, $\hbar$ is the reduced Planck constant, and $V$ the quantization volume. From Eqs. (15a) and (17),

$$(\hat{E}^{(0)} - \hat{E}^{(3)})|\psi\rangle = 0 \qquad (\hat{B}^{(0)} - \hat{B}^{(3)})|\psi\rangle = 0 \tag{18}$$

and from Eq. (15b),

$$\langle\psi|\hat{E}^{(0)+}\hat{E}^{(0)}|\psi\rangle = \langle\psi|\hat{E}^{(3)+}\hat{E}^{(3)}|\psi\rangle$$
$$\langle\psi|\hat{B}^{(0)+}\hat{B}^{(0)}|\psi\rangle = \langle\psi|\hat{B}^{(3)+}\hat{B}^{(3)}|\psi\rangle \tag{19}$$

The classical equivalent of Eq. (18) is

$$E^{(0)} - |\mathbf{E}^{(3)}| = 0 \qquad B^{(0)} - |\mathbf{B}^{(3)}| = 0 \tag{20a}$$

and that of Eqs. (19) is

$$\begin{aligned} \mathbf{E}^{(0)2} &= \mathbf{E}^{(3)} \cdot \mathbf{E}^{(3)} \\ \mathbf{B}^{(0)2} &= \mathbf{B}^{(3)} \cdot \mathbf{B}^{(3)} \end{aligned} \tag{20b}$$

where $\mathbf{E}^{(3)} = \langle \psi | \hat{\mathbf{E}}^{(3)} | \psi \rangle$, etc. Equation (20a) is identical with Eq. (11), and Eq. (20b) is consistent with Eqs. (11) and (12a). However, Eq. (19) was derived from a quantized counterpart, Eq. (15a), which is manifestly covariant in that physical photon states are admixtures of states (0) and (3) of the quantum field.[7] It follows that classical field states in vacuo are also admixtures of the classical (0) and (3) polarizations as defined by Eqs. (19) and (20). From this we arrive at two fundamentally important conclusions:

1. The electric and magnetic components of the electromagnetic field in vacuo are manifestly covariant four vectors in spacetime, $E_\mu$ and $B_\mu$, respectively, all of whose four components must be physically meaningful.
2. From Eq. (19) the fields $\mathbf{E}^{(3)} = E^{(0)}\hat{\mathbf{e}}^{(3)}$ and $\mathbf{B}^{(3)} = B^{(0)}\hat{\mathbf{e}}^{(3)}$ are independent of the phase of the traveling plane wave, which is consistent with Eq. (6) and the development thereof.

The four physical states of the classical, manifestly covariant, electromagnetic field are formed from the (0) and (3) admixtures $E^{(0)} - |\mathbf{E}^{(3)}|$ and $B^{(0)} - |\mathbf{B}^{(3)}|$ and from the well-known transverse ((1) and (2)) components.[8–15] Although Maxwell's phenomenological equations of the 1860s are conventionally accepted as being consistent with special relativity, the electric and magnetic fields that they relate in vacuo are purely spacelike. The field potentials in terms of which **E** and **B** are described in the conventional theory[8–15] are, on the other hand, taken to be components of the potential four vector in spacetime:

$$A_\mu \equiv (\mathbf{A}, +\mathrm{i}\phi) \tag{21}$$

l and $\phi$ is the timelike (scalar)



potential. The four-curl of $A_\mu$ conventionally produces the electromagnetic field tensor in spacetime (see Appendix A):

$$F_{\mu\nu} = -F_{\nu\mu} = \varepsilon_0 \begin{bmatrix} 0 & cB_Z & -cB_Y & -\mathrm{i}E_x \\ -cB_Z & 0 & cB_X & -\mathrm{i}E_Y \\ cB_Y & -cB_X & 0 & -\mathrm{i}E_Z \\ \mathrm{i}E_X & \mathrm{i}E_Y & \mathrm{i}E_Z & 0 \end{bmatrix} \tag{22}$$

The difficulty with $F_{\mu\nu}$ and with the conventional theory is that $F_{\mu\nu}$ contains no explicit reference to the timelike components of $E_\mu$ and $B_\mu$. These are removed by the mathematical nature of the four-curl:

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{23}$$

A method must be found to relate $E_\mu$ and $B_\mu$ to the four potential $A_\mu$, thus making the theory rigorously self-consistent and manifestly covariant.

It is reasonable to base this method in contemporary quantum field theory,[7] in which the electromagnetic field is an example of a massless gauge field (as opposed to a spinor field), described in general by the Poincaré group. The latter incorporates three Lorentz rotation generators ($J_i$), three Lorentz boost generators ($K_i$), and four generators of spacetime translation ($P_\mu$). Details are well summarized in Ref. 7. The difference between the Poincaré and Lorentz groups is that the former incorporates the generator of spacetime translations, defined by

$$P_\mu \equiv i\frac{\partial}{\partial x_\mu} \tag{24}$$

and which is proportional through a factor $\hbar$ to the momentum energy four vector operator. The Pauli-Lubansky pseudovector in spacetime, $W_\mu$, characterizes the Poincaré group by forming its second (spin) Casimir invariant $W_\mu W_\mu$. (The first (mass) Casimir invariant is formed from $P_\mu P_\mu$.) The Pauli Lubansky pseudovector is defined by

$$W_\mu = -\tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma \tag{25}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric spacetime tensor of rank four, and where the four tensor $J_{\nu\rho}$ is

$$J_{\mu\sigma}(\mu,\sigma=0,\ldots,3)\begin{bmatrix} J_{ij}=-J_{ji}=\varepsilon_{ijk}J_k \\ J_{i0}=-J_{0i}=-K_i \end{bmatrix}$$
$$i,j,k=1,2,3 \tag{26}$$

## III. THE LINK BETWEEN $E_\mu$, $B_\mu$, $F_{\mu\nu}$, AND $A_\mu$

The photon helicity is defined in the lightlike condition applied to Eq. (25), in which condition $W_\mu$ becomes proportional to $P_\mu$, so that the helicity is a number, $+1$ (Ref. 7). The opposite value, $-1$, is given by considerations[7] of parity inversion. We show in this section that $E_\mu$ and $B_\mu$ can be defined in terms of $F_{\mu\nu}$, and therefore of the four curl of $A_\mu$, by an equation whose structure is the same as that of Eq. (25). Thus, $E_\mu$ and $B_\mu$ are identified as a Pauli Lubansky vector and pseudovector, respectively. This procedure succeeds in expressing electric and magnetic four vectors in terms of a single potential four vector, and covariantly describes the conventional[8–15] relations between **E**, **B**, **A**, and $\phi$ in vacuo.

The primary basis of the derivation is the observation that $J_{\mu\nu}$ has the same structure as $F_{\mu\nu}$, both being antisymmetric four tensors of the type (26). The $ij=-ji$ components of $F_{\mu\nu}$ are therefore identified as being proportional to rotation generators of the Poincaré group.[7] With this observation, it becomes obvious that the spacelike electric components in Eq. (22) are proportional to boost generators of the Poincaré group.[7] Pure boost Lorentz transformations[7] are those connecting two inertial frames moving at a relative speed $v$. A Lorentz rotation is a four vector rotation in spacetime. Therefore, the conventional assertion that $cB^{(3)}$ and $E^{(3)}$ in $F_{\mu\nu}$ are physically meaningless is tantamount to asserting that one out of three rotation generators and one out of three boost generators are physically meaningless. This is a reductio ad absurdum, and a vivid demonstration of the fact that the conventional assertion that $E^{(3)}$ and $B^{(3)}$ are unphysical is flawed fundamentally, i.e.,is geometrically unsound.

Secondly, Eq. (3), which we have derived elsewhere[2] using independent considerations in the quantum field, shows that $\hat{B}^{(3)}$ is directly proportional to the photon's quantized angular momentum boson operator $\hat{\mathbf{J}}$. Classically, the rotation generators $\mathbf{J}_X$, $\mathbf{J}_Y$, and $\mathbf{J}_Z$ of the Poincaré group are matrices of numbers which obey the commutation relations

$$[\mathbf{J}_X,\mathbf{J}_Y]=\mathrm{i}\mathbf{J}_Z \tag{27}$$



and cyclic permutations thereof. These are immediately recognizable to be the commutators of quantized angular momentum within the factor $\hbar$. This suggests that $F_{\mu\nu}$ is proportional to $J_{\mu\nu}$ through a four scalar invariant of spacetime. For example, components 12 and 21 of $F_{\mu\nu}$ are respectively $cB^{(3)}$ and $-cB^{(3)}$, and all other off-diagonal components of $F_{\mu\nu}$ have the same dimensions as the 12 and 21 components. The 12 and 21 components of $J_{\mu\nu}$ are the angular momenta $J^{(3)}$ and $-J^{(3)}$ within a factor $\hbar$. The 21 and 12 component proportionality is therefore embodied in Eq. (3).

Thirdly, the contemporary quantum field description of photon helicity in terms of $W_\mu$ and $P_\mu$ clearly involves the concept of spacetime translation within the Poincaré group, introduced by Wigner[21] in 1939, and whose generator, as we have seen, is $P_\mu$. This is missing from the Lorentz group.[7] The concept of spacetime translation is also missing from the Maxwell equations, which do not deal explicitly in the timelike field polarization (0). Spacetime translation is implied in d'Alembert's equation (1),[7] but if and only if all four field polarizations are taken to be physically meaningful. To see this, recall (1) that the four-curl (23) removes the (0) polarization, and (2) that the Maxwell equations (2) are equations[7] in $F_{\mu\nu}$ and its dual $\tilde{F}_{\mu\nu}$. From the proportionality of $F_{\mu\nu}$ to $J_{\mu\nu}$ it becomes clear, however, that the components of $F_{\mu\nu}$ must be either rotation or boost generators of the Poincaré group, and there is no reference within $F_{\mu\nu}$ to spacetime translation. Photon helicity, on the other hand, is described in terms of the proportionality and orthogonality in spacetime of $W_\mu$ to $P_\mu$ (Ref. 7) in the lightlike condition. Therefore, the description of electric and magnetic components in vacuo in terms of $F_{\mu\nu}$ is inconsistent with the contemporary description of helicity. This inconsistency can be remedied if and only if electric and magnetic components of the electromagnetic field in vacuo are manifestly covariant four vectors $E_\mu$ and $B_\mu$.

Fourthly, defining a photon state $|k\rangle$ in the lightlike condition, the photon helicity ($\lambda$) in contemporary thought[7] is given by the condition

$$(W_\mu-\lambda P_\mu)|k\rangle=0 \tag{28}$$

so that for the massless photon, $\lambda$ is a number $(+1)$, which is the ratio of $W_\mu$ to $P_\mu$ and which has the dimensions of angular momentum,[7] provided that $P_\mu$ has the dimensions of linear momentum/energy by multiplication by $\hbar$. For lightlike particles with no mass, such as the photon[7],

$$k_\mu\equiv(0,0,k,-\mathrm{i}k) \tag{29}$$

which can be regarded as a unit four vector

$$\delta_\mu \equiv (0, 0, 1, -\mathrm{i}) \tag{30}$$

describing a massless particle moving at the speed of light in the $Z$ spacelike axis (the propagation axis of the electromagnetic wave). Equation (30) can be incorporated into Eq. (25) by dividing the left and right sides of Eq. (25) by $\hbar$, so that $W_\mu$, $J_{\nu\rho}$, and $P_\sigma$ become numbers. This is consistent with the definition of rotation and boost generators as matrices of numbers (Eqs. (2.65)–(2.67) of Ref. 7).

With these considerations, we are led to the following fundamental definitions of $B_\mu$ and $E_\mu$ in terms of $F_{\nu\rho}$ and $\delta_\sigma$:

$$cB_\mu = -\frac{\mathrm{i}}{2\varepsilon_0}\varepsilon_{\mu\nu\rho\sigma}F_{\nu\rho}\delta_\sigma \tag{31a}$$

$$E_\mu = \frac{1}{2\varepsilon_0}\varepsilon_{\mu\nu\rho\sigma}\tilde{F}_{\nu\rho}\delta_\sigma \tag{31b}$$

In these equations, we recall that if $\varepsilon_{0123} = 1$, then its other nonzero elements are $+1$ and $-1$, according as to whether $\varepsilon_{0123}$ can be generated by an even or odd number of subscript pair permutations. Thus, for example,

$$\begin{aligned} \varepsilon_{3120} &= -1 \qquad \varepsilon_{3210} = 1 \qquad \varepsilon_{2310} = -1 \\ \varepsilon_{3120} &= -1 \qquad \varepsilon_{1320} = 1 \qquad \varepsilon_{1230} = -1 \end{aligned} \tag{32}$$

and so on. All elements of $\varepsilon_{\mu\nu\rho\sigma}$ are zero in which two or more subscripts are equal. The elements of $F_{\nu\rho}$ are labeled explicitly as

$$F_{\nu\rho}(\nu, \rho = 0, 1, 2, 3) = \begin{bmatrix} 11 & 12 & 13 & 10 \\ 21 & 22 & 23 & 20 \\ 31 & 32 & 33 & 30 \\ 01 & 02 & 03 & 00 \end{bmatrix} \tag{33}$$

With these definitions it is verified by tensor algebra (Appendix B) that the real elements (labeled (1), (2), (3), and (0)) of the magnetic and electric field four vectors

$$\begin{aligned} E_\mu &\equiv (E^{(1)}, E^{(2)}, E^{(3)}, -\mathrm{i}E^{(0)}) \\ B_\mu &\equiv (B^{(1)}, B^{(2)}, B^{(3)}, -\mathrm{i}B^{(0)}) \end{aligned} \tag{34}$$



are given by Eqs. (31). (The dual $\tilde{F}_{\mu\nu}$ of $F_{\rho\sigma}$ is obtained by the well-known[7] dual transformation $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$.)

Equation (31a) covariantly defines $B_\mu$ as a Pauli-Lubansky pseudovector, and Eq. (31b) covariantly defines $E_\mu$ as a Pauli-Lubansky vector. These definitions imply several properties of both $B_\mu$ and $E_\mu$:

1. Since $F_{\mu\nu}$ is the four-curl of $A_\nu$ (Eq. (23)), Eqs (31) covariantly relate $B_\mu$ and $E_\mu$ to $A_\mu$ in spacetime.
2. Since $E_\mu$ and $B_\mu$ are four vectors in Minkowski spacetime, it follows in pseudo-Euclidean geometry that $E_\mu E_\mu$ and $B_\mu B_\mu$ are constants in spacetime, and that

$$\frac{\partial E_\mu}{\partial x_\mu} = \text{constant} \qquad \frac{\partial B_\mu}{\partial x_\mu} = \text{constant} \tag{35}$$

3. Equation (31a) is dual with Eq. (31b), because under the dual transformation of fields

$$F_{\rho\sigma} \to \tilde{F}_{\mu\nu} \qquad E_\mu \to -\mathrm{i}cB_\mu \tag{36}$$

4. The parity inversion $\hat{P}$ and motion reversal $\hat{T}$ symmetries of $B_\mu$ are consistent with those of $F_{\nu\rho}$ and $\delta_\sigma$, bearing in mind that the latter is the unit generator of spacetime translation. Since Eq. (31b) is dual with Eq. (31a), its symmetries are consistent with those of Eq. (31a). $B_\mu$ is a pseudovector because its spacelike component **B** is positive to $\hat{P}$ and negative to $\hat{T}$. $E_\mu$ is a vector because **E** is negative to $\hat{P}$ and positive to $\hat{T}$.
5. From the properties of the Pauli-Lubansky pseudovectors and vectors,[7] both $E_\mu$ and $B_\mu$ are orthogonal to $\delta_\mu$ in spacetime:

$$B_\mu\delta_\mu = 0 \qquad E_\mu\delta_\mu = 0 \tag{37}$$

6. Both $E_\mu$ and $B_\mu$ are defined in Eqs. (31) in terms of the unit generator of spacetime translations $\delta_\sigma$, allowing $E_\mu$ and $B_\mu$ to be covariantly and consistently interpreted in terms of helicity in the lightlike condition.[7]
7. Since $E_\mu$ and $B_\mu$ are defined covariantly, the timelike components $E^{(0)}$ and $B^{(0)}$, respectively, are both explicitly and implicitly stated to be physically meaningful in spacetime.
8. The products $B_\mu B_\mu$ and $E_\mu E_\mu$ are both Casimir invariants[7] of the Poincaré group, specifically Casimir invariants of the second kind, or

"spin" invariants. The product $\delta_\mu \delta_\mu$ is a Casimir invariant of the first kind ("mass" invariant). This deduction follows from the definition (31) of $B_\mu$ and $E_\mu$ as Pauli-Lubansky types. In the lightlike condition, i.e., for the massless electromagnetic gauge field,

$$\delta_\mu \delta_\mu = 0 \qquad E_\mu E_\mu = 0 \qquad B_\mu B_\mu = 0 \tag{38}$$

Equation (38) is a classical statement of the fact that the photon in the quantum field is massless and possesses spin.

9. From Eqs. (37) and (38), $E_\mu$ and $B_\mu$ are both orthogonal and proportional to $\delta_\mu$ in spacetime. The proportionality constant (a scalar in spacetime) expresses the helicity of the electromagnetic gauge field.

It can be verified explicitly that the fundamental conditions (37) and (38) are satisfied by the circularly polarized transverse components of Eq. (7) in combination with the longitudinal components of Eqs. (8) and (12b). For example,

$$\begin{aligned} E_\mu E_\mu &\equiv E^{(1)2} + E^{(2)2} + E^{(3)2} - E^{(0)2} \\ &= E^{(0)2}\left(\hat{\mathbf{e}}^{(1)} \cdot \hat{\mathbf{e}}^{(1)} e^{2i\phi} + \hat{\mathbf{e}}^{(2)} \cdot \hat{\mathbf{e}}^{(2)} e^{-2i\phi} + \hat{\mathbf{e}}^{(3)} \cdot \hat{\mathbf{e}}^{(3)} - 1\right) \\ &= E^{(0)2}\left(\hat{\mathbf{e}}^{(1)} \cdot \hat{\mathbf{e}}^{(1)} e^{2i\phi} + \hat{\mathbf{e}}^{(2)} \cdot \hat{\mathbf{e}}^{(2)} e^{-2i\phi}\right) \\ &= \frac{E^{(0)2}}{2}\left((\mathbf{i} - i\mathbf{j}) \cdot (\mathbf{i} - i\mathbf{j}) e^{2i\phi} + (\mathbf{i} + i\mathbf{j}) \cdot (\mathbf{i} - + i\mathbf{j}) e^{-2i\phi}\right) \\ &= 0 \end{aligned} \tag{39}$$

## IV. CONSEQUENCES FOR VACUUM ELECTRODYNAMICS

Equations (31) covariantly define the four vectors $E_\mu$ and $B_\mu$ in spacetime. This means that the fundamentals of vacuum electrodynamics are changed. One immediate consequence is that Eqs. (35) restate the Gauss theory in covariant form. Using the polarizations defined in Eqs. (7), (8), and (12b) it is clear that the constant in Eqs. (35) is zero and that the Gauss theorem in differential form is covariantly written as

$$\frac{\partial E_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{E} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t} = 0 \tag{40}$$



and

$$\frac{\partial B_\mu}{\partial x_\mu} = 0 \quad \text{or} \quad \nabla \cdot \mathbf{B} + \frac{1}{c}\frac{\partial B^{(0)}}{\partial t} = 0 \tag{41}$$

Equations (40) and (41) replace the conventional spacelike statements of the Gauss theorem in differential form in vacuo:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \tag{42}$$

Therefore, it becomes clear that the covariant definitions (30) lead to the following covariant statements of the Maxwell equations in vacuo:

$$\begin{aligned} \frac{\partial F_{\mu\nu}}{\partial x_\mu} &= 0 \qquad \frac{\partial \tilde{B}_\mu}{\partial x_\mu} = 0 \\ \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\mu} &= 0 \qquad \frac{\partial B_\mu}{\partial x_\mu} = 0 \end{aligned} \tag{43}$$

in which $\tilde{B}_\mu$ is the dual of $\mathbf{B}_\mu$, and $\tilde{F}_{\mu\nu}$ that of $\mathbf{F}_{\rho\sigma}$. The Maxwell equations in the form (43) are covariantly consistent with the d'Alembert equation (1), which was the starting point of our development.

The vacuum electromagnetic energy density is, from Eqs. (31), covariantly defined in S.I. units as (see Appendix C)

$$U = \frac{1}{2}\left(\varepsilon_0 E_\mu E_\mu + \frac{1}{\mu_0} B_\mu B_\mu\right) \tag{44}$$

where $\mu_0$ is the permeability in vacuo. (Note that it is not consistent to refer to the vacuum as "free space"; it is covariantly described as "free spacetime.") From the example of Eqs. (39), it is clear that field polarizations (0) and (3), although physically meaningful, do not contribute to $U$, so that Eq. (44) happens to reduce to the conventional[8–15] spacelike definition of $U$:

$$U(\text{conventional}) \equiv \frac{1}{2}\left(\varepsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B}\right) \tag{45}$$

This deduction is consistent with the quantum field theory leading[7] to Eq. (16).

Similarly, the vacuum electromagnetic flux density (the conventional, spacelike, Poynting vector[8–15]) is covariantly defined from Eqs. (30) as the four vector product of $E_\mu$ and $B_\mu$, the four tensor, in S.I. units:

$$S_{\mu\nu} = \frac{1}{\mu_0}(E_\mu B_\nu - B_\mu E_\nu) \tag{46}$$

The conventional statement of the law of conservation of electromagnetic energy in vacuo is the Poynting theorem,[8–15] expressed through the continuity equation:

$$\nabla \cdot \mathbf{S} + \frac{1}{c^2}\frac{\partial U}{\partial t} = 0 \tag{47}$$

This is already Lorentz covariant in structure, because it is an equation in the four divergence of the Poynting four vector $S_\mu = (S, -\mathrm{i}S^{(0)})$, i.e.,

$$\frac{\partial S_\mu}{\partial x_\mu} = 0 \tag{48}$$

However, the conventional definition (47) implies that the two spacelike components of the Poynting four vector $S_\mu$ orthogonal to the propagation direction of the electromagnetic wave in vacuo must vanish. The definition (47), although Lorentz covariant, is not necessarily manifestly covariant, because it is based on the conventional[8–15] assumption that **E** and **B** are spacelike and transverse.

In a manifestly covariant description it is necessary to relate the Poynting four vector of Eq. (48) to the Poynting four tensor $S_{\mu\nu}$ formed from the vector product in spacetime of the novel four vectors $E_\mu$ and $B_\mu$. It is reasonable to propose that this relation is

$$S_\mu = \frac{\mathrm{i}}{2}\varepsilon_{\mu\nu\rho\sigma}S_{\nu\rho}\delta_\sigma \tag{49}$$



where $\varepsilon_{\mu\nu\rho\sigma}$ and $\delta_\sigma$ have the same meaning as in Eq. (31). Explicitly,

$$S^{(1)} \equiv S_1 = \frac{\mathrm{i}}{2}(\varepsilon_{1230}S_{23}\delta_0 + \varepsilon_{1320}S_{32}\delta_0 + \varepsilon_{1203}S_{20}\delta_3 + \varepsilon_{1023}S_{02}\delta_3) \tag{50a}$$

$$S^{(2)} \equiv S_2 = \frac{\mathrm{i}}{2}(\varepsilon_{2310}S_{31}\delta_0 + \varepsilon_{2130}S_{13}\delta_0 + \varepsilon_{2013}S_{01}\delta_3 + \varepsilon_{2103}S_{10}\delta_3) \tag{50b}$$

$$S^{(3)} \equiv S_3 = \frac{\mathrm{i}}{2}(\varepsilon_{3210}S_{21}\delta_0 + \varepsilon_{3120}S_{12}\delta_0)$$

$$-\mathrm{i}S^{(0)} \equiv -\mathrm{i}S_0 = \frac{\mathrm{i}}{2}(\varepsilon_{0123}S_{12}\delta_3 + \varepsilon_{0213}S_{21}\delta_3) \tag{50c}$$

$$\text{with} \quad \delta_3 = 1 \quad \delta_0 = -\mathrm{i} \quad (\delta_1 = \delta_2 = 0)$$

$$\begin{array}{llll} \varepsilon_{0123} = 1 & \varepsilon_{0213} = -1 & \varepsilon_{3120} = -1 & \varepsilon_{3210} = -1 \\ \varepsilon_{2130} = 1 & \varepsilon_{2310} = -1 & \varepsilon_{2013} = 1 & \varepsilon_{2103} = -1 \\ \varepsilon_{1230} = -1 & \varepsilon_{1320} = 1 & \varepsilon_{1203} = 1 & \varepsilon_{1023} = -1 \end{array} \tag{50d}$$

Equations (50a) and (50b) show that in this definition, the Poynting four vector in spacetime develops components in the spacelike axes orthogonal to the propagation axis (3).

The definition (49) of the manifestly covariant Poynting vector introduces the unit generator of spacetime translations, $\delta_\sigma$, for an electromagnetic wave traveling in vacuo in the spacelike axis (3). In direct analogy with our fundamental definitions, Eqs. (31), of $E_\mu$ and $B_\mu$, $S_\mu$ is thereby defined within the Poincaré group rather than the Lorentz group, and spacetime translation is included explicitly in the definition. This means that the manifestly covariant Poynting vector is also a Pauli-Lubansky vector within the Poincaré group in spacetime. Note that from Eqs. (50c) and (50d),

$$\begin{aligned} S^{(3)} - |S^{(0)}| &= 0 \\ S^{(3)} \cdot S^{(3)} - |S^{(0)}|^2 &= 0 \end{aligned} \tag{51}$$

i.e., the conventional, spacelike Poynting vector, which has only one spacelike component, (3), and no timelike component, (0), becomes within the structure of Eq. (49) a physical state that is an admixture of (3) and (0) polarizations. The other spacelike components of $S_\mu$, i.e., (1) and (2), also become physically meaningful through Eqs. (50a) and (50b). At an instant in spacetime, these components (1) and (2) are experimental observables. However, observations (Appendix 3) of the electromagnetic energy flux density, known as the Poynting vector,[8–15] are made by the observer with an instrument such as a power meter, which gives only the time-averaged value of the Poynting vector. The components $S_{23}, S_{32}, S_{20}, S_{02}, S_{31}, S_{13}, S_{01}, S_{10}$ disappear upon time averaging (Appendix 3) because they are made up of products of one phase-dependent component and one that is phase independent. The components $S_{21}$ and $S_{12}$, on the other hand, are products of two phase-dependent components, one of which is the complex conjugate of the other, so that the phase disappears in the product, which is thereby nonzero after time averaging. For these reasons, the conventional Poynting theorem (47), which is not manifestly covariant, happens to be an adequate description of the law of conservation of electromagnetic energy, but only on a time-averaged basis. If it were experimentally possible to observe electromagnetic energy flux density in an instant in spacetime, then the components $S_{23}$ and so on would contribute explicitly to the law of conservation of energy. Clearly, if $S_\mu$ is a Pauli-Lubansky vector in the Poincaré group, then the product is a Casimir invariant of type two of the Poincaré group, and $S_\mu$ is orthogonal to $\delta_\mu$ in spacetime.

The description of the electromagnetic field polarization in vacuo through the four Stokes parameters in terms of $E_\mu$ and $B_\mu$ requires a modification[22] of the conventional description[23] based on Pauli matrices:

$$S_0 = [E_X E_Y]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix} \tag{52a}$$

$$S_1 = [E_X E_Y]\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix} \tag{52b}$$

$$S_2 = [E_X E_Y]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix} \tag{52c}$$

$$S_3 = [E_X E_Y]\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix} \tag{52d}$$

so that

$$S_0^2 = S_1^2 + S_2^2 + S_3^2 \tag{53a}$$

$$[S_1, S_2] = iS_3 \tag{53b}$$



Within a factor $\hbar$, the Stokes parameters obey the commutation rules of quantized angular momentum, and form a four vector $S_\mu \equiv (\mathbf{S}, iS_0)$. (The Stokes vector $S_\mu$ should not be confused with the Poynting vector, unfortunately also denoted $\mathbf{S}$ in the conventional literature.) It follows from Eq. (53b) that the Pauli matrices also obey the angular momentum commutation rules. Equations (52) omit the longitudinal and timelike polarizations of the four vector $E_\mu$, and for a manifestly covariant description, these must be included in the basic definition of the four Stokes parameters (real numbers in the conventional theory[8–15]). The generalization of $S_0$ to $S_3$ must conform with Eqs. (53), and $S_0$, which is proportional to the electromagnetic energy density in vacuo,[8–15] must conform to our earlier results (16) and (39). It is natural to propose the replacement in Eqs. (52) of the field vectors $[E_X, E_Y]$ and $\begin{bmatrix}E_X^*\\ E_Y^*\end{bmatrix}$ by their manifestly covariant equivalents (four vectors), and to replace the Pauli matrices by Dirac matrices (angular momentum operators[24]). The latter obey the same commutation rules and their structure is that of a "doubled" (4 × 4) Pauli matrix. The following generalization is manifestly covariant and conforms with Eq. (53):

$$S_0 = \left[E_X E_Y \frac{E_Z}{2} \frac{-iE^{(0)}}{2}\right]\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\\ E_Z/2\\ -iE^{(0)}/2\end{bmatrix} \tag{54a}$$

$$S_1 = \left[E_X E_Y \frac{E_Z}{2} \frac{-iE^{(0)}}{2}\right]\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\\ E_Z/2\\ -iE^{(0)}/2\end{bmatrix} \tag{54b}$$

$$S_2 = \left[E_X E_Y \frac{E_Z}{2} \frac{-iE^{(0)}}{2}\right]\begin{bmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\\ E_Z/2\\ -iE^{(0)}/2\end{bmatrix} \tag{54c}$$

$$S_3 = \left[E_X E_Y \frac{E_Z}{2} \frac{-iE^{(0)}}{2}\right]\begin{bmatrix}0 & -i & 0 & 0\\ i & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{bmatrix}\begin{bmatrix}E_X^*\\ E_Y^*\\ E_Z/2\\ -iE^{(0)}/2\end{bmatrix} \tag{54d}$$

(The factor $\frac{1}{2}$ follows from the definitions, Eqs. (31) and Appendix B, Eq. (B.7).) Explicitly written out, the covariant Stokes parameters for one sense of circular polarization become

$$S_0 = E_X E_X^* + E_Y E_Y^* = E^{(0)2} \tag{55a}$$

$$S_1 = E_X E_X^* - E_Y E_Y^* + \tfrac{1}{4}\left(E_Z^2 + E^{(0)2}\right) = \tfrac{1}{2}E^{(0)2} \tag{55b}$$

$$S_2 = E_X E_Y^* + E_Y E_X^* - \frac{i}{4}\left(E_Z E^{(0)} + E^{(0)} E_Z\right) = -\frac{1}{2} i E^{(0)2} \tag{55c}$$

$$S_3 = -i(E_X E_Y^* - E_Y E_X^*) = E^{(0)2} \tag{55d}$$

We find that the conventional result $S_1 = S_2 = 0$ in circular polarization[8–15] is replaced by

$$S_1 = iS_2 = \tfrac{1}{2}E^{(0)2} = \tfrac{1}{2}\mathbf{E}^{(3)} \cdot \mathbf{E}^{(3)} \tag{56}$$

Our covariant theory leaves the value of $S_0$ unchanged, as required, and finally, $S_3$ is also unchanged. Significantly, the results (55) can be expressed entirely in terms of $\mathbf{E}^{(3)}$ (or $\mathbf{B}^{(3)}$):

$$S_0 = |S_3| = \mathbf{E}^{(3)} \cdot \mathbf{E}^{(3)} \tag{57}$$

together with Eq. (56). Since $E^{(0)} = cB^{(0)}$ in free spacetime, Eqs. (56) and (57) can be represented in terms of $\mathbf{B}^{(3)}$. In particular,

$$|S_3| = c^2 \mathbf{B}^{(3)} \cdot \mathbf{B}^{(3)} = c^2 B^{(0)} |\mathbf{B}^{(3)}| \tag{58}$$

a result derived previously[1–5] through the relation

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{icE^{(0)}} \tag{59}$$

It is interesting to note that the following eigenvalue (operator type but classical) equation consistently reconciles the existence of only one photon helicity for one sense of circular polarization:

$$\begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \begin{bmatrix} E_X \\ E_Y \\ E_Z/2 \\ -iE^{(0)}/2 \end{bmatrix} = \lambda^{(i)} \begin{bmatrix} E_X \\ E_Y \\ E_Z/2 \\ -iE^{(0)}/2 \end{bmatrix} \tag{60}$$

The eigenvalues are

$$\lambda^{(1)} = \frac{-iE_Y}{E_X} = -1 \qquad \lambda^{(2)} = \frac{iE_X}{E_Y} = -1$$
$$\lambda^{(3)} = \frac{-E^{(0)}}{E_Z} = -1 \qquad \lambda^{(4)} = \frac{-E_Z}{E^{(0)}} = -1 \tag{61}$$

For the opposite sense of circular polarization, $E_Y$ and $E_Z$ change sign and the four eigenvalues $\lambda^{(i)}$ becomes $+1$. Equation (60) therefore provides the result

$$\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = \lambda^{(4)} = \mp 1 \tag{62}$$

for different senses of circular polarization, and reconciles the existence of four different field polarizations with only two different field helicities. (The four polarizations are right and left circular spacelike, longitudinal spacelike, and timelike. The two helicities are $+1$ and $-1$.) In the conventional theory,[8–15] Eq. (60) becomes

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} E_X \\ E_Y \end{bmatrix} = \lambda^{(i)} \begin{bmatrix} E_X \\ E_Y \end{bmatrix} \tag{63}$$

i.e., helicities $\lambda^{(3)}$ and $\lambda^{(4)}$ are missing, and the remaining two "transverse" helicities are generated by a Pauli matrix rather than a Dirac matrix.

It is therefore concluded that the structures of the Stokes parameters are changed in the manifestly covariant description of electrodynamics in vacuo, and therefore so is the fundamental specification of the polarization characteristics of light: the Hermitian polarization density, or coherency matrix of Born and Wolf[13] and the polarization tensor of Landau and Lifshitz.[12] Specifically, the Stokes parameters $S_1$ and $S_2$ no longer vanish in circular polarization, and this is a direct consequence of the covariant nature of $E_\mu$, in that its longitudinal and timelike components now contribute to a purely real, nonzero, $S_1$, and a purely imaginary $S_2$ with the opposite sign. Conventionally,[8–15] $S_1$ and $S_2$ are nonzero only in elliptical polarization. They can be described in terms of excess of linear polarization, and conventionally it is considered that there is no excess of linear polarization when the beam is fully right or left circularly polarized. However, in the covariant description, there is an additioinal longitudinal component in the propagation axis of the beam, even in a completely circularly polarized beam. The longitudinal components $\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$

vanish, however, if the beam has an equal amount of transverse right and transverse left circularly polarized components. In this state of transverse linear polarization, the imaginary contribution to $S_2$ vanishes, but the real contribution to $S_1$ doubles. This can be interpreted to mean that although $\mathbf{E}^{(3)}$ changes sign between right and left transverse circular polarization, its square $E^{(3)2}$ evidently does not. It is $E^{(3)2}$ that contributes to $S_1$. This emphasizes the fact that the Stokes parameters are quadratic in the electric part of the electromagnetic field.

$S_3$ is unchanged in the covariant description, because $S_3$ is defined in this description by

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{\mathrm{i}cE^{(0)}} = B^{(0)}\mathbf{k} \tag{64a}$$

$$B^{(0)} = \left(\frac{\varepsilon_0}{I_0 c}\right)^{1/2} |S_3| \tag{64b}$$

## V. DISCUSSION

The covariant description of the electromagnetic field in vacuo shows that there are physically meaningful fields $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ that satisfy Maxwell's equations. These fields do not appear explicitly in the conventional theory,[8–15] and are assumed to be physically meaningless. It is therefore necessary to identify experiments that can distinguish between the conventional theory and the manifestly covariant theory of the electromagnetic field. One immediately obvious consequence of $\mathbf{B}^{(3)}$ is that circularly polarized electromagnetic radiation can magnetize matter. Before embarking on a development of these properties, however, we show that effects such as natural optical activity, the electrical Kerr effect, and the development of ellipticity in an initially circularly polarized light beam can be explained in terms of changes in $\mathbf{B}^{(3)}$ as they traverse a sample. The essential reason for this is that whenever the Stokes parameter $S_3$ appears in physical optics, it signals (vide supra) the existence of $\mathbf{B}^{(3)}$, to whose magnitude it is directly proportional:

$$|\mathbf{B}^{(3)}| = \left(\frac{\varepsilon_0}{I_0 c}\right)^{1/2} |S_3| = \frac{|S_3|}{c^2 B^{(0)}} \tag{65}$$

Therefore, $S_3$ can be replaced whenever it occurs by the scalar quantity

$$\pm c^2 B_0 |\mathbf{B}^{(3)}| \equiv \pm c^2 \mathbf{B}^{(3)} \cdot \mathbf{B}^{(3)} \tag{66}$$

In material media, as opposed to free space, Kielich[25] has shown that linear and nonlinear optical activity depends on $S_3$, and in the Rayleigh theory[26] of natural optical activity in chiral media, it is well known that whatever the nature of the several molecular property tensors participating in the polarization and magnetization of the material, the observable of circular dichroism has pseudoscalar symmetry and is proportional to the third Stokes parameter. For different enantiomers for a given sense of transverse circular polarization, or for one enantiomer for different sense of transverse circular polarization,

$$\frac{I_R - I_L}{I_R + I_L} = \pm \frac{S_3}{S_0} \tag{67}$$

where $I_R$ and $I_L$ are the intensities of right and left components transmitted by structurally chiral material, with

$$I_0 = I_R + I_L \tag{68}$$

for the transmitted total beam intensity. From Eqs. (67) and (68) we derive the result (with $S_0 = c^2 B^{(0)2}$),

$$\pm \frac{S_3}{S_0} = \pm \frac{|\mathbf{B}^{(3)2}|}{B^{(0)2}} = \frac{I_R - I_L}{I_R + I_L} \tag{69}$$

which reveals the fundamental origin of the phenomenon of circular dichroism at all electromagnetic frequencies, because it shows that the observable $(I_R - I_L)$ is proportional to $|\mathbf{B}^{(3)2}|$.

The origin of circular dichroism, therefore, resides in the photon's longitudinal magnetostatic flux quantum $\hat{B}^{(3)}$, whose expectation value is $\mathbf{B}^{(3)}$.

The observable $I_R - I_L$ is therefore a spectral consequence of the interaction of $\mathbf{B}^{(3)}$ with structurally chiral material. From Eq. (69), $I_R - I_L$ is proportional to the real pseudoscalar $\pm|\mathbf{B}^{(3)}|$ after they emerge from the chiral material through which the beam has passed, i.e., after interaction has occurred between the flux quantum $\hat{B}^{(3)}$ and the appropriate molecular property tensors.[26] For one photon, the observable $I_R - I_L$ provides an experimental measure of the transmitted elementary $\mathbf{B}^{(3)}$ at each frequency. Although $\mathbf{B}^{(3)}$ itself is independent of frequency, the interacting molecular property tensor is not. Semiclassical perturbation theory[26] gives, for linear optical activity,

$$\frac{S_3}{S_0} = \frac{|\mathbf{B}^{(3)2}|}{B^{(0)2}} \tanh[\omega\mu_0 c l N \zeta''_{XYZ}(g)] \tag{70}$$

where $\mu_0$ is the permeability in vacuo, $\omega$ the angular frequency of the beam, $l$ the sample path length, and $\zeta''_{XYZ}$ a combination[26] of molecular property tensors, which may be electric and/or magnetic in nature. For nonlinear optical activity, Eq. (70), as shown by Kielich,[25] contains additional terms.

Therefore, every time natural optical activity is observed with $I_R - I_L$, as in circular dichroism, the quantity $\mathbf{B}^{(3)}$, has been measured. In this context, a covariant description of the electromagnetic field is one that identifies the phenomenon of circular dichroism with the longitudinal field $\mathbf{B}^{(3)}$, showing that the latter is physically meaningful and is, indeed, well measured in the literature although not explicitly recognized as a magnetic field. In the conventional description on the other hand, natural optical activity is measured by $S_3/S_0$, which is given by

$$\frac{S_3}{S_0} = \frac{-i(E_X E_Y^* - E_Y E_X^*)}{E_X E_X^* + E_Y E_Y^*} \tag{71}$$

and $\mathbf{E}^{(3)}$ and $\mathbf{B}^{(3)}$ are conventionally supposed to be physically meaningless. However, $S_3/S_0$ is, of course, also expressible in the covariant description by Eq. (71), showing that the covariant description is both simpler and more complete than the conventional one. The conventional assertion that $\mathbf{B}^{(3)}$ be physically meaningless conflicts with Eq. (6), and becomes unsustainable, because $\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}$ is a physically meaningful quantity directly proportional to $S_3$. It is more complete, more revealing, and more "natural" to describe optical activity as changes in $\mathbf{B}^{(3)}$ as a medium is traversed by a light beam. In other words, the phenomenon of natural optical activity is definitive experimental evidence for the existence of $\mathbf{B}^{(3)}$ in physical optics.

More generally, it can be shown that any phenomenon in optics that involves $S_3$ must involve $\mathbf{B}^{(3)}$ in its quantized or classical forms, whichever is the more appropriate to a given situation. Throughout the contemporary literature[27] that there are many of these optical phenomena, one commonplace example being the development of ellipticity in an initially circularly polarized light beam. For example, in the electric Kerr effect,[28] beam ellipticity ($\eta$) is expressed in terms of $S_3$, and is induced with an electric field in a probe laser. The electric Kerr effect is therefore

$$\frac{|\mathbf{B}^{(3)2}|}{B^{(0)2}} = \sin(2\eta) \tag{72}$$

where $\eta$ is the ellipticity developed in the transmitted probe as a result of the application of an electric field to a sample. This is experimental evidence for the existence of the longitudinal field $\mathbf{B}^{(3)}$. Note that for an initially linearly polarized beam, $\mathbf{B}^{(3)}$ is zero, and so Eq. (72) shows that the development of ellipticity in the Kerr effect is a direct consequence of the interaction of $\mathbf{B}^{(3)}$ with the medium through which the probe laser has passed.

Rayleigh refringent scattering theory[26] shows that the third Stokes parameter $S_3$ is associated with a change $d\eta/dZ$ in ellipticity in a beam passing through a sample of thickness $Z$. Therefore, $d\eta/dZ$ measures changes in $\mathbf{B}^{(3)}$ as it traverses the sample thickness $Z$. We arrive at the generally valid conclusion that ellipticity in the electromagnetic plane wave is directly related to $\mathbf{B}^{(3)}$, and that the development of ellipticity can be expressed in terms of these fields. The scalar magnitude of $\mathbf{B}^{(3)}$ is $|\mathbf{B}^{(3)}|$, respectively, associated with the timelike polarization (0) of the electromagnetic field. The timelike polarization always appears as an admixture with the longitudinal polarization, and both are physically meaningful because they are observed in fundamental optical phenomena. Equation (70), for example, shows that circular dichroism is related to the molecular property tensor sum represented by $\zeta''$, which is made up of the Rosenfeld tensor and the electric quadrupole tensor. Note carefully, however, that $\zeta''$ is a material property, while $\mathbf{B}^{(3)}$ is a property of free spacetime, which interacts with matter. The definition of $S_3$ in free spacetime in terms of $\mathbf{B}^{(3)}$ is obviously unaffected by any material property because $\mathbf{B}^{(3)}$ is associated with fundamental photon polarizations. In Eq. (70) we have used the result that $|\mathbf{B}^{(3)2}|$ is directly proportional to the Stokes parameter $S_3$ in free spacetime, and have replaced the Stokes parameter by a term proportional to $|\mathbf{B}^{(3)2}|$.

In summary, there is copious experimental evidence for the existence of $\mathbf{B}^{(3)}$, which is a physically meaningful magnetic field in free spacetime. Through Eq. (6), the conventional description is supplanted in physical optics by the more complete and more rigorous covariant description, i.e., by a description that is fully compatible with the theory of special relativity. Although the conventional description is self-consistent up to a point, the key equation (6) of this paper shows that it lacks the polarizations (3) and (0), which are present in the quantum field[7] but usually wrongly asserted to be physically meaningless. Note that Eq. (6) is invariant to the fundamental symmetries of physics[29]: charge conjugation $\hat{C}$, parity inversion $\hat{P}$, and motion reversal $\hat{T}$ and is therefore a rigorously self-consistent equation of electrodynamics in free spacetime.

It is also interesting to note[29] that in the field of high-energy particle physics, experimental evidence exists for timelike photons that can be produced in electron positron annihilation processes at extremely high energy. This presumably means that in such processes the concommitant magnetic and electric field amplitudes $B^{(0)}$ and $E^{(0)}$ exist independently, and are therefore physically meaningful.

Having established with available data the existence of $\mathbf{B}^{(3)}$, it is now possible to reinterpret known optical phenomena and to predict with some degree of confidence the existence of hitherto unmeasured optical phenomena based on $\mathbf{B}^{(3)}$. The everyday phenomenon of optical absorption is described by the Beer-Lambert law:

$$\alpha(\bar{\nu}) = \frac{1}{d} \log_e \frac{I_0}{I} \tag{73}$$

Here $I_0$ is the incident beam intensity, $I$ the transmitted beam intensity, and $d$ the sample length. $\alpha$ is the power absorption coefficient[30] in neper $m^{-1}$, and this quantity can be reinterpreted in terms of $\mathbf{B}^{(3)}$, because the zeroth Stokes parameter $S_0$ is proportional to beam intensity. Therefore, simple optical absorption is a process that can be interpreted in terms of the longitudinal electric and/or magnetic fields of the electromagnetic plane wave, an interpretation that is just as valid as the usual one[30] in terms of the transverse components $\mathbf{E}^{(1)}$ or $\mathbf{E}^{(2)}$. In general, since all four Stokes parameters in covariant electrodynamics can be expressed in terms of $\mathbf{B}^{(3)}$, all optical phenomena involving beam polarization or optical coherence processes in linear physical optics can also be described in terms of these longitudinal fields.

In nonlinear optics,[31] the light beam is used to induce phenomena in material media (e.g., molecular matter such as liquids), phenomena that depend nonlinearly on the electric and magnetic components of the intense laser beam. A large number of such phenomena have been observed in the past thirty years,[31] and the theory of such processes has been systematically developed by Kielich and coworkers,[32] following early inroads by Piekara and Kielich,[33] who were among the first to consider systematically statistical molecular theories of optically induced phenomena in isotropic dielectric and diamagnetic media. These earlier theories are, of course, formulated in terms of the transverse spacelike components of our covariant description, and should be modified to take into account the existence of $\mathbf{B}^{(3)}$ in free spacetime. These fields are expected to produce observable magnetization and polarization when they interact with matter. For laser beams that are intense enough, various optical saturation phenomena due to $\mathbf{B}^{(3)}$ should occur. A classic work such as the early paper by Kielich[34] on frequency and spatially variable electric and magnetic polarization induced in nonlinear media by electromagnetic fields should be covariantly developed, so that the Born-Infeld electrodynamics[35] to which it refers can be extended to include $\mathbf{B}^{(3)}$ within a manifestly covariant structure. Terms such as $\mathbf{E} \times \mathbf{E}^*$ in the work by Kielich[34] can be replaced by $\mathbf{B}^{(3)}$, for example, thus predicting birefringence effects proportional to the square root of intensity, in addition to the traditional effects proportional to intensity, such as the inverse Faraday effect.[36]

In another classic paper by Kielich,[37] on nonlinear processes resulting from multipole interaction between molecules and electromagnetic fields, it would be interesting to explore the role played by $\mathbf{B}^{(3)}$ in the various nonlinear optical phenomenon proposed in this work, for example, (1) a covariant reformulation of the Dirac theory to describe the absorption of a flux quantum $\hat{B}^{(3)}$; (2) a covariant scattering theory for $\mathbf{B}^{(3)}$; (3) the role of $\mathbf{B}^{(3)}$ in the nonlinear optical processes where linear superposition is lost; (4) investigations of the probability of an $n$ photon process with magnetic transitions involving an incoming $\hat{\mathbf{B}}^{(3)}$ flux quantum; (5) scattering theory involving the classical $\mathbf{B}^{(3)}$. Again, in the theory of nonlinear light scattering from colloidal media,[38] $\mathbf{B}^{(3)}$ is expected to play a basic part in defining the depolarization ratio, since, as we have seen, $\mathbf{B}^{(3)}$ is proportional to $I_R - I_L$. In general, in Rayleigh refringent scattering theory, the Stokes parameters in our covariant description enter in terms of $\mathbf{B}^{(3)}$, so that the longitudinal field is fundamental to any description. The role of $\mathbf{B}^{(3)}$ in the Majorana effect,[39] and intensity dependent optical circular birefringence[40] is also fundamental. The interesting phenomenon of ellipse self-rotation by a circularly polarized laser[41] is also fundamentally dependent on the longitudinal field $\mathbf{B}^{(3)}$.

More recently, the phenomena associated with light squeezing in quantum electrodynamics have become prevalent in the literature[42] and in this context Tanaś and Kielich[19] have systematically investigated the effect of squeezing on a large number of optical phenomena, including the effect on the four Stokes operators, the quantum equivalent of the four Stokes parameters.[43] It was deduced that the parameters $S_1$ and $S_2$ are in general affected by squeezing, and it would be interesting to develop this result in a manifestly covariant description, where classically, as we have seen, the four Stokes parameters are affected in basic structure, and new terms are added to $S_1$ and $S_2$. The field $\mathbf{B}^{(3)}$ also plays a role in light self-squeezing in Kerr media, discovered by Kielich et al.[44] and in general in all nonlinear quantum electrodynamics, fundamentally changing the structure of the theory.

For example, Frey et al.[45] have recently observed azimuth rotation due to an intense laser beam (the optical Faraday effect), and this has been shown by Farahi and Evans[5] to be a linear function of the square root of laser intensity, i.e., to be linearly dependent on the magnitude of $\mathbf{B}^{(3)}$. This is the first experimental evidence for the ability of $\mathbf{B}^{(3)}$ to magnetize a material, in this case a magnetic semiconductor.[45] Magnetization by a circularly polarized light beam has been observed as the inverse Faraday effect,[36] and recently, as laser-induced shifts in NMR spectra.[46] Light shifts in atomic spectra have also been observed experimentally[47] and can be reinterpreted in terms of $\mathbf{B}^{(3)}$. In general, a large number of phenomena can be reinterpreted in terms of longitudinal[48] fields in vacuo phenomena that are at present attributed solely to the transverse fields $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$. In theory, optical effects due to $\mathbf{B}^{(3)}$ can be identified and separated from the concommitant effects due to $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$, or $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$, because the former are expected to be proportional to the square root of laser intensity (and integral powers thereof), and the latter to even powers only of laser intensity.

## APPENDIX A: CARTESIAN AND CIRCULAR REPRESENTATIONS

The subscripts in the matrix in Eq. (22) are conventionally[8–15] given in the Cartesian basis, $(X, Y, Z)$, while circular polarization is described in the circular basis ((1), (2), (3)). Any physical phenomenon should be independent of the basis (i.e., laboratory frame of reference) used in its description, and in this paper the link between the two representations is given in terms of the following unit vector equations. Superscripts (1), (2), and (3) refer respectively to the first and second sense of transverse circular polarization, and the longitudinal polarization:

$$\hat{e}^{(1)} \equiv \frac{1}{\sqrt{2}}(\mathbf{i} - \mathrm{i}\mathbf{j}) \tag{A.1}$$

$$\hat{e}^{(2)} = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathrm{i}\mathbf{j}) \tag{A.2}$$

$$\hat{e}^{(3)} \equiv \mathbf{k} \tag{A.3}$$



where **i**, **j**, and **k** are Cartesian unit vectors in $X$, $Y$, and $Z$, respectively. Thus,

$$\hat{\mathbf{e}}^{(1)} \times \hat{\mathbf{e}}^{(2)} = \mathrm{i}\hat{\mathbf{e}}^{(3)} = \mathrm{i}\mathbf{k} \tag{A.4}$$

The circular basis is used in Eq. (34) to define $E_\mu$ and $B_\mu$ in terms of polarizations (0), (1), (2), and (3), which are respectively timelike, transverse circular spacelike (1) and (2), and longitudinal spacelike. In Eq. (33), $F_{\nu\rho}$ is accordingly defined in the circular basis. In Appendix B, however, the explicit demonstration of Eqs. (31) is carried out in the Cartesian basis. Equations (31) are, of course, valid in any frame of reference fixed in the laboratory. The longitudinal spacelike and timelike components are the same in the Cartesian and circular basis, while the transverse components can be interrelated with Eqs. (A.1) and (A.2). Equation (22) has been obtained from a Cartesian representation of $A_\mu$, the four potential, using a four curl, Eq. (23), in the Cartesian frame for the spacelike components.

## APPENDIX B: EXPLICIT DEMONSTRATION OF EQUATIONS (31)

In this Appendix we provide an explicit demonstration of the self-consistency of Eqs. (31), both for the $B_\mu$ and $E_\mu$ vectors, because these equations form the basis of our manifestly covariant theory of vacuum electrodynamics. From Eqs. (31a) and the definition of $\delta_\sigma$ in Eq. (30),

$$cB_1 = -\frac{\mathrm{i}}{2\varepsilon_0}(\varepsilon_{1230}F_{23}\delta_0 + \varepsilon_{1320}F_{32}\delta_0 + \varepsilon_{1203}F_{20}\delta_3 + \varepsilon_{1023}F_{02}\delta_3) \tag{B.1}$$

$$cB_2 = -\frac{\mathrm{i}}{2\varepsilon_0}(\varepsilon_{2310}F_{31}\delta_0 + \varepsilon_{2130}F_{13}\delta_0 + \varepsilon_{2013}F_{01}\delta_3 + \varepsilon_{2103}F_{10}\delta_3) \tag{B.2}$$

$$cB_3 = -\frac{\mathrm{i}}{2\varepsilon_0}(\varepsilon_{3210}F_{21}\delta_0 + \varepsilon_{3120}F_{12}\delta_0) \tag{B.3}$$

$$-\mathrm{i}cB_0 = -\frac{\mathrm{i}}{2\varepsilon_0}(\varepsilon_{0123}F_{12}\delta_3 + \varepsilon_{0213}F_{21}\delta_3) \tag{B.4}$$

with

$$\delta_1 = \delta_2 = 0 \qquad \delta_3 = 1 \qquad \delta_0 = -\mathrm{i}$$

$$\begin{array}{llll} \varepsilon_{0123} = 1 & \varepsilon_{0213} = -1 & \varepsilon_{3120} = -1 & \varepsilon_{3210} = 1 \\ \varepsilon_{2130} = 1 & \varepsilon_{2310} = -1 & \varepsilon_{2013} = 1 & \varepsilon_{2103} = -1 \\ \varepsilon_{1230} = -1 & \varepsilon_{1320} = 1 & \varepsilon_{1203} = 1 & \varepsilon_{1023} = -1 \end{array}$$

$$\begin{array}{ll} F_{23} = c\varepsilon_0 B_X = -F_{32} & F_{20} = -\mathrm{i}\varepsilon_0 E_Y = -F_{02} \\ F_{31} = c\varepsilon_0 B_Y = -F_{13} & F_{01} = \mathrm{i}\varepsilon_0 E_X = -F_{10} \\ F_{21} = -c\varepsilon_0 B_Z = -F_{12} & \end{array}$$

For one sense of circular polarization, we have

$$E_X = \frac{E_0}{\sqrt{2}} \qquad E_Y = -\frac{\mathrm{i}E_0}{\sqrt{2}} \qquad B_X = \frac{\mathrm{i}B_0}{\sqrt{2}} \qquad B_Y = \frac{B_0}{\sqrt{2}} \tag{B.5}$$

$$cB_Y = E_X \qquad cB_X = -E_Y \tag{B.6}$$

so that in Eqs. (B.1) and (B.2),

$$\begin{aligned} cB_1 &= cB_X - E_Y = 2cB_X \\ cB_2 &= cB_Y + E_X = 2cB_Y \end{aligned} \tag{B.7}$$

and so the left sides become magnetic components in vacuo with the vacuum relation $E_0 = cB_0$.

Similarly, the dual of $F_{\mu\nu}$ is the four tensor[8–15],

$$\tilde{F}_{\mu\nu} \equiv -\varepsilon_0 \begin{bmatrix} 0 & \mathrm{i}E_Z & -\mathrm{i}E_Y & -cB_X \\ -\mathrm{i}E_Z & 0 & \mathrm{i}E_X & -cB_Y \\ \mathrm{i}E_Y & -\mathrm{i}E_X & 0 & -cB_Z \\ cB_X & cB_Y & cB_Z & 0 \end{bmatrix} \tag{B.8}$$

and in Eq. (31b),

$$E_1 = \tfrac{1}{2}\left(\varepsilon_{1230}\tilde{F}_{23}\delta_0 + \varepsilon_{1320}\tilde{F}_{32}\delta_0 + \varepsilon_{1203}\tilde{F}_{20}\delta_3 + \varepsilon_{1023}\tilde{F}_{02}\delta_3\right) \tag{B.9}$$

$$E_2 = \tfrac{1}{2}\left(\varepsilon_{2310}\tilde{F}_{31}\delta_0 + \varepsilon_{2130}\tilde{F}_{13}\delta_0 + \varepsilon_{2013}\tilde{F}_{01}\delta_3 + \varepsilon_{2103}\tilde{F}_{10}\delta_3\right) \tag{B.10}$$

$$E_3 = +\tfrac{1}{2}\left(\varepsilon_{3210}\tilde{F}_{21}\delta_0 + \varepsilon_{3120}\tilde{F}_{12}\delta_0\right) \tag{B.11}$$

$$-\mathrm{i}E_0 = -\tfrac{1}{2}\left(\varepsilon_{0123}\tilde{F}_{12}\delta_3 + \varepsilon_{0213}\tilde{F}_{21}\delta_3\right) \tag{B.12}$$



With the relations (B.5) and (B.6) it can be shown that the components in Eqs. (B.9) and (B.10) are the electric components $2E_X$ and $2E_Y$, with

$$\tilde{F}_{12} = -\tilde{F}_{21} = \mathrm{i}\varepsilon_0 E_Z \qquad \tilde{F}_{13} = -\tilde{F}_{31} = -\mathrm{i}\varepsilon_0 E_Y$$

$$\tilde{F}_{10} = -\tilde{F}_{01} = -c\varepsilon_0 B_X \qquad \tilde{F}_{23} = -\tilde{F}_{32} = \mathrm{i}\varepsilon_0 E_X$$

Similarly, it may be checked explicitly that

$$\begin{aligned} B_\mu \delta_\mu &\equiv |B_1|\,|\delta_1| + |B_2|\,|\delta_2| + |B_3|\,|\delta_3| - |B_0|\,|\delta_0| \\ &= 0 + 0 + B_Z - B_Z \\ &= 0 \end{aligned}$$

## APPENDIX C: THE ELECTRODYNAMICAL ENERGY DENSITY AND TIME-AVERAGED ENERGY DENSITY, OR INTENSITY, $I_0$

Equation (38) produces the free spacetime result

$$E_\mu E_\mu = 0 \tag{C.1}$$

This is interpreted to mean that the scalar product of the two four vectors $E_\mu$ and $E_\mu$ is zero in the lightlike condition. In the conventional theory[8–15] the equivalent of Eq. (C.1) is

$$\mathbf{E} \cdot \mathbf{E} = 0 \tag{C.2}$$

Equations (C.1) and (C.2) do not mean, however, that the time-averaged electromagnetic energy density $I_0$ is zero in vacuo. The quantity $I_0$ (W m$^{-2}$) is defined covariantly by

$$\begin{aligned} I_0 &= \varepsilon_0 c E^{(0)2} \\ &\equiv \tfrac{1}{2}\varepsilon_0 c E_\mu E_\mu^* \end{aligned} \tag{C.3}$$

where $E_\mu^*$ is the complex conjugate of $E_\mu$ in vacuo. Explicitly,

$$\begin{aligned} E_\mu &\equiv (E^{(1)}, E^{(2)}, E^{(3)}, -\mathrm{i}E^{(0)}) \\ E_\mu^* &\equiv (E^{(1)*}, E^{(2)*}, E^{(3)}, -\mathrm{i}E^{(0)}) \end{aligned} \tag{C.4}$$

and

$$E_\mu E_\mu^* = \frac{E^{(0)2}}{2}((\mathbf{i} - \mathrm{i}\mathbf{j}) \cdot (\mathbf{i} + \mathrm{i}\mathbf{j}) + (\mathbf{i} + \mathrm{i}\mathbf{j}) \cdot (\mathbf{i} - \mathrm{i}\mathbf{j})) \tag{C.5}$$
$$= 2E^{(0)2}$$

This result shows that $E_\mu E_\mu^*$ is covariantly described because it is a constant in free spacetime. Equation (C.3) is known as the time-averaged energy density[8–15] or beam *intensity*. This is invariant to Lorentz transformation and is a scalar quantity. Note that although $E_\mu^*$ is defined in Eq. (C.4) as the complex conjugate of $E_\mu$, the sign of the timelike component $-\mathrm{i}E^{(0)}$ does not change, because the operation $E_\mu \to E_\mu^*$ takes place in a fixed frame of reference $(X, Y, Z, -\mathrm{i}ct)$ in pseudo-Euclidean spacetime. Finally, $E^{(3)}$ is defined as having no imaginary part, and is invariant under $E_\mu \to E_\mu^*$. Thus, $E^{(3)}$ and $-\mathrm{i}E^{(0)}$ do not contribute to $I_0$.

## APPENDIX D: SIMPLE LORENTZ TRANSFORMATION OF $E_\mu$ AND $B_\mu$

The simple Lorentz transformation of the four vector $E_\mu$ is given covariantly by

$$E'^{(0)} = E^{(0)}\xi; \qquad E'^{(3)} = E^{(3)}\xi \qquad E^{(2)} = E^{(2)\prime} \qquad E^{(1)} = E^{(1)\prime} \tag{D.1}$$

and, similarly, the transformation of $B_\mu$ is

$$B'^{(0)} = B^{(0)}\xi \qquad B'^{(3)} = B^{(3)}\xi \qquad B^{(2)} = B^{(2)\prime} \qquad B^{(1)} = B^{(1)\prime} \tag{D.2}$$

The transform is from the covariantly defined frame $(X, Y, Z, -\mathrm{i}ct)$ to $(X', Y', Z', -\mathrm{i}ct')$; which translates along $Z$ at speed $v$ relative to the former. In Eqs. (D.1) and (D.2),

$$\xi = \frac{1 - v/c}{(1 - v^2/c^2)^{1/2}} \tag{D.3}$$

This is referred to as a simple Lorentz transformation because there is no rotation and no translation generator considered. In other words, the origin of frame $(X, Y, Z, -\mathrm{i}ct)$ does not translate, and no rotations are considered in spacetime. For the electromagnetic plane wave in vacuo,



$v = c$, and Eqs. (D.1) and (D.2) give

$$E^{(0)} = E^{(0)\prime} \qquad B^{(0)} = B^{(0)\prime}$$
$$E^{(3)} = E^{(3)\prime} \qquad B^{(3)} = B^{(3)\prime} \tag{D.4}$$

which confirm that the equations

$$|\mathbf{E}^{(3)}| = E^{(0)} \qquad |\mathbf{B}^{(3)}| = B^{(0)} \tag{D.5}$$

are invariant to the simple Lorentz transformation. The results (D.1) to (D.4) confirm that the $E_\mu$ and $B_\mu$ fields are the same for all $v$, because $v = c$ in vacuo, and $c$ is the universal constant of special relativity. Therefore, all four components of both $E_\mu$ and $B_\mu$ are formally invariant to the simple Lorentz transformation.

It is important to note that this result is fully consistent with, but contains additional information compared with, the standard approach,[12] which applies the simple Lorentz transformation to the four potential vector $A_\mu$ and to the second rank tensor $F_{\mu\nu}$. In S.I. units the standard approach gives the well-known result

$$E'_X = \frac{E_X - vB_Y}{(1 - v^2/c^2)^{1/2}}$$
$$E'_Y = \frac{E_Y + vB_X}{(1 - v^2/c^2)^{1/2}} \tag{D.6}$$
$$E'_Z = E_Z$$

and using the free space relations

$$cB_Y = E_X \qquad cB_X = -E_Y \tag{D.7}$$

we obtain

$$E'_X = \xi E_X \qquad E'_Y = \xi E_Y \qquad E_Z = E'_Z \tag{D.8}$$

For $v = 0$, these equations show that the three spacelike components of $E_\mu$ (and of $B_\mu$) are separately invariant to Lorentz transformation, but say nothing about the timelike component $\mathbf{E}^{(0)}$ or its relation to $\mathbf{E}^{(3)}$. For this, a more complete theory, as in this paper, is needed. Since the simple Lorentz transformation does not involve the generator of translations, it is an incomplete description of the properties of the electromagnetic field.

The photon is never at rest, but being massless, always moves at the velocity of light $c$, implying that the origin of frame $(X, Y, Z, -ict)$ also moves at $c$. The generator of spacetime translations is automatically required, therefore, for a description of the photon, since the latter always translates at $c$ in any frame of reference. Since $c$ is a universal constant, the assumption that there is a frame $(X', Y', Z', -ict')$ which moves at $v$ relative to $(X, Y, Z, -ict)$ conflicts with Einstein's second principle. In other words it is not possible for the photon to define a frame moving at a speed $v$ relative to one that is moving at speed $c$.

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